Volume 10,Number 1 January 2008
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

A quarterly international publication of Eudoxus Press, LLC

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,7-15,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

ON INVARIANT APPROXIMATION FOR NONCOMMUTATIVE MAPPINGS IN LOCALLY CONVEX SPACES

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ABSTRACT

The aim of this paper is to generalize the results related to invariant approximation by weakening commutativity hypothesis and by increasing the number of mappings involved.

2000 Mathematics Subject Classification. 41A50, 46A03, 47H10.

Key Words and Phrases: Best approximant, common fixed points, commutating mappings, compatible mapping, demiclosed mapping, locally convex space.

1. INTRODUCTION

During the last four decades several interesting and valuable results as an application of fixed point theorems were studied extensively in the field of an invariant approximation theory. An excellent reference can be seen in [16].

Meinardus [9] was the first who employed a fixed-point theorem to establish the existence of an invariant approximation. Further, Brosowski [1] obtained a celebrated result and generalized the Meinardus's result. Afterwards, a number of results exist has been proved in the direction of Brosowski [1] (see in [3, 12, 13, 17]). In a paper, Jungck and Sessa [5] further weakened the hypothesis of Sahab, Khan and Sessa [12] by replacing the weak and strong topology for relatively nonexpansive commutative maps.

Recently, Nashine [10] obtained invariant approximation results for a class of contraction commutative three mappings in locally convex space and he extended all the previous related results. Some works on best approximation for weakly commutative mappings in locally convex spaces are done by Khan and Hussain [7, 11].

In [4] the concept of compatible mappings was introduced as a generalization of commuting mappings. The purpose of this paper is to further emulate the compatible mapping concept. We extend the result of Nashine [10] by employing compatible mappings in lieu of commuting mappings, and by using four mappings as opposed to three in the setup of in locally convex space. To achieve our goal, we use the concept given by Köthe [8], Tarafdar [18] and the result of Jungck [5]. In this way, we give new direction to the line of investigation given by Brosowski [1]. Finally, we derive some consequences from our main result.

2. PRELIMINARIES

Before we prove our main result, let us recall following definitions:

Definition 2.1. [8, 7, 11]. In the sequel $(\mathcal{E},\tau)$ will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in \mathcal{I}\}$ of seminorms defined on $\mathcal{E}$ is said to be an associated family of seminorms for τ if the family $\{\gamma\mathcal{U} : \gamma > 0\}$, where $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_{\alpha_i}$ and $\mathcal{U}_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighbourhood of zero for τ. A family $\{p_\alpha : \alpha \in \mathcal{I}\}$ of seminorms defined on $\mathcal{E}$ is called an augmented associated family for τ if $\{p_\alpha : \alpha \in \mathcal{I}\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in \mathcal{I}\}$ for any $\alpha, \beta \in \mathcal{I}$. The associated and augmented families of seminorms will be denoted by $\mathcal{A}(\tau)$ and $\mathcal{A}^*(\tau)$, respectively. It is well known that if given a locally convex space $(\mathcal{E},\tau)$, there always exists a family $\{p_\alpha : \alpha \in \mathcal{I}\}$ of seminorms defined of $\mathcal{E}$ such that $\{p_\alpha : \alpha \in \mathcal{I}\} = \mathcal{A}^*(\tau)$. A subset $\mathcal{M}$ of $\mathcal{E}$ is τ-bounded if and only if each p_α is bounded on $\mathcal{M}$.

The following construction will be crucial. Suppose that $\mathcal{M}$ is a τ-bounded subset of $\mathcal{E}$. For this set $\mathcal{M}$, we can select a number $\lambda_\alpha > 0$ for each $\alpha \in \mathcal{I}$ such that $\mathcal{M} \subset \lambda_\alpha \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \{x : p_\alpha(x) \le 1\}$. Clearly, $\mathcal{B} = \bigcap_\alpha \lambda_\alpha \mathcal{U}_\alpha$ is τ- bounded, τ-closed, absolutely convex and contains $\mathcal{M}$. The linear span $\mathcal{E}_\mathcal{B}$ of $\mathcal{B}$ in $\mathcal{E}$ is $\bigcup_{n=1}^{\infty} n\mathcal{B}$. The Minkowski functional of $\mathcal{B}$ is a norm $\|.\|_\mathcal{B}$ on $\mathcal{E}_\mathcal{B}$. Thus, $(\mathcal{E}_\mathcal{B}, \|.\|_\mathcal{B})$ is a normed space with $\mathcal{B}$ as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_\mathcal{B}$ for each $x \in \mathcal{E}_\mathcal{B}$.

Definition 2.2. Let $\mathcal{I}$ and $\mathcal{T}$ be selfmaps on $\mathcal{M}$. The map $\mathcal{T}$ is called

(i) $\mathcal{A}^*(\tau)$-nonexpansive if for all $x, y \in \mathcal{M}$

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \le p_\alpha(x - y),$$

for each $p_\alpha \in A^*(\tau)$.

(ii)$\mathcal{A}^*(\tau)$-$\mathcal{I}-$nonexpansive if for all $x, y \in \mathcal{M}$

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \le p_\alpha(\mathcal{I}x - \mathcal{I}y),$$

for each $p_\alpha \in \mathcal{A}^*(\tau)$.

For simplicity, we shall call $\mathcal{A}^*(\tau)$-nonexpansive ($\mathcal{A}^*(\tau) - \mathcal{I}$- nonexpansive) maps to be nonexpansive ($\mathcal{I}-$nonexpansive).

Following the concept of compatible due to Jungck [4], we have

Definition 2.3. [4]. A pair of self-mappings $(\mathcal{T},\mathcal{I})$ of a locally convex space $(\mathcal{E},\tau)$ is said to be compatible, if $p_\alpha(\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{E}$ such that $\mathcal{T}x_n, \mathcal{I}x_n \to t \in \mathcal{E}$.

Every commuting pair of mappings is compatible but the converse is not true in general.

Definition 2.4. [10]. Let $x_0 \in \mathcal{M}$. We denote by $\mathcal{P}_{\mathcal{M}}(x_0)$ the set of best $\mathcal{M}-$ approximant to x_0, i.e., if $\mathcal{P}_{\mathcal{M}}(x_0) = \{y \in \mathcal{M} : p_\alpha(y - x_0) = d_{p_\alpha}(x_0, \mathcal{M})$ for all $p_\alpha \in \mathcal{A}^*(\tau)\}$, where

$$d_{p_\alpha}(x_0, \mathcal{M}) = \inf\{p_\alpha(x_0 - z) : z \in \mathcal{M}\}.$$

Definition 2.5. [10]. The map $\mathcal{T} : \mathcal{M} \to \mathcal{E}$ is said to be demiclosed at 0 if for every net $\{x_n\}$ in $\mathcal{M}$ converging weakly to x and $\{\mathcal{T}x_n\}$ converging strongly to 0, we have $\mathcal{T}x = 0$.

Throughout this paper $\mathcal{F}(\mathcal{T})$(resp.$\mathcal{F}(\mathcal{I})$) denotes the set of fixed point of mapping $\mathcal{T}$(resp.$\mathcal{I}$).

The following result of Jungck [5] is needed in the sequel:

Theorem 2.6. [5]. *Let $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ be continuous self mappings of a compact metric space $(\mathcal{X}, d)$ with $\mathcal{A}(\mathcal{X}) \subseteq \mathcal{J}(\mathcal{X})$ and $\mathcal{S}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$. If $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{S}, \mathcal{J})$ are compatible pairs and satisfying*

$$d(\mathcal{A}x, \mathcal{S}y) < \max\{d(\mathcal{I}x, \mathcal{J}y), d(\mathcal{I}x, \mathcal{A}x), d(\mathcal{J}y, \mathcal{S}y), \frac{1}{2}[d(\mathcal{I}x, \mathcal{S}y)+d(\mathcal{J}y, \mathcal{A}x)]\} > 0,$$

then $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ have a unique common fixed point in $\mathcal{X}$.

3. MAIN RESULT

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{I}$ be compatible self-maps of a τ-bounded subset $\mathcal{M}$ of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Then $\mathcal{A}$ and $\mathcal{I}$ be two compatible on $\mathcal{M}$ with respect to $\|.\|_{\mathcal{B}}$.*

Proof. By hypothesis for each $p_\alpha \in \mathcal{A}^*(\tau)$,

$$p_\alpha(\mathcal{A}\mathcal{I}x_n - \mathcal{I}\mathcal{A}x_n) \to 0, \tag{3.1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that $\mathcal{A}x_n, \mathcal{I}x_n \to t \in \mathcal{M}$.

Taking supremum on both sides, we get

$$\sup_\alpha p_\alpha\left(\frac{\mathcal{A}\mathcal{I}x_n - \mathcal{I}\mathcal{A}x_n}{\lambda_\alpha}\right) \to 0$$

$$\|\mathcal{A}\mathcal{I}x_n - \mathcal{I}\mathcal{A}x_n\|_{\mathcal{B}} \to 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that $\mathcal{A}x_n, \mathcal{I}x_n \to t \in \mathcal{M}$. $\square$

We use a technique of Tarafdar [18] to obtain the following common fixed point theorems which generalize Theorem 2.6.

Theorem 3.2. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially compact subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{J}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$. If $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{S}, \mathcal{J})$ are compatible pairs, $\mathcal{A}$ and $\mathcal{S}$ are continuous, $\mathcal{I}$ and $\mathcal{J}$ are nonexpansive, and satisfying*

$$p_\alpha(\mathcal{A}x - \mathcal{S}y) < \mathbf{N}(x, y) \tag{3.2}$$

where

$$\mathbf{N}(x, y) = \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}y), \frac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}y) + p_\alpha(\mathcal{J}y - \mathcal{A}x)]\}$$

for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, then $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ have a unique common fixed point in $\mathcal{M}$.

Proof. Since the norm topology on $\mathcal{E}_\mathcal{B}$ has a base of neighbourhood of zero consisting of τ-closed sets and $\mathcal{M}$ is τ-sequentially compact, therefore, $\mathcal{M}$ is a $\|.\|_\mathcal{B}$-sequentially compact subset of $(\mathcal{E}_\mathcal{B}, \|.\|_\mathcal{B})$ (Theorem 1.2, [18]). By Lemma 3.1, $\mathcal{A}$ and $\mathcal{I}$ are $\|.\|_\mathcal{B}$-compatible maps of $\mathcal{M}$. Similarly, by the Lemma 3.1, $\mathcal{S}$ and $\mathcal{J}$ are $\|.\|_\mathcal{B}$-compatible maps of $\mathcal{M}$. From (3.2) we obtain for $x, y \in \mathcal{M}$,

$$\sup_\alpha p_\alpha(\tfrac{\mathcal{A}x - \mathcal{S}y}{\lambda_\alpha}) < \max\{\sup_\alpha p_\alpha(\tfrac{\mathcal{I}x - \mathcal{J}y}{\lambda_\alpha}), \sup_\alpha p_\alpha(\tfrac{\mathcal{I}x - \mathcal{A}x}{\lambda_\alpha}), \sup_\alpha p_\alpha(\tfrac{\mathcal{J}y - \mathcal{S}y}{\lambda_\alpha}), \frac{1}{2}[\sup_\alpha p_\alpha(\tfrac{\mathcal{I}x - \mathcal{S}y}{\lambda_\alpha}) + \sup_\alpha p_\alpha(\tfrac{\mathcal{J}y - \mathcal{A}x}{\lambda_\alpha})]\}.$$

Thus

$$\|\mathcal{A}x - \mathcal{S}y\|_\mathcal{B} < \max\{\|\mathcal{I}x - \mathcal{J}y\|_\mathcal{B}, \|\mathcal{I}x - \mathcal{A}x\|_\mathcal{B}, \|\mathcal{J}y - \mathcal{S}y\|_\mathcal{B}, \frac{1}{2}[\ \|\mathcal{I}x - \mathcal{S}y\|_\mathcal{B} + \|\mathcal{J}y - \mathcal{A}x\|_\mathcal{B}]\}. \tag{3.3}$$

Note that, if $\mathcal{I}$ and $\mathcal{J}$ are nonexpansive on a τ-bounded, τ-sequentially compact subset $\mathcal{M}$ of $\mathcal{E}$, then $\mathcal{I}$ and $\mathcal{J}$ are also nonexpansive with respect to $\|.\|_\mathcal{B}$ and hence $\|.\|_\mathcal{B}$-continuous ([8]). A comparison of our hypothesis with that of Theorem 2.6 tells that we can apply Theorem 2.6 to $\mathcal{M}$ as a subset of $(\mathcal{E}_\mathcal{B}, \|.\|_\mathcal{B})$ to conclude that there exists a unique $v \in \mathcal{M}$ such that $v = \mathcal{A}v = \mathcal{S}v = \mathcal{I}v = \mathcal{J}v$. □

Theorem 3.3. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and $p-$starshaped subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{J}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$. Suppose $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{S}, \mathcal{J})$ are compatible pairs, $\mathcal{A}$ and $\mathcal{S}$ are continuous, $\mathcal{I}$ and $\mathcal{J}$ are nonexpansive, and affine, $\mathcal{I}(\mathcal{M}) = \mathcal{M} = \mathcal{J}(\mathcal{M})$, $p \in \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$. If $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ satisfy the following:*

$$p_\alpha(\mathcal{A}x - \mathcal{S}y) < \mathbf{N}(x, y) \tag{3.4}$$

where

$$\mathbf{N}(x, y) = \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}y), \frac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}y) + p_\alpha(\mathcal{J}y - \mathcal{A}x)]\}$$

for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, then $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ have a common fixed point in $\mathcal{M}$ provided one of the following conditions hold:

(i) *$\mathcal{M}$ is τ-sequentially compact;*

(ii) *$\mathcal{M}$ is weakly compact in $(\mathcal{E}, \tau)$, $\mathcal{I}$ and $\mathcal{J}$ are weakly continuous and $\mathcal{I} - \mathcal{A}$ and $\mathcal{J} - \mathcal{S}$ are demiclosed at 0.*

Proof. Choose a monotonically nondecreasing sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. For each $n \in \mathbb{N}$, define $\mathcal{A}_n, \mathcal{S}_n : \mathcal{M} \to \mathcal{M}$ as follows:

$$\mathcal{A}_n x = k_n \mathcal{A}x + (1 - k_n)p, \quad \mathcal{S}_n x = k_n \mathcal{S}x + (1 - k_n)p. \tag{3.5}$$

Obviously, for each n, $\mathcal{A}_n$ and $\mathcal{S}_n$ map $\mathcal{M}$ into itself since $\mathcal{M}$ is $p-$starshaped. As $\mathcal{I}$ is affine, $(\mathcal{A}, \mathcal{I})$ is compatible and $p \in \mathcal{F}(\mathcal{I})$, so

$$\mathcal{A}_n \mathcal{I}x = k_n \mathcal{A}\mathcal{I}x + (1 - k_n)p$$

$$\mathcal{I}\mathcal{A}_n x_n = \mathcal{I}(k_n \mathcal{A}x + (1 - k_n)p = k_n \mathcal{I}\mathcal{A}x + (1 - k_n)\mathcal{I}p.$$

Since $(\mathcal{A}, \mathcal{I})$ is compatible, therefore

$$\begin{aligned} 0 &\le \lim_n p_\alpha(\mathcal{A}_m \mathcal{I}x_n - \mathcal{I}\mathcal{A}_m x_n) \\ &\le \lim_n p_\alpha(\mathcal{A}\mathcal{I}x_n - \mathcal{I}\mathcal{A}x_n) + \lim_n (1 - k_m) p_\alpha(p - \mathcal{I}p) \\ &= 0, \end{aligned}$$

whenever $\lim_n \mathcal{I}_n x_n = \lim_n \mathcal{A}x_n = t \in \mathcal{M}$ for all n, for each $x \in \mathcal{M}$. Hence $\{\mathcal{A}_n\}$ and $\mathcal{I}$ are compatible for each n and $\mathcal{A}_n(\mathcal{M}) \subseteq \mathcal{M} = \mathcal{J}(\mathcal{M})$. Similarly, we can prove $\mathcal{S}_n$ and $\mathcal{J}$ are compatible for each n and $\mathcal{S}_n(\mathcal{M}) \subseteq \mathcal{M} = \mathcal{I}(\mathcal{M})$.

For all $x, y \in \mathcal{M}, p_\alpha \in \mathcal{A}^*(\tau)$ and for all $j \ge n$, (n fixed), we obtain from (3.4) and (3.10) that

$$\begin{aligned} p_\alpha(\mathcal{A}_n x - \mathcal{S}_n y) &= k_n p_\alpha(\mathcal{A}x - \mathcal{S}y) \le k_j p_\alpha(\mathcal{A}x - \mathcal{S}y) \\ &\le p_\alpha(\mathcal{A}x - \mathcal{S}y) \\ &< \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}y), \\ &\qquad \tfrac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}y) + p_\alpha(\mathcal{J}y - \mathcal{A}x)]\} \\ &< \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}_n x) + p_\alpha(\mathcal{A}_n x - \mathcal{A}x), \\ &\qquad p_\alpha(\mathcal{J}y - \mathcal{S}_n y) + p_\alpha(\mathcal{S}_n y - \mathcal{S}y), \\ &\qquad \tfrac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}_n y) + p_\alpha(\mathcal{S}_n y - \mathcal{S}y) + \\ &\qquad p_\alpha(\mathcal{J}y - \mathcal{A}_n x) + p_\alpha(\mathcal{A}_n x - \mathcal{A}x)]\} \end{aligned}$$

$$< \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}_n x) + (1-k_n)p_\alpha(p - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}_n y) + (1-k_n)p_\alpha(p - \mathcal{J}y), \tfrac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}_n y) + (1-k_n)p_\alpha(p - \mathcal{S}y) + p_\alpha(\mathcal{J}y - \mathcal{A}_n x) + (1-k_n)p_\alpha(p - \mathcal{A}x)]\}.$$

Hence for all $j \geq n$, we have

(3.6)

$$p_\alpha(\mathcal{A}_n x - \mathcal{S}_n y) < \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}_n x) + (1-k_j)p_\alpha(p - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}_n y) + (1-k_j)p_\alpha(p - \mathcal{S}y), \tfrac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}_n y) + (1-k_j)p_\alpha(p - \mathcal{S}y) + p_\alpha(\mathcal{J}y - \mathcal{A}_n x) + (1-k_j)p_\alpha(p - \mathcal{A}x)]\}.$$

As $\lim k_j = 1$, from (3.6), for every $n \in \mathbb{N}$, we have

(3.7)

$$\begin{aligned} p_\alpha(\mathcal{A}_n x - \mathcal{S}_n y) &= \lim_j p_\alpha(\mathcal{A}_n x - \mathcal{S}_n y) \\ &< \lim_j\{\max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}_n x) + (1-k_j)p_\alpha(p - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}_n y) \\ &\quad + (1-k_j)p_\alpha(p - \mathcal{S}y), \tfrac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}_n y) + (1-k_j)p_\alpha(p - \mathcal{S}y) + p_\alpha(\mathcal{J}y - \mathcal{A}_n x) + (1-k_j)p_\alpha(p - \mathcal{A}x)]\}. \end{aligned}$$

This implies that for every $n \in \mathbb{N}$,

(3.8)

$$p_\alpha(\mathcal{A}_n x - \mathcal{S}_n y) < \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}_n x), p_\alpha(\mathcal{J}y - \mathcal{S}_n y), \tfrac{1}{2}[p_\alpha(\mathcal{I}x - \mathcal{S}_n y) + p_\alpha(\mathcal{J}y - \mathcal{A}_n x)]\},$$

for all $x, y \in \mathcal{M}$ and for all $p_\alpha \in \mathcal{A}^*(\tau)$.

Moreover, $\mathcal{I}$ and $\mathcal{J}$ being nonexpansive on $\mathcal{M}$, implies that $\mathcal{I}$ and $\mathcal{J}$ are $\|.\|_\mathcal{B}$-nonexpansive and, hence, $\|.\|_\mathcal{B}$-continuous. Since the norm topology on $\mathcal{E}_\mathcal{B}$ has a base of neighbourhood of zero consisting of τ-closed sets and $\mathcal{M}$ is τ-sequentially complete, therefore, $\mathcal{M}$ is a $\|.\|_\mathcal{B}$-sequentially complete subset of

$(\mathcal{E}_{\mathcal{B}}, \|.\|_{\mathcal{B}})$ (see proof of Theorem 1.2 in [18]). Thus from Theorem 3.2 with the condition (a) or (b), for every $n \in \mathbb{N}$, $\mathcal{A}_n, \mathcal{S}_n$, $\mathcal{I}$ and $\mathcal{J}$ have unique common fixed point x_n in $\mathcal{M}$, i.e.,

$$x_n = \mathcal{A}_n x_n = \mathcal{S}_n x_n = \mathcal{I} x_n = \mathcal{J} x_n, \tag{3.9}$$

for each $n \in \mathbb{N}$.

(i) As $\mathcal{M}$ is τ-sequentially compact and $\{x_n\}$ is a sequence in $\mathcal{M}$, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to y \in \mathcal{M}$. As $\mathcal{I}$ and $\mathcal{S}, \mathcal{T}$ are continuous and

$$x_m = \mathcal{I} x_m = \mathcal{A}_m x_m = k_m \mathcal{A} x_m + (1 - k_m)p,$$

$$x_m = \mathcal{J} x_m = \mathcal{S}_m x_m = k_m \mathcal{S} x_m + (1 - k_m)p,$$

so it follows that $y = \mathcal{T} y = \mathcal{S} y = \mathcal{I} y = \mathcal{J} y$.

(ii) The sequence $\{x_n\}$ has a subsequence $\{x_m\}$ converges to $u \in \mathcal{M}$. Since $\mathcal{I}$ is weakly continuous and so as in (i), we have $\mathcal{I} u = u$. Now,

$$x_m = \mathcal{I} x_m = \mathcal{A}_m x_m = k_m \mathcal{A} x_m + (1 - k_m)p$$

implies that

$$\mathcal{I} x_m - \mathcal{A} x_m = (1 - k_m)[p - \mathcal{A} x_m] \to 0$$

as $m \to \infty$. The demiclosedness of $\mathcal{I} - \mathcal{A}$ at 0 implies that $(\mathcal{I} - \mathcal{A})u = 0$. Hence $\mathcal{I} u = u = \mathcal{A} u$. Similarly, we can show $\mathcal{S} u = u = \mathcal{J} u$,when $\mathcal{J} - \mathcal{S}$ is demiclosed at 0. This completes the proof.

□

An immediate consequence of the Theorem 3.3 is as follows:

Corollary 3.4. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and $p-$starshaped subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{J}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$. Suppose $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{S}, \mathcal{J})$ are compatible pairs, $\mathcal{A}$ and $\mathcal{S}$ are continuous, $\mathcal{I}$ and $\mathcal{J}$ are nonexpansive, and affine, $\mathcal{I}(\mathcal{M}) = \mathcal{M} = \mathcal{J}(\mathcal{M})$, $p \in \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$. If $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ satisfy the following:*

$$p_\alpha(\mathcal{A}x - \mathcal{S}y) < \mathbf{N}(x, y) \tag{3.10}$$

where

$$\mathbf{N}(x, y) = \max\{p_\alpha(\mathcal{I}x - \mathcal{J}y), p_\alpha(\mathcal{I}x - \mathcal{A}x), p_\alpha(\mathcal{J}y - \mathcal{S}y),$$
$$\tfrac{1}{2}p_\alpha(\mathcal{I}x - \mathcal{S}y), \tfrac{1}{2}p_\alpha(\mathcal{J}y - \mathcal{A}x)\}$$

for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, then $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ have a common fixed point in $\mathcal{M}$ under each of the conditions $(i) - (ii)$ of Theorem 3.3.

An immediate consequence of the Theorem 3.3 and Corollary 3.4 is as follows:

Corollary 3.5. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and $p-$starshaped subset of a Hausdorff locally convex space $(\mathcal{E},\tau)$. Let $\mathcal{A},\mathcal{S},\mathcal{I}$ and $\mathcal{J}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subseteq \mathcal{J}(\mathcal{M})$ and $\mathcal{S}(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$. Suppose $\mathcal{A},\mathcal{I},\mathcal{S}$ and $\mathcal{J}$ are commutative, $\mathcal{A}$ and $\mathcal{S}$ are continuous, $\mathcal{I}$ and $\mathcal{J}$ are non-expansive, and affine, $\mathcal{I}(\mathcal{M}) = \mathcal{M} = \mathcal{J}(\mathcal{M})$, $p \in \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$. If $\mathcal{A},\mathcal{S},\mathcal{I}$ and $\mathcal{J}$ satisfy (3.4) or (3.10) for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, then $\mathcal{A},\mathcal{S},\mathcal{I}$ and $\mathcal{J}$ have a common fixed point in $\mathcal{M}$ under each of the conditions $(i)-(ii)$ of Theorem 3.3.*

An application of Theorem 3.3, we prove the following more general result in invariant approximation theory:

Theorem 3.6. *Let $\mathcal{A},\mathcal{S},\mathcal{I}$ and $\mathcal{J}$ be selfmaps of a Hausdorff locally convex space $(\mathcal{E},\tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{A},\mathcal{S}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$. Suppose that $\mathcal{A}$ and $\mathcal{S}$ are continuous, $(\mathcal{A},\mathcal{I})$ and $(\mathcal{S},\mathcal{J})$ are compatible pairs, $\mathcal{I}$ and $\mathcal{J}$ are nonexpansive and affine on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$. Further, suppose $\mathcal{A},\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ satisfy (3.4) for each $x, y \in \mathcal{D}, p_\alpha \in A^*(\tau)$. If $\mathcal{D}$ is nonempty $p-$starshaped with $p \in \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$ and $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$, then $\mathcal{A},\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ have a common fixed point in $\mathcal{D}$ provided one of the following conditions hold:*

(i) *$\mathcal{D}$ is τ-sequentially compact;*

(ii) *$\mathcal{D}$ is weakly compact in $(\mathcal{E},\tau)$, $\mathcal{I}$ and $\mathcal{J}$ are weakly continuous and $\mathcal{I}-\mathcal{A}$ and $\mathcal{J}-\mathcal{S}$ are demiclosed at 0.*

Proof. First, we show that $\mathcal{A}$ and $\mathcal{S}$ are self map on $\mathcal{D}$, i.e., $\mathcal{A},\mathcal{S} : \mathcal{D} \to \mathcal{D}$. Let $y \in \mathcal{D}$, then $\mathcal{I}y, \mathcal{J}y \in \mathcal{D}$, since $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$. Also, if $y \in \partial\mathcal{M}$, then $\mathcal{A}y \in \mathcal{M}$, since $\mathcal{A}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{A}x_0 = \mathcal{S}x_0 = x_0 = \mathcal{I}x_0 = \mathcal{J}x_0$, so for each $p_\alpha \in \mathcal{A}^*(\tau)$, we have from (3.4)

$$p_\alpha(\mathcal{A}y - x_0) = p_\alpha(\mathcal{A}y - \mathcal{S}x_0) \leq \mathbf{N}(y, x_0).$$

Now, $\mathcal{A}y \in \mathcal{M}$ and $\mathcal{I}y \in \mathcal{D}$, this imply that $\mathcal{A}y$ is also closest to x_0, so $\mathcal{A}y \in \mathcal{D}$. Similarly $\mathcal{S}y \in \mathcal{D}$. Consequently $\mathcal{A},\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ are selfmaps on $\mathcal{D}$. The conditions of Theorem 3.3 ($(i)-(ii)$) are satisfied and, hence, there exists a $w \in \mathcal{D}$ such that $\mathcal{A}w = \mathcal{S}w = w = \mathcal{I}w = \mathcal{J}w$. This completes the proof. □

An immediate consequence of the Theorem 3.6 is as follows:

Corollary 3.7. *Let $\mathcal{A},\mathcal{S},\mathcal{I}$ and $\mathcal{J}$ be selfmaps of a Hausdorff locally convex space $(\mathcal{E},\tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{A},\mathcal{S}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{I})$. Suppose that $\mathcal{A},\mathcal{S}$ are continuous, $(\mathcal{A},\mathcal{I})$ and $(\mathcal{S},\mathcal{J})$ are compatible pairs, $\mathcal{I}$ and $\mathcal{J}$ are non-expansive and affine on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$. Further, suppose $\mathcal{A},\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ satisfy (3.4) for each $x, y \in \mathcal{D}, p_\alpha \in \mathcal{A}^*(\tau)$. If $\mathcal{D}$ is nonempty $p-$starshaped with $p \in \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$ and $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$, then $\mathcal{A},\mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ have a common fixed point in $\mathcal{D}$ under each of the conditions $(i)-(ii)$ of Theorem 3.6.*

An immediate consequence of the Theorem 3.6 and Corollary 3.7 is as follows:

Corollary 3.8. *Let $\mathcal{A}, \mathcal{S}, \mathcal{I}$ and $\mathcal{J}$ be selfmaps of a Hausdorff locally convex space $(\mathcal{E}, \tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{A}, \mathcal{S}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{I})$. Suppose that $\mathcal{A}, \mathcal{S}$ are continuous, $\mathcal{A}, \mathcal{I}, \mathcal{S}$ and $\mathcal{J}$ are commutative, $\mathcal{I}$ and $\mathcal{J}$ are nonexpansive and affine on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$. Further, suppose $\mathcal{A}, \mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ satisfy (3.4) or (3.10) for each $x, y \in \mathcal{D}, p_\alpha \in \mathcal{A}^*(\tau)$. If $\mathcal{D}$ is nonempty $p-$starshaped with $p \in \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{J})$ and $\mathcal{I}(\mathcal{D}) = \mathcal{D} = \mathcal{J}(\mathcal{D})$, then $\mathcal{A}, \mathcal{S}$, $\mathcal{I}$ and $\mathcal{J}$ have a common fixed point in $\mathcal{D}$ under each of the conditions $(i) - (ii)$ of Theorem 3.6.*

Remark 3.9. With the remark given by Jungck [4] that every commuting pair of mappings is compatible but the converse is not true in general, and by using four mappings as opposed to three, our results generalize the results of Nashine [10] and consequently other related results (see in [1, 2, 3, 6, 9, 12, 13, 14, 15, 17]).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,17-23,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

A new application of almost increasing sequences

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Abstract

In the present paper, a theorem on $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series has been proved under weaker conditions. Also we have obtained a new result concerning the $\mid C, 1 \mid_k$ summability factors.

1 Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the n-th Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad and \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{v=1}^{\infty} \frac{\mid t_n^\alpha \mid^k}{n} < \infty, \tag{4}$$

2000 AMS Subject Classification: 40D15, 40F05, 40G99.

Key Words: Absolute summability, almost increasing sequence.

where $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [7]).

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{5}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{6}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2], [3])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{7}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1. \tag{8}$$

In the special case $p_n = 1$ for all values of n, $\mid \bar{N}, p_n \mid_k$ summability is the same as $\mid C, 1 \mid_k$ summability.

2. Known Results. Mishra and Srivastava [9] have proved the following theorem for $\mid \bar{N}, p_n \mid$ summability.

Theorem A. Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{9}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{11}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{12}$$

If

$$\sum_{v=1}^{n} \frac{\mid s_v \mid}{v} = O(X_n) \quad as \quad n \to \infty \tag{13}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{14}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{15}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid$.

Later on Bor [4] has proved Theorem A for $\mid \bar{N}, p_n \mid_k$ summability in the following form.

Theorem B. Let (X_n) be a positive non-decreasing sequence and the sequences (β_n) and (λ_n) are such that conditions (9)-(15) of Theorem A are satisfied with the condition (13) replaced by:

$$\sum_{v=1}^{n} \frac{\mid s_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty. \tag{16}$$

Then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

It may be noted that if we take $k = 1$ in Theorem B, then we get Theorem A.

3. Main result. The aim of this paper is to prove Theorem B under weaker conditions. For this we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$.

Now, we shall prove the following theorem.

Theorem. Let (X_n) be an almost increasing sequence. If the conditions (9)-(12) and (14)-(16) are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

Remark. It should be noted that, from the hypotheses of the Theorem, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [4]).

We require the following lemma for the proof of the theorem.

Lemma ([8]). If (X_n) be an almost increasing sequence, then under the conditions (10)-(11) we have that

$$nX_n\beta_n = O(1), \tag{17}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{18}$$

4. Proof of the Theorem. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=1}^{n} p_v \sum_{r=1}^{v} \frac{a_r P_r \lambda_r}{r p_r} = \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) \frac{a_v P_v \lambda_v}{v p_v}. \tag{19}$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} P_v a_v \lambda_v}{v p_v}, \; n \geq 1. \tag{20}$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} s_v \Delta \left(\frac{P_{v-1} P_v \lambda_v}{v p_v} \right) + \frac{\lambda_n s_n}{n} \\
&= \frac{s_n \lambda_n}{n} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v \frac{P_{v+1} P_v \Delta \lambda_v}{(v+1) p_{v+1}} \\
&+ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta \left(\frac{P_v}{v p_v} \right) - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} s_v P_v \lambda_v \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say.
\end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{21}$$

Firstly by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} \mid T_{n,1} \mid^k &= \sum_{n=1}^{m} \left(\frac{P_n}{n p_n} \right)^{k-1} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid \frac{\mid s_n \mid^k}{n} \\
&= O(1) \sum_{n=1}^{m} \mid \lambda_n \mid \frac{\mid s_n \mid^k}{n} \\
&= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \frac{\mid s_v \mid^k}{v} \\
&+ O(1) \mid \lambda_m \mid \sum_{n=1}^{m} \frac{\mid s_n \mid^k}{n} \\
&= O(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma.

Now, using the fact that $P_{v+1} = O((v+1)p_{v+1})$ by (14), we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \mid \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \frac{P_v}{p_v} \mid s_v \mid p_v \mid \Delta\lambda_v \mid \right\}^k
\end{aligned}$$

Now applying Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \mid \Delta\lambda_v \mid^k \\
&\times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v \mid \Delta\lambda_v \mid^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v \mid \Delta\lambda_v \mid}{p_v}\right)^{k-1} \mid s_v \mid^k \mid \Delta\lambda_v \mid \\
&= O(1) \sum_{v=1}^{m} \mid s_v \mid^k \mid \Delta\lambda_v \mid \left(\frac{P_v}{v p_v}\right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} v\beta_v \frac{\mid s_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{\mid s_r \mid^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} \frac{\mid s_v \mid^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid \beta_v \mid X_v + O(1) m\beta_m X_m = O(1)
\end{aligned}$$

as $m \to \infty$, in view of the hypotheses of the Theorem and Lemma.

Again, since $\Delta(\frac{P_v}{v p_v}) = O(\frac{1}{v})$, by (14) and (15) (see [9]), as in $T_{n,1}$ we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,3} \mid^k &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right) p_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v} \right\}^k
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k p_v \mid s_v \mid^k\mid \lambda_v \mid^k \\
&\times \quad \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k \mid s_v \mid^k p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k p_v \mid s_v \mid^k\mid \lambda_v \mid^k \frac{1}{P_v}.\frac{v}{v} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1} \mid \lambda_v \mid^{k-1}\mid \lambda_v \mid \frac{\mid s_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m}\mid \lambda_v \mid \frac{\mid s_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}X_v\beta_v + O(1)X_m \mid \lambda_m \mid= O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

Finally, using Hölder's inequality, as in $T_{n,3}$ we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,4}\mid^k &= \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\mid\sum_{v=1}^{n-1}s_v\frac{P_v}{v}\lambda_v\mid^k \\
&= \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\mid\sum_{v=1}^{n-1}s_v\frac{P_v}{vp_v}p_v\lambda_v\mid^k \\
&\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\mid s_v\mid^k\left(\frac{P_v}{vp_v}\right)^k p_v\mid\lambda_v\mid^k \\
&\times \quad \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1} \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k \mid s_v \mid^k p_v \mid \lambda_v \mid^k \frac{1}{P_v}.\frac{v}{v} \\
&= O(1)\sum_{v=1}^{m}\mid \lambda_v \mid \frac{\mid s_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}X_v\beta_v + O(1)X_m \mid \lambda_m \mid= O(1) \quad as \quad m\to\infty.
\end{aligned}
$$

Therefore we get

$$
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,r}\mid^k= O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.
$$

This completes the proof of the Theorem.

Finally if we take $p_n = 1$ for all values of n in this theorem, then we get a new result concerning the $\mid C, 1 \mid_k$ summability factors.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,25-37,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

On $*$-Homomorphisms between JC^*-Algebras

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Abstract. It is shown that every almost unital almost linear mapping $f : \mathcal{A} \to \mathcal{B}$ of JC^*-algebra $\mathcal{A}$ to a JC^*-algebra $\mathcal{B}$ is a homomorphism when $f(2^n u \circ y) = f(2^n u) \circ f(y)$ holds for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, and that every almost unital almost linear continuous mapping $f : \mathcal{A} \to \mathcal{B}$ of a JC^*-algebra $\mathcal{A}$ of real rank zero to a JC^*-algebra $\mathcal{B}$ is a homomorphism when $f(2^n u \circ y) = f(2^n u) \circ f(y)$ holds for all $u \in \{v \in \mathcal{A} \mid v = v^*, \|v\| = 1, v \text{ is invertible}\}$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$.

Furthermore, we are going to prove the generalized Hyers–Ulam–Rassias stability of $*$-homomorphisms between JC^*-algebras, and $\mathbb{C}$-linear $*$-derivations on JC^*-algebras.

2000 *Mathematics Subject Classification.* Primary 47E10, 39B52, 17Cxx, 46L05.

Key words and Phrases. Hyers–Ulam–Rassias stability, homomorphism in JC^*-algebra, real rank 0, linear derivation

1. Introduction

Our knowledge concerning the continuity properties of epimorphisms on Banach algebras, Jordan–Banach algebras, and, more generally, nonassociative complete normed algebras, is now fairly complete and satisfactory (see [9] and [10]). A basic continuity problem consists in determining algebraic conditions on a Banach algebra A which ensure

This work was supported by Korea Research Foundation Grant KRF-2005-041-C00027.

that derivations on A are continuous. In 1996, Villena [10] proved that derivations on semisimple Jordan–Banach algebras are continuous.

Let E_1 and E_2 be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p}||x||^p$$

for all $x \in E_1$. Găvruta [2] generalized the Rassias' result.

Jun, Kim and Shin [4] proved the following: Let X and Y be Banach spaces. Denote by $\varphi : X \times X \to [0, \infty)$ a function such that

$$\varepsilon(x) := \sum_{j=1}^{\infty} 2^{-j}(\varphi(2^{j-1}x, 0) + \varphi(0, 2^{j-1}x) + \varphi(2^{j-1}x, 2^{j-1}x)) < \infty$$

for all $x \in X$. Suppose that $f, g, h : X \to Y$ are mappings satisfying

$$\|2f(\frac{x+y}{2}) - g(x) - h(y)\| \leq \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\begin{aligned}
&\|2f(\frac{x}{2}) - T(x)\| \leq \|g(0)\| + \|h(0)\| + \varepsilon(x),\\
&\|g(x) - T(x)\| \leq \|g(0)\| + 2\|h(0)\| + \varphi(x, 0) + \varepsilon(x),\\
&\|h(x) - T(x)\| \leq 2\|g(0)\| + \|h(0)\| + \varphi(0, x) + \varepsilon(x)
\end{aligned}$$

for all $x \in X$.

B.E. Johnson [3], Theorem 7.2 also investigated almost algebra $*$-homomorphisms between Banach $*$-algebras : Suppose that $\mathcal{U}$ and $\mathcal{B}$ are Banach $*$-algebras which satisfy the conditions of [3], Theorem 3.1. Then for each positive ϵ and K there is a positive δ such that if $T \in L(\mathcal{U}, \mathcal{B})$ with $\|T\| < K$, $\|T^\vee\| < \delta$ and $\|T(x^*)^* - T(x)\| \leq \delta\|x\| \quad (x \in \mathcal{U})$ then there is a $*$-homomorphism $T' : \mathcal{U} \to \mathcal{B}$ with $\|T - T'\| < \epsilon$. Here $L(\mathcal{U}, \mathcal{B})$ is the space of bounded linear maps from $\mathcal{U}$ into $\mathcal{B}$, and $T^\vee(x, y) = T(xy) - T(x)T(y) \quad (x, y \in \mathcal{U})$. See [3] for details.

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [9]). Let $\mathcal{H}$ be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds.

A complex Jordan algebra $\mathcal{C}$ with product $x \circ y$ and involution $x \mapsto x^*$ is called a JB^*-algebra if $\mathcal{C}$ carries a Banach space norm $\| \cdot \|$ satisfying $\|x \circ y\| \leq \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$. Here $\{xy^*z\} := x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*)$ denotes the *Jordan triple product* of $x, y, z \in \mathcal{C}$. A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra*.

Throughout this paper, let $\mathcal{A}$ be a JC^*-algebra with norm $|| \cdot ||$ and unit e, and $\mathcal{B}$ a JC^*-algebra with norm $\| \cdot \|$ and unit e'. Let $\mathcal{U}(\mathcal{A}) = \{ u \in \mathcal{A} \mid u^*u = uu^* = e\}$, $\mathcal{A}_{sa} = \{x \in \mathcal{A} \mid x = x^*\}$, and $I_1(\mathcal{A}_{sa}) = \{v \in \mathcal{A}_{sa} \mid \|v\| = 1, v \text{ is invertible}\}$.

In this paper, we prove that every almost unital almost linear mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(3^n u \circ y) = h(3^n u) \circ h(y)$ holds for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, and that for a JC^*-algebra $\mathcal{A}$ of real rank zero (see [1]), every almost unital almost linear continuous mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(3^n u \circ y) = h(3^n u) \circ h(y)$ holds for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$.

Furthermore, we are going to prove the generalized Hyers–Ulam–Rassias stability of $*$-homomorphisms between JC^*-algebras, and $\mathbb{C}$-linear $*$-derivations on JC^*-algebras.

2. $*$-homomorphisms between JC^*-algebras

We are going to investigate $*$-homomorphisms between JC^*-algebras.

THEOREM 2.1. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be mappings satisfying $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$, and let $f(2^n u \circ y) = f(2^n u) \circ f(y)$, $g(2^n u \circ y) = g(2^n u) \circ g(y)$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A} \setminus \{0\} \times \mathcal{A} \setminus \{0\} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^{j-1}x, 2^{j-1}y) \quad < \quad \infty, \tag{1}$$

$$\|2f(\frac{\mu x+\mu y}{2})-\mu g(x)-\mu h(y)\| \leq \varphi(x,y), \tag{2}$$
$$\|f(2^n u^*)-f(2^n u)^*\| \leq \varphi(2^n u, 2^n u) \tag{3}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, *all* $u \in \mathcal{U}(\mathcal{A})$, *all* $x, y \in \mathcal{A}$, *and all* $n = 0, 1, 2, \cdots$. *Assume that*

$$\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = e'. \tag{4}$$

Then the mappings $f, g, h : \mathcal{A} \to \mathcal{B}$ *are* $*$*-homomorphisms.*

Proof. Put $\mu = 1 \in \mathbb{T}^1$. It follows from Corollary 2.5 of [4] that there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{aligned}
&\|2f(\frac{x}{2}) - H(x)\| \leq \varepsilon(x),\\
&\|g(x) - H(x)\| \leq \varphi(x,0) + \varepsilon(x),\\
&\|h(x) - H(x)\| \leq \varphi(0,x) + \varepsilon(x)
\end{aligned} \tag{5}$$

for all $x \in \mathcal{A} \setminus \{0\}$, where

$$\varepsilon(x) := \sum_{j=1}^{\infty} 2^{-j}(\varphi(2^{j-1}x, 0) + \varphi(0, 2^{j-1}x) + \varphi(2^{j-1}x, 2^{j-1}x)) < \infty$$

for all $x \in \mathcal{A} \setminus \{0\}$. The additive mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathcal{A}$, and

$$\lim_{n\to\infty} 2^{-n} f(2^n x) = \lim_{n\to\infty} 2^{-n} g(2^n x) = \lim_{n\to\infty} 2^{-n} h(2^n x)$$

for all $x \in \mathcal{A}$. Let $\widetilde{f}(x) = 2f(\frac{x}{2})$ for all $x \in \mathcal{A}$, then

$$\lim_{n\to\infty} \frac{1}{2^n} \widetilde{f}(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in \mathcal{A}$.

By the assumption,

$$\begin{aligned}
&\|f(2^n \mu x) - \mu f(2^n x)\|\\
&= \|f(2^n \mu x) - \tfrac{1}{2}\mu g(2^n x) - \tfrac{1}{2}\mu h(2^n x) + \tfrac{1}{2}\mu g(2^n x) + \tfrac{1}{2}\mu h(2^n x) - \mu f(2^n x)\|\\
&\leq \tfrac{1}{2}\varphi(2^n x, 2^n x) + \tfrac{1}{2}|\mu|\varphi(2^n x, 2^n x) = \varphi(2^n x, 2^n x)
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$. Thus $2^{-n}\|f(2^n\mu x) - \mu f(2^n x)\| \to 0$ as $n \to \infty$ for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$. Hence

$$H(\mu x) = \lim_{n\to\infty} \frac{f(2^n\mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu f(2^n x)}{2^n} = \mu H(x) \tag{6}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A} \setminus \{0\}$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $2|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{2} = 1 - \frac{2}{4}$. By Theorem 1 of [5], there exist four elements $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{T}^1$ such that $4\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3 + \mu_4$. And $H(x) = H(2 \cdot \frac{1}{2}x) = 2H(\frac{1}{2}x)$ for all $x \in \mathcal{A}$. So $H(\frac{1}{2}x) = \frac{1}{2}H(x)$ for all $x \in \mathcal{A}$. Thus by (6)

$$\begin{aligned} H(\lambda x) &= H(\frac{M}{4} \cdot 4\frac{\lambda}{M}x) = M \cdot H(\frac{1}{4} \cdot 4\frac{\lambda}{M}x) \\ &= \frac{M}{4}H(4\frac{\lambda}{M}x) = \frac{M}{4}H(\mu_1 x + \mu_2 x + \mu_3 x + \mu_4 x) \\ &= \frac{M}{4}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x) + H(\mu_4 x)) \\ &= \frac{M}{4}(\mu_1 + \mu_2 + \mu_3 + \mu_4)H(x) = \frac{M}{4} \cdot 4\frac{\lambda}{M}H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C} \setminus \{0\}$ and all $x, y \in \mathcal{A}$. And $H(0x) = 0 = 0H(x)$ for all $x \in \mathcal{A}$. So the unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping.

By (1) and (3), we get

$$H(u^*) = \lim_{n\to\infty} \frac{f(2^n u^*)}{2^n} = \lim_{n\to\infty} \frac{f(2^n u)^*}{2^n} = (\lim_{n\to\infty} \frac{f(2^n u)}{2^n})^* = H(u)^*$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (cf. [6]), say, $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* \\ &= (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* \\ &= H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Since $f(2^n u \circ y) = f(2^n u) \circ f(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n u) \circ f(y) = H(u) \circ f(y) \tag{7}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (7),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ f(2^n y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ f(2^n y) = H(u) \circ \frac{1}{2^n} f(2^n y) \tag{8}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (8) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{9}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{m} \lambda_j u_j \quad (\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$, it follows from (9) that

$$\begin{aligned} H(x \circ y) &= H(\sum_{j=1}^{m} \lambda_j u_j \circ y) = \sum_{j=1}^{m} \lambda_j H(u_j \circ y) \\ &= \sum_{j=1}^{m} \lambda_j H(u_j) \circ H(y) = H(\sum_{j=1}^{m} \lambda_j u_j) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. By (7) and (9),

$$H(e) \circ H(y) = H(e \circ y) = H(e) \circ f(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = H(e) = e'$,

$$H(y) = f(y)$$

for all $y \in \mathcal{A}$. Similarly, $H(y) = g(y) = h(y)$ for all $y \in \mathcal{A}$. Therefore, the mapping $f, g, h : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms, as desired. □

Corollary 2.2. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be mappings satisfying $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$ and let $f(2^n u \circ y) = f(2^n u) \circ f(y)$, $g(2^n u \circ y) = g(2^n u) \circ g(y)$ and $h(2^n u \circ y) =$*

$h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\begin{aligned} \|2f(\frac{\mu x + \mu y}{2}) - \mu g(x) - \mu h(y)\| &\leq \theta(||x||^p + ||y||^p), \\ \|f(2^n u^*) - f(2^n u)^*\| &\leq 2^{np+1}\theta \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $n = 0, 1, \cdots$, and all $x, y \in \mathcal{A} \setminus \{0\}$. Assume that $\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = e'$. Then the mappings f, g and h are $*$-homomorphisms.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in \mathcal{A} \setminus \{0\}$, and then apply Theorem 2.1. □

THEOREM 2.3. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be mappings satisfying $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$ and let $f(2^n u \circ y) = f(2^n u) \circ f(y)$, $g(2^n u \circ y) = g(2^n u) \circ g(y)$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying* (1), (3), *and* (4) *such that*

$$\|2f(\frac{\mu x + \mu y}{2}) - \mu g(x) - \mu h(y)\| \leq \varphi(x, y) \tag{10}$$

for $\mu = 1, i$, and all $x, y \in \mathcal{A} \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the mappings $f, g, h : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms.

Proof. Put $\mu = 1$ in (10). By the same reasoning as the proof of Theorem 2.1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (5). By the same reasoning as the proof of [8], Theorem, the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (10). By the same method as the proof of Theorem 2.1, one can obtain that

$$H(ix) = \lim_{n\to\infty} \frac{f(2^n ix)}{2^n} = \lim_{n\to\infty} \frac{if(2^n x)}{2^n} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) = (s + it)H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 2.1. □

From now on, assume that $\mathcal{A}$ is a JC^*-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [1]).

Now we are going to investigate continuous $*$-homomorphisms between JC^*-algebras.

THEOREM 2.4. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be continuous mappings satisfying $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$ and let $f(2^n u \circ y) = f(2^n u) \circ f(y)$, $g(2^n u \circ y) = g(2^n u) \circ g(y)$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying (1), (2), (3), and (4). Then the mappings $f, g, h : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms.*

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (5).

Since $f(2^n u \circ y) = f(2^n u) \circ f(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$,

$$H(u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n u \circ y) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n u) \circ f(y) = H(u) \circ f(y) \qquad (11)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. By the additivity of H and (11),

$$2^n H(u \circ y) = H(2^n u \circ y) = H(u \circ (2^n y)) = H(u) \circ f(2^n y)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Hence

$$H(u \circ y) = \frac{1}{2^n} H(u) \circ f(2^n y) = H(u) \circ \frac{1}{2^n} f(2^n y) \qquad (12)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$. Taking the limit in (12) as $n \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \qquad (13)$$

for all $u \in I_1(\mathcal{A}_{sa})$ and all $y \in \mathcal{A}$.

By (11) and (13),

$$H(e) \circ H(y) = H(e \circ y) = H(e) \circ f(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = H(e) = e'$,

$$H(y) = f(y)$$

for all $y \in \mathcal{A}$. Similarly, $H(y) = g(y) = h(y)$ for all $y \in \mathcal{A}$. So $H : \mathcal{A} \to \mathcal{B}$ is continuous. But by the assumption that $\mathcal{A}$ has real rank zero, it is easy to show that $I_1(\mathcal{A}_{sa})$ is dense in $\{x \in \mathcal{A}_{sa} \mid ||x|| = 1\}$. So for each $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$, there is a sequence $\{\kappa_j\}$ such that $\kappa_j \to w$ as $j \to \infty$ and $\kappa_j \in I_1(\mathcal{A}_{sa})$. Since $H : \mathcal{A} \to \mathcal{B}$ is continuous, it follows from (13) that

$$\begin{aligned} H(w \circ y) &= H(\lim_{j\to\infty} \kappa_j \circ y) = \lim_{j\to\infty} H(\kappa_j \circ y) \\ &= \lim_{j\to\infty} H(\kappa_j) \circ H(y) = H(\lim_{j\to\infty} \kappa_j) \circ H(y) \\ &= H(w) \circ H(y) \end{aligned} \tag{14}$$

for all $w \in \{z \in \mathcal{A}_{sa} \mid ||z|| = 1\}$ and all $y \in \mathcal{A}$.

For each $x \in \mathcal{A}$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 := \frac{x-x^*}{2i}$ are self-adjoint.

First, consider the case that $x_1 \neq 0, x_2 \neq 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (14) that

$$\begin{aligned} H(x \circ y) &= H(x_1 \circ y + ix_2 \circ y) = H(||x_1||\frac{x_1}{||x_1||} \circ y + i||x_2||\frac{x_2}{||x_2||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||} \circ y) + i||x_2||H(\frac{x_2}{||x_2||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}) \circ H(y) + i||x_2||H(\frac{x_2}{||x_2||}) \circ H(y) \\ &= \{H(||x_1||\frac{x_1}{||x_1||}) + iH(||x_2||\frac{x_2}{||x_2||})\} \circ H(y) = H(x_1 + ix_2) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$.

Next, consider the case that $x_1 \neq 0, x_2 = 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (14) that

$$\begin{aligned} H(x \circ y) &= H(x_1 \circ y) = H(||x_1||\frac{x_1}{||x_1||} \circ y) = ||x_1||H(\frac{x_1}{||x_1||} \circ y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}) \circ H(y) = H(||x_1||\frac{x_1}{||x_1||}) \circ H(y) = H(x_1) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $y \in \mathcal{A}$.

Finally, consider the case that $x_1 = 0, x_2 \neq 0$. Since $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear, it follows from (14) that

$$H(x \circ y) = H(ix_2 \circ y) = H(i||x_2||\frac{x_2}{||x_2||} \circ y) = i||x_2||H(\frac{x_2}{||x_2||} \circ y)$$

$$
\begin{aligned}
&= i||x_2||H(\frac{x_2}{||x_2||}) \circ H(y) = H(i||x_2||\frac{x_2}{||x_2||}) \circ H(y) = H(ix_2) \circ H(y) \\
&= H(x) \circ H(y)
\end{aligned}
$$

for all $y \in \mathcal{A}$. Hence

$$H(x \circ y) = H(x) \circ H(y)$$

for all $x, y \in \mathcal{A}$.

Therefore, the mappings $f, g, h : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms, as desired. □

Corollary 2.5. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be continuous mappings satisfying $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$, and let $f(2^n u \circ y) = f(2^n u) \circ f(y)$, $g(2^n u \circ y) = g(2^n u) \circ g(y)$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$
\begin{aligned}
\|2f(\frac{\mu x + \mu y}{2}) - \mu g(x) - \mu h(y)\| &\leq \theta(||x||^p + ||y||^p), \\
\|f(2^n u^*) - f(2^n u)^*\| &\leq 2^{np+1}\theta
\end{aligned}
$$

for all $\mu \in \mathbb{T}^1$, all $u \in I_1(\mathcal{A}_{sa})$, all $x, y \in \mathcal{A} \backslash \{0\}$, and all $n = 0, 1, 2, \cdots$. If $\lim_{n\to\infty} \frac{f(2^n e)}{2^n} = e'$, the mappings $f, g, h : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in \mathcal{A} \setminus \{0\}$, and then apply Theorem 2.4. □

Theorem 2.6. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be continuous mappings satisfying $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$, and let $f(2^n u \circ y) = f(2^n u) \circ f(y)$, $g(2^n u \circ y) = g(2^n u) \circ g(y)$ and $h(2^n u \circ y) = h(2^n u) \circ h(y)$ for all $u \in I_1(\mathcal{A}_{sa})$, all $y \in \mathcal{A}$, and all $n = 0, 1, 2, \cdots$, for which there exists a function $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfying* (1), (3), (4), *and (10). Then the mappings $f, g, h : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms.*

Proof. By the same reasoning as the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the system of the inequalities (5).

The rest of the proof is the same as the proofs of Theorems 2.1 and 2.4. □

3. Stability of $*$-homomorphisms in JC^*-algebras

We prove the generalized Hyers–Ulam–Rassias stability of $*$-homomorphisms in JC^*-algebras.

THEOREM 3.1. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be mappings with $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, z, w) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y, 2^j z, 2^j w) < \infty, \tag{15}$$

$$\|2f(\frac{\mu x + \mu y + z \circ w}{2}) - \mu g(x) - \mu h(y) - f(z) \circ f(w)\| \tag{16}$$
$$\leq \varphi(x, y, z, w),$$

$$\|f(2^n u^*) - f(2^n u)^*\| \leq \varphi(2^n u, 2^n u, 0, 0) \tag{17}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $x, y, z, w \in \mathcal{A} \setminus \{0\}$, and all $n = 0, 1, 2, \cdots$. Then there exists a unique $*$-homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\|2f(\tfrac{x}{2}) - H(x)\| \leq \varepsilon(x),$$
$$\|g(x) - H(x)\| \leq \varphi(x, 0, 0, 0) + \varepsilon(x),$$
$$\|h(x) - H(x)\| \leq \varphi(0, x, 0, 0) + \varepsilon(x) \tag{18}$$

for all $x \in \mathcal{A} \setminus \{0\}$, where

$$\varepsilon(x) := \sum_{j=1}^{\infty} 2^{-j} (\varphi(2^{j-1} x, 0, 0, 0) + \varphi(0, 2^{j-1} x, 0, 0) + \varphi(2^{j-1} x, 2^{j-1} x, 0, 0)) < \infty.$$

Proof. Put $z = w = 0$ and $\mu = 1 \in \mathbb{T}^1$ in (16). By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (18). The $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) \tag{19}$$

for all $x \in \mathcal{A}$.

It follows from (19) that

$$H = \lim_{n \to \infty} \frac{f(2^{2n} x)}{2^{2n}} \tag{20}$$

for all $x \in \mathcal{A}$. Let $x = y = 0$ in (16). Then we get

$$\|2f(\frac{z \circ w}{2}) - f(z) \circ f(w)\| \leq \varphi(0, 0, z, w)$$

for all $z, w \in \mathcal{A}$. Since

$$\frac{1}{2^{2n}} \varphi(0, 0, 2^n z, 2^n w) \leq \frac{1}{2^n} \varphi(0, 0, 2^n z, 2^n w),$$

$$\begin{aligned}\frac{1}{2^{2n}}\|2f(\frac{1}{2}2^n z\circ 2^n w)-f(2^n z)\circ f(2^n w)\| &\le \frac{1}{2^{2n}}\varphi(0,0,2^n z,2^n w)\\ &\le \frac{1}{2^n}\varphi(0,0,2^n z,2^n w) \qquad (21)\end{aligned}$$

for all $z, w \in \mathcal{A}$. By (19), (20), and (21),

$$\begin{aligned}2H(\frac{z\circ w}{2}) &= \lim_{n\to\infty}\frac{2f(\frac{1}{2}2^{2n}z\circ w)}{2^{2n}} = \lim_{n\to\infty}\frac{2f(\frac{1}{2}2^n z\circ 2^n w)}{2^n\cdot 2^n}\\ &= \lim_{n\to\infty}(\frac{f(2^n z)}{2^n}\circ\frac{f(2^n w)}{2^n}) = \lim_{n\to\infty}\frac{f(2^n z)}{2^n}\circ\lim_{n\to\infty}\frac{f(2^n w)}{2^n}\\ &= H(z)\circ H(w)\end{aligned}$$

for all $z, w \in \mathcal{A}$. But since H is $\mathbb{C}$-linear,

$$H(z\circ w) = 2H(\frac{z\circ w}{2}) = H(z)\circ H(w)$$

for all $z, w \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism satisfying the inequality (18) as desired. □

COROLLARY 3.2. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}&\|2f(\frac{\mu x+\mu y+z\circ w}{2})-\mu g(x)-\mu h(y)-f(z)\circ f(w)\|\\ &\qquad\qquad \le \theta(\|x\|^p+\|y\|^p+\|z\|^p+\|w\|^p),\\ &\|f(2^n u^*)-f(2^n u)^*\| \le 2^{np+1}\theta\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, all $x, y, z, w \in \mathcal{A}\setminus\{0\}$, and all $n = 0, 1, 2, \cdots$. Then there exists a unique $*$-homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

$$\begin{aligned}\|2f(\tfrac{x}{2})-H(x)\| &\le \tfrac{1}{2-2^p}\theta\|x\|^p\\ \|g(x)-H(x)\| &\le \tfrac{3-2^p}{2-2^p}\theta\|x\|^p\\ \|h(x)-H(x)\| &\le \tfrac{3-2^p}{2-2^p}\theta\|x\|^p\end{aligned}$$

for all $x \in \mathcal{A}\setminus\{0\}$.

Proof. Define $\varphi(x, y, z, w) = \theta(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$, and then apply Theorem 3.1. □

THEOREM 3.3. *Let $f, g, h : \mathcal{A} \to \mathcal{B}$ be a mapping with $f(0) = 0$, $g(0) = 0$ and $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^4 \to [0, \infty)$ satisfying* (15) *and* (16) *such that*

$$\|2f(\frac{\mu x + \mu y + z \circ w}{2}) - \mu g(x) - \mu h(y) - f(z) \circ f(w)\| \leq \varphi(x, y, z, w)$$

for $\mu = 1, i$, and all $x, y, z, w \in \mathcal{A} \setminus \{0\}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then there exists a unique $*$-homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (18).

Proof. By the same reasoning as the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (18).

The rest of the proof is the same as the proofs of Theorems 2.1 and 3.1. $\square$

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,39-51,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Hyers–Ulam–Rassias Stability of Isometric Homomorphisms in Quasi-Banach Algebras

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Abstract. In this paper, we prove the Hyers–Ulam–Rassias stability of isometric homomorphisms in quasi-Banach algebras. This is applied to investigate isometric isomorphisms between quasi-Banach algebras.

2000 *Mathematics Subject Classification.* Primary 39B52, 46B03, 46B04, 47B48, 47Jxx.

Key words and Phrases. Hyers–Ulam–Rassias stability, additive mapping, quasi-Banach algebra, isometric isomorphism, p-Banach algebra

1 Introduction and preliminaries

The stability problem of functional equations originated from a question of S.M. Ulam [39] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

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[*]The corresponding author was supported by the research fund of Hanyang University (HY-2006-N).

If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D.H. Hyers [13] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [28] introduced the following inequality: Assume that there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [28] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in X$. The above inequality has provided a lot of influence in the development of what is known as *Hyers–Ulam–Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [11] generalized the Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [5]–[6], [10], [11], [14]–[18], [20]–[26], [29]–[33], [36], [37]).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 ([4, 35]) *Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:*

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki–Rolewicz theorem [35] (see also [4]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.2 ([1]) *Let $(A, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(A, \|\cdot\|)$ is called a quasi-normed algebra if A is an algebra and there is a constant $C > 0$ such that $\|xy\| \leq C\|x\| \cdot \|y\|$ for all $x, y \in A$.*

A quasi-Banach algebra is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a p-norm then the quasi-Banach algebra is called a p-Banach algebra.

Definition 1.3 *Let A and B be quasi-Banach algebras with norms $\|\cdot\|_A$ and $\|\cdot\|_B$. An algebra homomorphism $H : A \to B$ is called an isometric homomorphism if the algebra homomorphism $H : A \to B$ satisfies*

$$\|H(x) - H(y)\|_B = \|x - y\|_A$$

for all $x, y \in A$. If, in addition, the algebra homomorphism $H : A \to B$ is bijective, then the algebra homomorphism $H : A \to B$ is called an isometric isomorphism.

The stability of isometries in normed spaces and Banach algebras have been investigated in several papers (see [3, 8, 9, 12, 19]).

The paper is organized as follows: In Section 2, we prove the Hyers–Ulam–Rassias stability of isometric homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation and the Jensen functional equation.

In Section 3, we investigate isometric isomorphisms between quasi-Banach algebras.

2 Stability of isometric homomorphisms in quasi-Banach algebras

Throughout this section, assume that A is a quasi-normed algebra with quasi-norm $\|\cdot\|_A$ and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$. Let K be the modulus of concavity of $\|\cdot\|_B$.

We prove the Hyers–Ulam–Rassias stability of isometric homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation.

Theorem 2.1 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\|_B &\leq \theta(\|x\|_A^r + \|y\|_A^r), &(1)\\ \|f(xy) - f(x)f(y)\|_B &\leq \theta(\|x\|_A^r + \|y\|_A^r), &(2)\\ |\ \|f(x)\|_B - \|x\|_A\ | &\leq \theta\|x\|_A^r &(3) \end{aligned}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique isometric homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2\theta}{(2^{pr} - 2^p)^{\frac{1}{p}}}\|x\|_A^r \tag{4}$$

for all $x \in A$.

Proof. Letting $y = x$ in (1), we get

$$\|f(2x) - 2f(x)\|_B \leq 2\theta\|x\|_A^r \tag{5}$$

for all $x \in A$. So

$$\|f(x) - 2f(\frac{x}{2})\|_B \leq \frac{2\theta}{2^r}\|x\|_A^r$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\|_B^p \le \sum_{j=l}^{m-1} \|2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}})\|_B^p \le \frac{2^p\theta^p}{2^{pr}} \sum_{j=l}^{m-1} \frac{2^{pj}}{2^{prj}} \|x\|_A^{pr} \tag{6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

It follows from (1) that

$$\begin{aligned} \|H(x+y) - H(x) - H(y)\|_B &= \lim_{n\to\infty} 2^n \|f(\frac{x+y}{2^n}) - f(\frac{x}{2^n}) - f(\frac{y}{2^n})\|_B \\ &\le \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}} (\|x\|_A^r + \|y\|_A^r) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$H(x+y) = H(x) + H(y)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (6), we get (4).

By the same reasoning as in the proof of Theorem of [28], the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} 4^n \|f(\frac{xy}{2^n \cdot 2^n}) - f(\frac{x}{2^n})f(\frac{y}{2^n})\|_B \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{2^{nr}} (\|x\|_A^r + \|y\|_A^r) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Now, let $T : A \to B$ be another Cauchy additive mapping satisfying (4). Then we have

$$\begin{aligned} \|H(x) - T(x)\|_B &= 2^n \|H(\frac{x}{2^n}) - T(\frac{x}{2^n})\|_B \\ &\le 2^n K(\|H(\frac{x}{2^n}) - f(\frac{x}{2^n})\|_B + \|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\|_B) \\ &\le \frac{2^{n+2}K\theta}{(2^{pr} - 2^p)^{\frac{1}{p}} 2^{nr}} \|x\|_A^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H.

It follows from (3) that

$$| \, \|2^n f(\frac{x}{2^n})\|_B - \|x\|_A \, | = \frac{2^n \theta}{2^{nr}} \|x\|_A^r,$$

which tends to zero as $n \to \infty$ for all $x \in A$. So

$$\|H(x)\|_B = \lim_{n\to\infty} \|2^n f(\frac{x}{2^n})\|_B = \|x\|_A$$

for all $x \in A$. Hence

$$\|H(x) - H(y)\|_B = \|H(x-y)\|_B = \|x-y\|_A$$

for all $x, y \in A$. So the mapping $H : A \to B$ is an isometry. Thus the mapping $H : A \to B$ is a unique isometric homomorphism satisfying (4). □

Theorem 2.2 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping satisfying* (1), (2) *and* (3). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique isometric homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{2\theta}{(2^p - 2^{pr})^{\frac{1}{p}}} \|x\|_A^r \tag{7}$$

for all $x \in A$.

Proof. It follows from (5) that

$$\|f(x) - \frac{1}{2} f(2x)\|_B \le \theta \|x\|_A^r$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\|_B^p \le \sum_{j=l}^{m-1} \|\frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x)\|_B^p \le \theta^p \sum_{j=l}^{m-1} \frac{2^{prj}}{2^{pj}} \|x\|_A^{pr} \tag{8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (8) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. □

We prove the Hyers–Ulam–Rassias stability of isometric homomorphisms in quasi-Banach algebras, associated to the Jensen functional equation.

Theorem 2.3 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping with $f(0) = 0$ satisfying (2) and (3) such that*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r) \tag{9}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique isometric homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{K(3+3^r)\theta}{(3^p - 3^{pr})^{\frac{1}{p}}}\|x\|_A^r \tag{10}$$

for all $x \in A$.

Proof. Letting $y = -x$ in (9), we get

$$\| - f(x) - f(-x)\|_B \leq 2\theta\|x\|_A^r$$

for all $x \in A$. Letting $y = 3x$ and replacing x by $-x$ in (9), we get

$$\|2f(x) - f(-x) - f(3x)\|_B \leq (3^r + 1)\theta\|x\|_A^r$$

for all $x \in A$. Thus

$$\|3f(x) - f(3x)\|_B \leq K(3^r + 3)\theta\|x\|_A^r \tag{11}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{3}f(3x)\|_B \leq \frac{K(3^r+3)\theta}{3}\|x\|_A^r$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned} \|\frac{1}{3^l}f(3^l x) - \frac{1}{3^m}f(3^m x)\|_B^p &\leq \sum_{j=l}^{m-1} \|\frac{1}{3^j}f(3^j x) - \frac{1}{3^{j+1}}f(3^{j+1}x)\|_B^p \\ &\leq \frac{K^p(3^r+3)^p\theta^p}{3^p}\sum_{j=l}^{m-1}\frac{3^{prj}}{3^{pj}}\|x\|_A^{pr} \end{aligned} \tag{12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (12) that the sequence $\{\frac{1}{3^n}f(3^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{3^n}f(3^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{3^n}f(3^n x)$$

for all $x \in A$.

By (9),

$$\begin{aligned}\|2H(\frac{x+y}{2}) - H(x) - H(y)\|_B &= \lim_{n\to\infty} \frac{1}{3^n}\|2f(3^n \cdot \frac{x+y}{2}) - f(3^n x) - f(3^n y)\|_B \\ &\leq \lim_{n\to\infty} \frac{3^{rn}}{3^n}\theta(\|x\|_A^r + \|y\|_A^r) = 0\end{aligned}$$

for all $x, y \in A$. So

$$2H(\frac{x+y}{2}) = H(x) + H(y)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (12), we get (10).

It follows from (2) that

$$\begin{aligned}\|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \frac{1}{9^n}\|f(9^n xy) - f(3^n x)f(3^n y)\|_B \\ &\leq \lim_{n\to\infty} \frac{3^{nr}\theta}{9^n}(\|x\|_A^r + \|y\|_A^r) = 0\end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Now, let $T : A \to B$ be another Jensen additive mapping satisfying (10). Then we have

$$\begin{aligned}\|H(x) - T(x)\|_B^p &= \frac{1}{3^{pn}}\|H(3^n x) - T(3^n x)\|_B^p \\ &\leq \frac{1}{3^{pn}}(\|H(3^n x) - f(3^n x)\|_B^p + \|T(3^n x) - f(3^n x)\|_B^p) \\ &\leq 2 \cdot \frac{3^{prn}}{3^{pn}} \cdot \frac{K^p(3+3^r)^p\theta^p}{3^p - 3^{pr}}\|x\|_A^{pr},\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a mapping with $f(0) = 0$ satisfying (2), (3) and (9). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique isometric homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{K(3^r+3)\theta}{(3^{pr} - 3^p)^{\frac{1}{p}}}\|x\|_A^r \tag{13}$$

for all $x \in A$.

Proof. It follows from (11) that

$$\|f(x) - 3f(\frac{x}{3})\|_B \le \frac{K(3^r+3)\theta}{3^r}\|x\|_A^r$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned}\|3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m})\|_B^p &\le \sum_{j=l}^{m-1} \|3^j f(\frac{x}{3^j}) - 3^{j+1} f(\frac{x}{3^{j+1}})\|_B^p \\ &\le \frac{K^p(3^r+3)^p\theta^p}{3^{pr}} \sum_{j=l}^{m-1} \frac{3^{pj}}{3^{prj}}\|x\|_A^{pr} \qquad (14)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (14) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3. □

3 Isometric isomorphisms between quasi-Banach algebras

Throughout this section, assume that A is a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and unit e and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$ and unit e'. Let K be the modulus of concavity of $\|\cdot\|_B$.

We investigate isometric isomorphisms between quasi-Banach algebras, associated to the Cauchy functional equation.

Theorem 3.1 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (1) *and* (3) *such that*

$$f(xy) = f(x)f(y) \qquad (15)$$

for all $x, y \in A$. If $lim_{n\to\infty} 2^n f(\frac{e}{2^n}) = e'$ and $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is an isometric isomorphism.

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2). By Theorem 2.1, there exists an isometric homomorphism $H : A \to B$ satisfying (4). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

It follows from (15) that

$$\begin{aligned} H(x) &= H(ex) = \lim_{n\to\infty} 2^n f(\frac{ex}{2^n}) = \lim_{n\to\infty} 2^n f(\frac{e}{2^n} \cdot x) = \lim_{n\to\infty} 2^n f(\frac{e}{2^n}) f(x) \\ &= e' f(x) = f(x) \end{aligned}$$

for all $x \in A$. So the bijective mapping $f : A \to B$ is an isometric isomorphism. □

Theorem 3.2 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (1), (3) *and* (15). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is an isometric isomorphism.*

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2). By Theorem 2.2, there exists an isometric homomorphism $H : A \to B$ satisfying (7). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

We investigate isometric isomorphisms between quasi-Banach algebras, associated to the Jensen functional equation.

Theorem 3.3 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping with $f(0) = 0$ satisfying* (3), (9) *and* (15). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{3^n} f(3^n e) = e'$, then the mapping $f : A \to B$ is an isometric isomorphism.*

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2). By Theorem 2.3, there exists an isometric homomorphism $H : A \to B$ satisfying (10). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

Theorem 3.4 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping with $f(0) = 0$ satisfying (3), (9) and (15). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} 3^n f(\frac{e}{3^n}) = e'$, then the mapping $f : A \to B$ is an isometric isomorphism.*

Proof. Since $f(xy) - f(x)f(y) = 0$ for all $x, y \in A$, the mapping $f : A \to B$ satisfies (2). By Theorem 2.4, there exists an isometric homomorphism $H : A \to B$ satisfying (13). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} 3^n f(\frac{x}{3^n})$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,53-59,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

BETTER ERROR ESTIMATION FOR SZÁSZ-MIRAKJAN-BETA OPERATORS

OKTAY DUMAN, MEHMET ALI ÖZARSLAN, AND HÜSEYIN AKTUĞLU

ABSTRACT. In this paper, we present a modification of a sequence of mixed summation-integral type operators having Szász and Beta basis functions in summation and integration, the so-called Szász-Mirakjan-Beta operators. Then we show that our modified operators have a better error estimation on the interval $[0, 2]$. Furthermore, we give an r-th order generalization of the modified Szász-Mirakjan-Beta operators and investigate their approximation properties.

1. Introduction

The classical Szász-Mirakjan operators are defined by

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

where $f \in C[0,\infty)$, $x \geq 0$ and $n \in \mathbb{N}$.

Some approximation properties of the Szász-Mirakjan operators and their modifications were studied by Agrawal and Kasana [1], Duman and Özarslan [3], Finta [4], Finta, Govil and Gupta [5], Gupta [6], Gupta and Noor [7], Gupta, Noor and Beniwal [8], Gupta and Pant [9], Srivastava and Gupta [11], Totik [12], Zeng and Piriou [13]. Further properties and general approximation results on these operators may be found in the monograph by Altomore and Campiti [2].

Recently Gupta and Noor [7] have proposed a sequence of mixed summation-integral type operators, the so-called Szász-Mirakjan-Beta operators, as follows:

(1.1)

$$U_n(f;x) = e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!B(n+1,k)} \int_0^{\infty} f(t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt \; + e^{-nx} f(0),$$

where $f \in C[0,\infty)$ such that $|f(t)| \leq M(1+t)^{\gamma}$ for some $M > 0$, $\gamma > 0$.

Now, for the operators U_n given by (1.1), the following lemma follows from [7] immediately.

Key words and phrases. Szász-Mirakjan-Beta operators; positive linear operators; the Korovkin-type approximation theorem; modulus of continuity; Lipschitz class functionals.

2000 *Mathematics Subject Classification.* 41A25, 41A36.

Lemma A [7]. *Let $e_i(x) = x^i$, $i = 0, 1, 2$. Then, for each $x \geq 0$ and $n > 1$, we have*

(*a*) $U_n(e_0; x) = 1$,
(*b*) $U_n(e_1; x) = x$,
(*c*) $U_n(e_2; x) = \dfrac{1}{n-1}\left(nx^2 + 2x\right)$.

Lemma A shows that the operators U_n preserve the test functions $e_0(x) = 1$ and $e_1(x) = x$. Actually, many well-known approximating operators preserve these test functions, such as Bernstein polynomials, Meyer-König and Zeller operators, Szász-Mirakjan operators, Baskakov operators etc. Observe that these operators do not preserve the test function $e_2(x) = x^2$. However, by modifying the Bernstein polynomials, King [10] presented a non-trivial sequence of positive linear operators which approximate each continuous function on $[0, 1]$ while preserving the functions e_0 and e_2. Then it is proved that these modified operators have a better rate of convergence than the classical Bernstein polynomials on the interval $[0, 1/3]$. Thus a natural question arises: can we construct a sequence of positive linear operators preserving the test functions e_0 and e_2 so that our modified operators have a better error estimation than Szász-Mirakjan-Beta operators. In the present paper we mainly focus on this problem.

2. Construction of the operators

We first consider the Banach lattice

$$C_\gamma[0, \infty) := \{f \in C[0, +\infty) : |f(x)| \leq M(1 + x)^\gamma \text{ for some } M > 0,\ \gamma > 0\}$$

endowed with the norm

$$\|f\|_\gamma := \sup_{x\in[0,+\infty)} \frac{|f(x)|}{(1 + x)^\gamma}.$$

Then, the set $\{e_0, e_1, e_2\}$ is a K_+-subset of $C_\gamma[0, \infty)$; also the space $C_\gamma[0, \infty)$ is isomorphic to $C[0, 1]$ (see, for details, [2]).

Let $\{r_n(x)\}$ be sequence of real-valued continuous functions defined on $[0, \infty)$ with $0 \leq r_n(x) < \infty$. Then we have

$$U_n(f; r_n(x)) = e^{-nr_n(x)} \sum_{k=1}^{\infty} \frac{(nr_n(x))^k}{k!B(n + 1, k)} \int_0^\infty f(t) \frac{t^{k-1}}{(1 + t)^{n+k+1}} dt + e^{-nr_n(x)} f(0),$$

where $x \in [0, \infty)$, $f \in C_\gamma[0, \infty)$, $\gamma > 0$ and $n \in \mathbb{N}$. Now, if we replace $r_n(x)$ by $r^*_n(x)$ defined as

$$r^*_n(x) := \frac{1}{n}\left(-1 + \sqrt{1 + n(n-1)x^2}\right), \quad x \geq 0 \text{ and } n \in \mathbb{N}, \tag{2.1}$$

then we get the following positive linear operators

(2.2)

$$U^*_n(f; x) := e^{-nr^*_n(x)} \sum_{k=1}^{\infty} \frac{(nr^*_n(x))^k}{k!B(n + 1, k)} \int_0^\infty f(t) \frac{t^{k-1}}{(1 + t)^{n+k+1}} dt + e^{-nr^*_n(x)} f(0),$$

where $f \in C_\gamma[0,\infty)$, $\gamma > 0$ and $x \geq 0$.

Then, observe that every U_n^* maps $C_B[0,+\infty)$, the space of all bounded and continuous functions on $[0,+\infty)$, into itself.

On the other hand, from Lemma A we obtain the following result at once.

Lemma 2.1. *For each $x \geq 0$, we have*

(a) $U_n^*(e_0;x) = 1$,

(b) $U_n^*(e_1;x) = \dfrac{1}{n}\left(-1+\sqrt{1+n(n-1)x^2}\right)$,

(c) $U_n^*(e_2;x) = x^2$.

Now, fix $b > 0$ and consider the lattice homomorphism $T_b : C[0,+\infty) \to C[0,b]$ defined by $T_b(f) := f|_{[0,b]}$ for every $f \in C[0,+\infty)$. In this case, we see that, for each $i = 0,1,2$,

$$\lim_{n\to\infty} T_b\left(U_n^*(e_i)\right) = T_b(e_i) \quad \text{uniformly on } [0,b]. \tag{2.3}$$

Thus, with the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4 (vi) of [2, p. 199]) we have the following: "Let X be a compact set and H be a cofinal subspace of $C(X)$. If E is a Banach lattice, $S : C(X) \to E$ is a lattice homomorphism and if $\{L_n\}$ is a sequence of positive linear operators from $C(X)$ into E such that $\lim_{n\to\infty} L_n(h) = S(h)$ for all $h \in H$, then $\lim_{n\to\infty} L_n(f) = f$ provided that f belongs to the Korovkin closure of H".

Hence, by using (2.3) and the above property we obtain the following Korovkin-type approximation result.

Theorem 2.2. $\lim_{n\to\infty} U_n^*(f;x) = f(x)$ *uniformly with respect to $x \in [0,b]$ provided $f \in C_\gamma[0,\infty)$, $\gamma > 0$ and $b > 0$.*

3. Better error estimation

In this section we compute the rate of convergence of the operators U_n^* defined by (2.2). Then, we will show that our operators has better error estimation on the interval $[0,2]$ than that of the Szász-Mirakjan-Beta operators U_n given by (1.1). To achieve this we use the modulus of continuity and the elements of Lipschitz class functionals.

If we define the function ψ_x, $(x \geq 0)$, by $\psi_x(y) = y - x$, then by Lemma 2.1 one can get the following result, immediately.

Lemma 3.1. *For every $x \geq 0$, we have*

(a) $U_n^*(\psi_x;x) = -x + \dfrac{1}{n}\left(-1+\sqrt{1+n(n-1)x^2}\right)$,

(b) $U_n^*(\psi_x^2;x) = 2x\left(x+\dfrac{1}{n}-\dfrac{\sqrt{1+n(n-1)x^2}}{n}\right)$.

Let $f \in C_B[0,+\infty)$ and $x \geq 0$. Then, the modulus of continuity of f denoted by $\omega(f,\delta)$, is defined to be

$$\omega(f,\delta) = \sup_{|y-x|\leq\delta;\ x,y\in[0,+\infty)} |f(y)-f(x)|.$$

Then we have the following

Theorem 3.2. *For every* $f \in C_B[0,+\infty)$, $x \geq 0$ *and* $n > 1$, *we have*

$$|U_n^*(f;x) - f(x)| \leq 2\omega(f, \delta_{n,x}),$$

where $\delta_{n,x} := \sqrt{2x\left(x - r_n^*(x)\right)}$ *and* $r_n^*(x)$ *is given by* (2.1).

Proof. Now, let $f \in C_B[0,+\infty)$ and $x \geq 0$. Using linearity and monotonicity of U_n^* we easily get, for every $\delta > 0$ and $n \in \mathbb{N}$, that

$$|U_n^*(f;x) - f(x)| \leq \omega(f,\delta)\left\{1 + \frac{1}{\delta}\sqrt{U_n^*\left(\psi_x^2;x\right)}\right\}.$$

Now applying Lemma 3.1 (*b*) and choosing $\delta = \delta_{n,x}$ the proof is completed. □

Remark. For the Szász-Mirakjan-Beta operators given by (1.1) we may write that, for every $f \in C_B[0,+\infty)$, $x \geq 0$ and $n > 1$,

$$|U_n(f;x) - f(x)| \leq 2\omega(f, \alpha_{n,x}), \tag{3.1}$$

where $\alpha_{n,x} := \sqrt{\frac{x(2+x)}{n-1}}$ (see [7]).

Now we claim that the error estimation in Theorem 3.2 is better than that of (3.1) provided $f \in C_B[0,+\infty)$ and $x \in [0,2]$. Indeed, for $0 \leq x \leq 2$, we have $\frac{x^2}{4} \leq 1$. Also since $(n-\frac{1}{2})^2 - n(n-1) = \frac{1}{4}$, we can write that

$$x^2\left[(n-\frac{1}{2})^2 - n(n-1)\right] \leq 1,$$

or

$$1 + n(n-1)x^2 \geq (n-\frac{1}{2})^2x^2$$

which gives

$$\sqrt{1+n(n-1)x^2} \geq \left(\frac{2n-1}{2}\right)x.$$

Then we obtain

$$-\frac{1}{n} + \frac{1}{n}\sqrt{1+n(n-1)x^2} \geq -\frac{1}{n} + \left(\frac{2n-1}{2n}\right)x.$$

Using the above inequality we have

$$x - r_n^*(x) \leq \frac{2+x}{2n}$$

or

$$2x\left(x - r_n^*(x)\right) \leq \frac{x(2+x)}{n} \leq \frac{x(2+x)}{n-1} \tag{3.2}$$

for $x \in [0,2]$ and $n > 1$. This guarantees that $\delta_{n,x} \leq \alpha_{n,x}$ for $x \in [0,2]$ and $n > 1$, which corrects our claim.

Now we can also compute the rate of convergence of the operators U_n^* by means of the elements of the Lipschitz class $Lip_M(\alpha)$, $(\alpha \in (0,1])$. To

get this, we recall that a function $f \in C_B[0,\infty)$ belongs to $Lip_M(\alpha)$ if the inequality

$$|f(y) - f(x)| \leq M\,|y-x|^{\alpha} \quad (x, y \in [0,\infty)) \tag{3.3}$$

holds.

Theorem 3.3. *For every $f \in Lip_M(\alpha)$, $x \geq 0$ and $n > 1$, we have*

$$|U_n^*(f;x) - f(x)| \leq M\left\{2x\left(x - r_n^*(x)\right)\right\}^{\frac{\alpha}{2}},$$

where $r_n^*(x)$ is given by (2.1).

Proof. Since $f \in Lip_M(\alpha)$ and $x \geq 0$, using inequality (3.3) and then applying the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we get

$$\begin{aligned}
|U_n^*(f;x) - f(x)| &\leq U_n^*\left(|f(y) - f(x)|\,;x\right) \\
&\leq M\,U_n^*\left(|y-x|^{\alpha}\,;x\right) \\
&\leq M\,\left\{U_n^*\left(\psi_x^2;x\right)\right\}^{\frac{\alpha}{2}} \\
&\leq M\left\{2x\left(x - r_n^*(x)\right)\right\}^{\frac{\alpha}{2}},
\end{aligned}$$

whence the result. □

Notice that as in the proof of Theorem 3.2, since $U_n(\psi_x^2;x) = \frac{x(2+x)}{n-1}$, the Szász-Mirakjan-Beta operators defined by (1.1) satisfy

$$|U_n(f;x) - f(x)| \leq M\left\{\frac{x(2+x)}{n-1}\right\}^{\frac{\alpha}{2}} \tag{3.4}$$

for every $f \in Lip_M(\alpha)$, $x \geq 0$ and $n > 1$. So, it follows from (3.2) that the above claim also holds for Theorem 3.2, i.e., the rate of convergence of the operators U_n^* by means of the elements of the Lipschitz class functionals is better than the ordinary error estimation given by (3.4) whenever $x \in [0,2]$.

4. $r-th$ ORDER GENERALIZATION OF THE OPERATORS U_n^*

Let $C_\gamma^{(r)}[0,\infty)$, $r = 0,1,2,\ldots$, denote the space of all functions $f \in C_\gamma[0,\infty)$ such that the r-th derivative $f^{(r)} \in C_\gamma[0,\infty)$ (for some $\gamma > 0$) with $f^{(0)}(x) := f(x)$. In the case of $r = 0$, the space $C_\gamma^{(0)}[0,\infty)$ coincides with $C_\gamma[0,\infty)$. Now we consider the following $r-$th order generalization of the positive linear operators U_n^* defined by (2.2):

$$\begin{aligned}
U_{n,r}^*(f;x) &= e^{-nr_n^*(x)} \sum_{k=0}^{\infty}\sum_{i=0}^{r} \frac{(nr_n^*(x))^k}{k!B(n+1,k)} \int_0^{\infty} f^{(i)}(t)\frac{(t-x)^i}{i!}dt, \\
&+e^{-nr_n^*(x)} \sum_{i=0}^{r} \frac{(-1)^i x^i f^i(0)}{i!}
\end{aligned} \tag{4.1}$$

where $f \in C_\gamma^{(r)}[0,\infty)$, $\gamma > 0$, $r = 0,1,2,\ldots$, $n \in \mathbb{N}$ and $r_n^*(x)$ is given by (2.1). Observe that $U_{n,0}^* = U_n^*$.

Now using the definition of the operators $U_{n,r}^*$ we may write that

$$U_{n,r}^*(f;x)=\int_0^\infty \sum_{i=0}^r W_n(x,t)f^{(i)}(t)\frac{(t-x)^i}{i!}dt, \tag{4.2}$$

where

$$W_n(x,t)=e^{-nr_n^*(x)}\sum_{k=1}^\infty \frac{(nr_n^*(x))^k}{k!B(n+1,k)}\frac{t^{k-1}}{(1+t)^{n+k+1}}+e^{-nr_n^*(x)}\delta(t)$$

and $\delta(t)$ is the Dirac delta function.

Thus we have the following

Theorem 4.1. *For all $f\in C_\gamma^{(r)}[0,\infty)$, $\gamma>0$, such that $f^{(r)}\in Lip_M(\alpha)$, and for every $x\geq 0$ we have*

$$\left|U_{n,r}^*(f;x)-f(x)\right|\leq \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left|U_n^*\left(|t-x|^{r+\alpha};x\right)\right|,$$

where $r=1,2,...$ and $B(\alpha,r)$ is the beta function.

Proof. By (4.2) and Lemma 2.1 (a) one can write that

$$f(x)-U_{n,r}^*(f;x)=\int_0^\infty W_n(x,t)\left\{f(x)-\sum_{i=0}^r f^{(i)}(t)\frac{(t-x)^i}{i!}\right\}dt \tag{4.3}$$

Then we known from Taylor's formula that

$$\begin{aligned} f(x)-\sum_{i=0}^r f^{(i)}(t)\frac{(x-t)^i}{i!} &= \frac{(x-t)^r}{(r-1)!} \\ \times\int_0^1(1-s)^{r-1}&\left\{f^{(r)}\left(t+s(x-t)\right)-f^{(r)}(t)\right\}ds. \end{aligned} \tag{4.4}$$

Since $f^{(r)}\in Lip_M(\alpha)$,

$$\left|f^{(r)}\left(t+s(x-t)\right)-f^{(r)}(t)\right|\leq M\,s^\alpha\,|t-x|^\alpha. \tag{4.5}$$

Using (4.5) and the usual definition of the beta integral in (4.4) we conclude that

$$\left|f(x)-\sum_{i=0}^r f^{(i)}(t)\frac{(x-t)^i}{i!}\right|\leq \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\,|t-x|^{r+\alpha}. \tag{4.6}$$

Thus, the proof is completed by considering (4.3) and (4.6). □

Finally, for the uniform convergence of the operators $U_{n,r}^*$ given by (4.2) we obtain the next result.

Theorem 4.2. *For every $f\in C_\gamma^{(r)}[0,\infty),\gamma>0$, $r=1,2,...$, such that $f^{(r)}\in Lip_M(\alpha)$, we have*

$$\lim_{n\to\infty}U_{n,r}^*(f;x)=f(x) \text{ uniformly with respect to } x\in[0,b],\ b>0.$$

Proof. Let $x \in [0,b]$ and define the function g by $g(t) = |t-x|^{r+\alpha}$. Then, from Theorem 2.2, it is clear that

$$\lim_{n\to\infty} U_n^*(g;x) = g(x) = 0 \text{ uniformly with respect to } x \in [0,b].$$

So the proof follows from Theorem 4.1 immediately. □

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,61-74,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

SOME THEOREMS ON IF-COMPACT LINEAR OPERATORS

HAKAN EFE

Abstract. The purpose of this paper to introduce IF-compact operators in IF-normed linear spaces in the sense of Lael and Nourouzi [7]. Also classical compact operator by means of IF-concept have investigated.

1. Introduction

The notion of fuzzy norm on a linear space was introduced by Katsaras [8] in 1984 firstly. In 1992, Felbin [4] gave a definition of a fuzzy norm on a linear space whose associated metric is Kaleva type [5]. In 1994, Chang and Mordeson [3] introduced another idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [9]. Xiao and Zhu [11] redefined the idea of Felbin's [4] definition of fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space. In 2003, Bag and Samanta [1] introduced a definition of a fuzzy norm and proved a decomposition theorem of a fuzzy norm into a family of crisp norms. In 2005, Bag and Samanta [2] gave an idea of fuzzy norm of a linear operator from a fuzzy normed linear space to another fuzzy normed linear space. They also defined various notion of continuities operators and boundedness of linear operators over fuzzy normed linear spaces. In 2004, Park [10] using the idea of intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space. In 2006, Lael and Nourouzi [6] introduced fuzzy compact operators between fuzzy normed spaces. Very recently Lael and Nourouzi [7] gave a new definition for IF-normed linear space and proved some theorems: open mapping, closed graph and uniform boundedness in IF-normed linear spaces.

In this paper, we introduce IF-compact operators in IF-normed linear spaces in the sense of Lael and Nourouzi. We have investigated classical compact operator by means of IF-concept.

2. Preliminaries

Definition 1 ([7]). *The 3-tuple (X,μ,ν) is said to be an IF-normed linear space if X is a real vector space, and μ, ν are F-sets of $X\times\mathbb{R}$ satisfying the following conditions for every $x,y\in X$ and $t,s\in\mathbb{R}$,*

(i) $\mu(x,t)=0$, for all non-positive real number t,
(ii) $\mu(x,t)=1$ for all $t\in\mathbb{R}^+$ if and only if $x=0$,
(iii) $\mu(cx,t)=\mu(x,\frac{t}{|c|})$, for all $t\in\mathbb{R}^+$ and $c\neq 0$,
(iv) $\mu(x+y,s+t)\geq\min\{\mu(x,t),\mu(y,s)\}$,
(v) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,

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2000 *Mathematics Subject Classification.* 46S40, 47A30, 47B07.
Key words and phrases. IF-normed linear space, IF-compact operator.

(vi) $\nu(x,t)=1$, for all non-positive real number t,
(vii) $\nu(x,t)=0$ for all $t\in\mathbb{R}^+$ if and only if $x=0$,
(viii) $\nu(cx,t)=\nu(x,\frac{t}{|c|})$, for all $t\in\mathbb{R}^+$ and $c\neq 0$,
(ix) $\nu(x+y,s+t)\leq\max\{\nu(x,t),\nu(y,s)\}$,
(x) $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case, we will call (μ,ν) an IF-norm on X. In addition, (X,μ) is called an F-normed space.

It is easy to see that for every $x\in X$, the functions $\mu(x,\cdot)$ and $\nu(x,\cdot)$ are nondecreasing and nonincreasing on $\mathbb{R}$, respectively.

Lemma 1 ([7]). *Let (X,μ) be an F-normed linear space and $\nu(x,t)=1-\mu(x,t)$ for all $x\in X$ and $t\in\mathbb{R}$. Then (X,μ,ν) is an IF-normed linear space.*

Example 1. *Let $(X,||\cdot||)$ be a normed space and that μ_0,ν_0,μ_1 and ν_1 be F-sets on $X\times\mathbb{R}$ defined by*

$$\mu_0(x,t)=\begin{cases}\frac{t}{t+||x||} & if \quad t>0\\ 0 & if \quad t\leq 0\end{cases} \quad and\ \nu_0(x,t)=\begin{cases}\frac{||x||}{t+||x||} & if \quad t>0\\ 1 & if \quad t\leq 0\end{cases},$$

$$\mu_1(x,t)=\begin{cases}\exp\left(-\frac{||x||}{t}\right) & if \quad t>0\\ 0 & if \quad t\leq 0\end{cases} \quad and\ \nu_1(x,t)=\begin{cases}1-\exp\left(-\frac{||x||}{t}\right) & if \quad t>0\\ 1 & if \quad t\leq 0\end{cases}.$$

Then (μ_0,ν_0) and (μ_1,ν_1) ae two IF-norms on X.

Definition 2 ([7]). *Let (X,μ,ν) be an IF-normed linear space and $(x_n)_n$ be a sequence in X. Then $(x_n)_n$ is said to be convergent to $x\in X$ if $\lim_{n\to\infty}\mu(x_n-x,t)=1$ and $\lim_{n\to\infty}\nu(x_n-x,t)=0$, for all $t>0$. We denote it by $x_n\to x$.*

Definition 3. *Let (X,μ,ν) be an IF-normed linear space and $(x_n)_n$ be a sequence in X. Then $(x_n)_n$ is said to be a Cauchy sequence, if $\lim_{n\to\infty}\mu(x_{n+p}-x_n,t)=1$ and $\lim_{n\to\infty}\nu(x_{n+p}-x_n,t)=0$ for all $t>0$, $p\in\mathbb{N}$.*

Theorem 1 ([7]). *Let (X,μ,ν) be an IF-normed linear space. Assume further that*

$$(\bigstar) \qquad \mu(x,t)=0 \text{ for all } t>0 \text{ implies } x=0.$$

Define

$$||x||_\alpha=\wedge\{t>0:\mu(x,t)\geq\alpha,\ \nu(x,t)\leq 1-\alpha\},$$

where $\alpha\in(0,1)$. Then $\{||\cdot||_\alpha:\alpha\in(0,1)\}$ is an ascending family of norms on X, and they are called α-norms on X corresponding to (or induced by) the IF-norm (μ,ν) on X.

Remark 1. *Let (X,μ,ν) be an IF-normed linear space. Assume further that, for $x\neq 0$, $\mu(x,\cdot)$ and $\nu(x,\cdot)$ are continuous functions of $\mathbb{R}$ and μ is strictly increasing on $\{t>0:0<\mu(x,t)<1\}$ and ν is strictly decreasing on $\{t>0:0<\nu(x,t)<1\}$. Let we show this ($\bigstar\bigstar$).*

Lemma 2 ([7]). *Let (X,μ,ν) be an IF-normed linear space satisfying ($\bigstar$) and $(x_n)_n$ be a sequence in X. Then $x_n\to x$ iff $\lim_{n\to\infty}||x_n-x||_\alpha=0$ for all $\alpha\in(0,1)$.*

3. Main Results

Definition 4. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. The operator T is called IF-continuous at $z \in X$ if for any $\varepsilon > 0$, and $\alpha \in (0, 1)$ there exist $\delta > 0$ and $\beta \in (0, 1)$ such that for all $x \in X$, if $\mu_1(x - z, \delta) > \beta$ and $\nu_1(x - z, \delta) < 1 - \beta$ then $\mu_2(T(x) - T(z), \varepsilon) > \alpha$ and $\nu_2(T(x) - T(z), \varepsilon) < 1 - \alpha$.*

If T is IF-continuous at each point of X, then T is said to be IF-continuous on X.

Definition 5. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. The operator T is called strongly IF-continuous at $z \in X$ if for any $\varepsilon > 0$, there exist $\delta > 0$ such that for all $x \in X$, $\mu_2(T(x) - T(z), \varepsilon) \geq \mu_1(x - z, \delta)$ and $\nu_2(T(x) - T(z), \varepsilon) \leq \nu_1(x - z, \delta)$.*

If T is strongly IF-continuous at each point of X, then T is said to be strongly IF-continuous on X.

Definition 6. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. The operator T is called weakly IF-continuous at $z \in X$ if for any $\varepsilon > 0$, and $\alpha \in (0, 1)$ there exist $\delta > 0$ such that for all $x \in X$, if $\mu_1(x - z, \delta) \geq \alpha$ and $\nu_1(x - z, \delta) \leq 1 - \alpha$ then $\mu_2(T(x) - T(z), \varepsilon) \geq \alpha$ and $\nu_2(T(x) - T(z), \varepsilon) \leq 1 - \alpha$.*

If T is weakly IF-continuous at each point of X, then T is said to be weakly IF-continuous on X.

Definition 7. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. The operator T is called sequentially IF-continuous at $x \in X$ if for any sequence $(x_n)_n$, $x_n \in X$ for all n, with $x_n \to x$ implies $T(x_n) \to T(x)$. I.e., for all $t > 0$, if $\lim_{n\to\infty} \mu_1(x_n - x, t) = 1$ and $\lim_{n\to\infty} \nu_1(x_n - x, t) = 0$ then $\lim_{n\to\infty} \mu_2(T(x_n) - T(x), t) = 1$ and $\lim_{n\to\infty} \nu_2(T(x_n) - T(x), t) = 0$.*

If T is sequentially IF-continuous at each point of X, then T is said to be sequentially IF-continuous on X.

Remark 2. *If a linear operator is strongly IF-continuous then it is weakly IF-continuous.*

Theorem 2. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. If T is strongly IF-continuous then it is sequentially IF-continuous but not conversely.*

Proof. We assume that T is strongly IF-continuous at $z \in X$. Then for any $\varepsilon > 0$, there exist $\delta > 0$ such that for all $x \in X$,

((i)) $\quad \mu_2(T(x) - T(z), \varepsilon) \geq \mu_1(x - z, \delta)$ and $\nu_2(T(x) - T(z), \varepsilon) \leq \nu_1(x - z, \delta)$.

Let $(x_n)_n$ be a sequence in X such that $x_n \to z$, i.e.,

((ii)) $$\lim_{n\to\infty} \mu_1(x_n - z, t) = 1 \text{ and } \lim_{n\to\infty} \nu_1(x_n - z, t) = 0$$

for all $t > 0$. Now from (i), $\mu_2(T(x_n) - T(z), \varepsilon) \geq \mu_1(x_n - z, \delta)$ and $\nu_2(T(x_n) - T(z), \varepsilon) \leq \nu_1(x_n - z, \delta)$ for $n = 1, 2, ...$ This implies,

$$\begin{aligned} \lim_{n\to\infty} \mu_2(T(x_n) - T(z), \varepsilon) &\geq \lim_{n\to\infty} \mu_1(x_n - z, \delta) \text{ and} \\ \lim_{n\to\infty} \nu_2(T(x_n) - T(z), \varepsilon) &\leq \lim_{n\to\infty} \nu_1(x_n - z, \delta). \end{aligned}$$

Hence, $\lim_{n\to\infty} \mu_2(T(x_n) - T(z), \varepsilon) = 1$ and $\lim_{n\to\infty} \nu_2(T(x_n) - T(z), \varepsilon) = 0$ by (ii). Since ε is a small arbitrary number, it follows that $T(x_n) \to T(z)$. □

To show that sequential IF-continuity of T does not imply strong IF-continuity of T, consider the following example:

Example 2. *Let $(X = \mathbb{R}, ||\cdot||)$ be a normed linear space where $||x|| = |x|$ for all $x \in \mathbb{R}$. Define μ_1, μ_2, ν_1 and $\nu_2 : X \times \mathbb{R} \to [0,1]$ by*

$$\mu_1(x,t) = \begin{cases} \frac{t}{t+|x|} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases} \quad \text{and } \nu_1(x,t) = \begin{cases} \frac{|x|}{t+|x|} & if \quad t > 0 \\ 1 & if \quad t \leq 0 \end{cases},$$

$$\mu_2(x,t) = \begin{cases} \frac{t}{t+k|x|} & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases} \quad \text{and } \nu_2(x,t) = \begin{cases} \frac{|x|}{t+k|x|} & if \quad t > 0 \\ 1 & if \quad t \leq 0 \end{cases},$$

where $k > 0$ is a constant. Then (X, μ_1, ν_1) and (Y, μ_2, ν_2) are IF-normed linear spaces. If we consider the function $T(x) = \frac{x^4}{1+x^2}$, then T is sequentially IF-continuous but not strongly IF-continuous.

Theorem 3. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. Then T is IF-continuous iff it is sequentially IF-continuous.*

Proof. Suppose T is IF-continuous at $z \in X$. Let $(x_n)_n$ be a sequence in X such that $x_n \to z$. Let $\varepsilon > 0$ be given and choose $\alpha \in (0,1)$. Since T is IF-continuous at $z \in X$, then there exist $\delta > 0$ and $\beta \in (0,1)$ such that for all $x \in X$, if $\mu_1(x - z, \delta) > \beta$ and $\nu_1(x - z, \delta) < 1 - \beta$ then $\mu_2(T(x) - T(z), \varepsilon) > \alpha$ and $\nu_2(T(x) - T(z), \varepsilon) < 1 - \alpha$. Since $x_n \to z$ in X, there exists pozitive integer n_0 such that $\mu_1(x_n - z, \delta) > \beta$ and $\nu_1(x_n - z, \delta) < 1 - \beta$ for all $n \geq n_0$. Then $\mu_2(T(x_n) - T(z), \varepsilon) > \alpha$ and $\nu_2(T(x_n) - T(z), \varepsilon) < 1 - \alpha$ for all $n \geq n_0$. So for a given $\varepsilon > 0$ and for any $\alpha \in (0,1)$, there exists pozitive integer n_0 such that $\mu_2(T(x_n) - T(z), \varepsilon) > \alpha$ and $\nu_2(T(x_n) - T(z), \varepsilon) < 1 - \alpha$ for all $n \geq n_0$. This implies $\lim_{n\to\infty} \mu_2(T(x_n) - T(z), \varepsilon) = 1$ and $\lim_{n\to\infty} \nu_2(T(x_n) - T(z), \varepsilon) = 0$. Since $\varepsilon > 0$ is arbitrary, thus $T(x_n) \to T(z)$ in (Y, μ_2, ν_2).

Next we suppose that T is sequentially IF-continuous at $z \in X$. If possible assume that T is not IF-continuous at $z \in X$. Thus there exists $\varepsilon > 0$ and $\alpha \in (0,1)$ such that for any $\delta > 0$ and $\beta \in (0,1)$, there exists y(depending on δ, β) such that

$$\begin{aligned} \mu_1(z - y, \delta) &> \beta \text{ and } \nu_1(z - y, \delta) < 1 - \beta \text{ but} \qquad \text{(i)} \\ \mu_2(T(z) - T(y), \varepsilon) &\leq \alpha \text{ and } \nu_2(T(z) - T(y), \varepsilon) \geq 1 - \alpha. \end{aligned}$$

Thus for $\beta = 1 - \frac{1}{n+1}$, $\delta = \frac{1}{n+1}$, $n = 1, 2, ...$, there exists y_n such that $\mu_1(z - y_n, \frac{1}{n+1}) > 1 - \frac{1}{n+1}$ and $\nu_1(z - y_n, \frac{1}{n+1}) < \frac{1}{n+1}$ but $\mu_2(T(z) - T(y_n), \varepsilon) \leq \alpha$ and $\nu_2(T(z) - T(y_n), \varepsilon) \geq 1 - \alpha$. Taking $\delta > 0$, there exists n_0 such that $\frac{1}{n+1} < \delta$ for all $n \geq n_0$. Then $\mu_1(z - y_n, \delta) \geq \mu_1(z - y_n, \frac{1}{n+1}) > 1 - \frac{1}{n+1}$ and $\nu_1(z - y_n, \delta) \leq$

$\nu_1(z-y_n, \frac{1}{n+1}) < \frac{1}{n+1}$ for all $n \geq n_0$. This implies $\lim_{n\to\infty} \mu_1(z-y_n, \delta) \geq 1$ and $\lim_{n\to\infty} \nu_1(z-y_n, \delta) \leq 0$. Hence $y_n \to z$. But from (i), $\mu_2(T(z)-T(y_n), \varepsilon) \leq \alpha$ and $\nu_2(T(z)-T(y_n), \varepsilon) \geq 1-\alpha$ so $\mu_2(T(z)-T(y_n), \varepsilon) \nrightarrow 1$ and $\nu_2(T(z)-T(y_n), \varepsilon) \nrightarrow 0$ as $n \to \infty$. Thus $T(y_n)$ does not convergence to $T(z)$ whereas $y_n \to z$ (w.r.t. (μ_1, ν_1)), which is a contradiction to our assuption. Hence T is IF-continuous at z. □

Definition 8 ([7])**.** *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. The operator T is called weakly IF-bounded if for any $\alpha \in (0,1)$, there exists constant $h_\alpha > 0$ such that for every $x \in X$ and $\varepsilon > 0$, $\mu_1(h_\alpha x, \varepsilon) \geq \alpha$ and $\nu_1(h_\alpha x, \varepsilon) \leq 1-\alpha \Rightarrow \mu_2(T(x), \varepsilon) \geq \alpha$ and $\nu_2(T(x), \varepsilon) \leq 1-\alpha$.*

Definition 9 ([7])**.** *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. The operator T is called strongly IF-bounded if there exists constant $h > 0$ such that for every $x \in X$ and $\varepsilon > 0$, $\mu_2(T(x), \varepsilon) \geq \mu_1(hx, \varepsilon)$ and $\nu_2(T(x), \varepsilon) \leq \nu_1(hx, \varepsilon)$.*

Theorem 4 ([7])**.** *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces satisfying ($\bigstar$), where $\mu_2(x, \cdot)$ and $\nu_2(x, \cdot)$ are continuous function on $\mathbb{R}$ for all $x \in X$. If the linear operator $T : (X, \mu_1, \nu_1) \to (Y, \mu_2, \nu_2)$ is bounded w.r.t. α-norms of (μ_1, ν_1) and (μ_2, ν_2), then T is weakly IF-bounded.*

Theorem 5. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. Then*

(i) T is strongly IF-continuous on X iff T is strongly IF-continuous at a point $x_0 \in X$.
(ii) T is strongly IF-continuous iff T is strongly IF-bounded.

Proof. (i) Since T is strongly IF-continuous at $x_0 \in X$, thus for each $\varepsilon > 0$, there exist $\delta > 0$ such that for all $x \in X$, $\mu_2(T(x) - T(x_0), \varepsilon) \geq \mu_1(x - x_0, \delta)$ and $\nu_2(T(x) - T(x_0), \varepsilon) \leq \nu_1(x - x_0, \delta)$. Taking any $y \in X$ and replacing x by $x + x_0 - y$ we get

$$\begin{aligned}
\mu_2(T(x+x_0-y) - T(x_0), \varepsilon) &\geq \mu_1(x + x_0 - y - x_0, \delta) \\
&\Rightarrow \mu_2(T(x) + T(x_0) - T(y) - T(x_0), \varepsilon) \geq \mu_1(x-y, \delta) \\
&\Rightarrow \mu_2(T(x) - T(y), \varepsilon) \geq \mu_1(x-y, \delta),
\end{aligned}$$

and

$$\begin{aligned}
\nu_2(T(x+x_0-y) - T(x_0), \varepsilon) &\leq \nu_1(x + x_0 - y - x_0, \delta) \\
&\Rightarrow \nu_2(T(x) + T(x_0) - T(y) - T(x_0), \varepsilon) \leq \nu_1(x-y, \delta) \\
&\Rightarrow \nu_2(T(x) - T(y), \varepsilon) \leq \nu_1(x-y, \delta).
\end{aligned}$$

Since $y \in X$ is arbitrary, it follows that T is strongly IF-continuous on X.

(ii) We suppose that T is strongly IF-continuous. Using continuity of T at $x = 0$, for $\varepsilon = 1$, there exists $\delta > 0$ such that $\mu_2(T(x) - T(0), 1) \geq \mu_1(x - 0, \delta)$ and $\nu_2(T(x) - T(0), \varepsilon) \leq \nu_1(x - 0, \delta)$, for all $x \in X$.

Suppose that $x \neq 0$ and $t > 0$. Putting $u = x/t$ then

$$\mu_2(T(x), t) = \mu_2(tT(u), t) = \mu_2(T(u), 1) \geq \mu_1(u, \delta) = \mu_1\left(\frac{x}{t}, \delta\right) = \mu_1\left(x, \frac{t}{h}\right)$$

and

$$\nu_2(T(x),t) = \nu_2(tT(u),t) = \nu_2(T(u),1) \le \nu_1(u,\delta) = \nu_1\left(\frac{x}{t},\delta\right) = \nu_1\left(x,\frac{t}{h}\right)$$

where $h = 1/\delta$. So $\mu_2(T(x),t) \ge \mu_1(hx,t)$ and $\nu_2(T(x),t) \le \nu_1(hx,t)$.

If $x \neq 0$ and $t \le 0$ then $\mu_2(T(x),t) = 0 = \mu_1(hx,t)$ and $\nu_2(T(x),t) = 0 = \nu_1(hx,t)$.

If $x = 0$ and $t \in \mathbb{R}$ then $T(0) = 0$ and

$$\begin{aligned}\mu_2(0,t) &= \mu_1\left(0,\frac{t}{h}\right) = 1 \text{ and } \nu_2(0,t) = \nu_1\left(0,\frac{t}{h}\right) = 0 \text{ if } t > 0,\\ \mu_2(0,t) &= \mu_1\left(0,\frac{t}{h}\right) = 0 \text{ and } \nu_2(0,t) = \nu_1\left(0,\frac{t}{h}\right) = 1 \text{ if } t \le 0.\end{aligned}$$

Hence T is strongly IF-bounded.

Conversely suppose that T is strongly IF-bounded. Then there exists $h > 0$ such that $\mu_2(T(x),\varepsilon) \ge \mu_1(hx,\varepsilon)$ and $\nu_2(T(x),\varepsilon) \le \nu_1(hx,\varepsilon)$ for all $x \in X$ and for all $\varepsilon > 0$. This implies

$$\mu_2(T(x)-T(0),\varepsilon) \ge \mu_1(x-0,\frac{\varepsilon}{h}) \text{ and } \nu_2(T(x)-T(0),\varepsilon) \le \nu_1(x-0,\frac{\varepsilon}{h})$$

for all $x \in X$ and for all $\varepsilon > 0$. Hence

$$\mu_2(T(x)-T(0),\varepsilon) \ge \mu_1(x-0,\delta) \text{ and } \nu_2(T(x)-T(0),\varepsilon) \le \nu_1(x-0,\delta)$$

where $\delta = \frac{\varepsilon}{h}$. This implies that T is strongly IF-continuous at 0 and hence it is strongly IF-continuous on X. □

Theorem 6. *Let (X,μ_1,ν_1) and (Y,μ_2,ν_2) be two IF-normed linear spaces and $T : X \to Y$ be a linear operator. Then*

(i) T is weakly IF-continuous on X iff T is weakly IF-continuous at a point $x_0 \in X$.
(ii) T is weakly IF-continuous iff T is weakly IF-bounded.

Proof. (i) Since T is weakly IF-continuous at $x_0 \in X$, thus for each $\varepsilon > 0$, there exist $\delta > 0$ such that for all $x \in X$, $\mu_1(x-x_0,\delta) \ge \alpha$ and $\nu_1(x-x_0,\delta) \le 1-\alpha \Rightarrow \mu_2(T(x)-T(x_0),\varepsilon) \ge \alpha$ and $\nu_2(T(x)-T(x_0),\varepsilon) \le 1-\alpha$. Taking any $y \in X$ and replacing x by $x+x_0-y$ we get

$$\mu_1(x+x_0-y-x_0,\delta) \ge \alpha \text{ and } \nu_1(x+x_0-y-x_0,\delta) \le 1-\alpha$$

$$\Rightarrow \mu_2(T(x+x_0-y)-T(x_0),\varepsilon) \ge \alpha \text{ and } \nu_2(T(x+x_0-y)-T(x_0),\varepsilon) \le 1-\alpha.$$

I.e.,

$$\begin{aligned}\mu_1(x-y,\delta) &\ge \alpha \text{ and } \nu_1(x-y,\delta) \le 1-\alpha\\ &\Rightarrow \mu_2(T(x)-T(y),\varepsilon) \ge \alpha \text{ and } \nu_2(T(x)-T(y),\varepsilon) \le 1-\alpha.\end{aligned}$$

Since $y \in X$ is arbitrary, it follows that T is weakly IF-continuous on X.

(ii) We suppose that T is weakly IF-continuous. Using continuity of T at $x = 0$, for $\varepsilon = 1$, there exists $\delta > 0$ such that $\mu_1(x-0,\delta) \ge \alpha$ and $\nu_1(x-0,\delta) \le 1-\alpha$ implies $\mu_2(T(x)-T(0),1) \ge \alpha$ and $\nu_2(T(x)-T(0),1) \le 1-\alpha$, for all $x \in X$. I.e., $\mu_1(x,\delta) \ge \alpha$ and $\nu_1(x,\delta) \le 1-\alpha$ implies $\mu_2(T(x),1) \ge \alpha$ and $\nu_2(T(x),1) \le 1-\alpha$, for all $x \in X$.

Suppose that $x \neq 0$ and $t > 0$. Putting $x = u/t$ then

$$\begin{aligned}
\mu_1(u/t, \delta) &\geq \alpha \text{ and } \nu_1(u/t, \delta) \leq 1-\alpha \\
&\Rightarrow \mu_2(T(u/t), 1) \geq \alpha \text{ and } \nu_2(T(u/t), 1) \leq 1-\alpha, \text{ i.e.,} \\
\mu_1(u, t\delta) &\geq \alpha \text{ and } \nu_1(u, t\delta) \leq 1-\alpha \\
&\Rightarrow \mu_2(T(u), t) \geq \alpha \text{ and } \nu_2(T(u), t) \leq 1-\alpha, \text{ i.e.,} \\
\mu_1(h_\alpha u, t) &\geq \alpha \text{ and } \nu_1(h_\alpha u, t) \leq 1-\alpha \\
&\Rightarrow \mu_2(T(u), t) \geq \alpha \text{ and } \nu_2(T(u), t) \leq 1-\alpha
\end{aligned}$$

where $h_\alpha = \frac{1}{\delta}$. This implies T is weakly IF-bounded.

If $x \neq 0$ and $t \leq 0$ then $\mu_1(h_\alpha x, t) = \mu_2(T(x), t) = 0$ and $\nu_1(h_\alpha x, t) = \nu_2(T(x), t) = 1$ for any $h_\alpha > 0$.

If $x = 0$ and $t \in \mathbb{R}$ then for $h_\alpha > 0$,

$$\begin{aligned}
\mu_1(h_\alpha \cdot 0, t) &= \mu_2(T(0), t) = 1 \text{ and } \nu_1(h_\alpha \cdot 0, t) = \nu_2(T(0), t) = 0 \text{ if } t > 0, \\
\mu_1(h_\alpha \cdot 0, t) &= \mu_2(T(0), t) = 0 \text{ and } \nu_1(h_\alpha \cdot 0, t) = \nu_2(T(0), t) = 1 \text{ if } t \leq 0.
\end{aligned}$$

Hence T is weakly IF-bounded.

Conversely suppose that T is weakly IF-bounded. Then there exists $h_\alpha > 0$ such that $\mu_1(h_\alpha x, t) \geq \alpha$ and $\nu_1(h_\alpha x, t) \leq 1-\alpha \Rightarrow \mu_2(T(x), t) \geq \alpha$ and $\nu_2(T(x), t) \leq 1-\alpha$ for all $x \in X$ and for all $t \in \mathbb{R}$. This implies

$$\begin{aligned}
\mu_1\left(x-0, \frac{t}{h_\alpha}\right) &\geq \alpha \text{ and } \nu_1\left(x-0, \frac{t}{h_\alpha}\right) \leq 1-\alpha \\
&\Rightarrow \mu_2(T(x)-T(0), t) \geq \alpha \text{ and } \nu_2(T(x)-T(0), t) \leq 1-\alpha.
\end{aligned}$$

Then,

$$\begin{aligned}
\mu_1\left(x-0, \frac{\varepsilon}{h_\alpha}\right) &\geq \alpha \text{ and } \nu_1\left(x-0, \frac{\varepsilon}{h_\alpha}\right) \leq 1-\alpha \\
&\Rightarrow \mu_2(T(x)-T(0), \varepsilon) \geq \alpha \text{ and } \nu_2(T(x)-T(0), \varepsilon) \leq 1-\alpha
\end{aligned}$$

for $\varepsilon > 0$. Hence

$$\begin{aligned}
\mu_1(x-0, \delta) &\geq \alpha \text{ and } \nu_1(x-0, \delta) \leq 1-\alpha \\
&\Rightarrow \mu_2(T(x)-T(0), \varepsilon) \geq \alpha \text{ and } \nu_2(T(x)-T(0), \varepsilon) \leq 1-\alpha
\end{aligned}$$

where $\delta = \frac{\varepsilon}{h_\alpha}$. This implies that T is weakly IF-continuous at $x = 0$ and hence it is weakly IF-continuous on X. □

Lemma 3. *Let (X, μ, ν) be an IF-normed linear spaces satisfying (★) and (★★) and $\{||\cdot||_\alpha : \alpha \in (0,1)\}$ be the family of corresponding α-norms of (μ, ν) on X. Then for $x_0 \in X$, $x_0 \neq 0$, $\mu(x_0, ||x_0||_\alpha) \geq \alpha$ and $\nu(x_0, ||x_0||_\alpha) \leq 1-\alpha$ for all $\alpha \in (0,1)$.*

Proof. Let $||x_0||_\alpha = \delta$, then $\delta > 0$. There exists a sequence $(t_n)_n$, $t_n > 0$, $n \in \mathbb{N}$ such that $\mu(x_0, t_n) \geq \alpha$, $\nu(x_0, t_n) \leq 1-\alpha$ and $t_n \downarrow \delta$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \mu(x_0, t_n) &\geq \alpha \text{ and } \lim_{n\to\infty} \nu(x_0, t_n) \leq 1-\alpha \\
&\Rightarrow \mu(x_0, \lim_{n\to\infty} t_n) \geq \alpha \text{ and } \nu(x_0, \lim_{n\to\infty} t_n) \leq 1-\alpha \text{ by } (\bigstar\bigstar) \\
&\Rightarrow \mu(x_0, ||x_0||_\alpha) \geq \alpha \text{ and } \nu(x_0, ||x_0||_\alpha) \leq 1-\alpha \text{ forall } \alpha \in (0,1).
\end{aligned}$$

□

Lemma 4. *Let (X, μ, ν) be an IF-normed linear spaces satisfying (★) and (★★) and $\{||\cdot||_\alpha : \alpha \in (0,1)\}$ be the family of corresponding α-norms of (μ, ν) on X. Then for $x_0 \neq 0$, $\alpha \in (0,1)$ and $t' > 0$, $||x_0||_\alpha = t'$ iff $\mu(x_0, t') = \alpha$ and $\nu(x_0, t') = 1 - \alpha$.*

Proof. Let $\alpha \in (0,1)$, $x_0 \neq 0$ and $t' = ||x_0||_\alpha = \wedge\{s : \mu(x_0, s) \geq \alpha,\ \nu(x_0, s) \leq 1 - \alpha\}$. Since $\mu(x, \cdot)$ and $\nu(x, \cdot)$ are continuous (by (★★)), from Lemma 3 we have $\mu(x_0, t') \geq \alpha$ and $\nu(x_0, t') \leq 1 - \alpha$. Also $\mu(x_0, t') \leq \mu(x_0, s)$ if $\mu(x_0, s) \geq \alpha$ and $\nu(x_0, t') \geq \nu(x_0, s)$ if $\nu(x_0, s) \leq 1 - \alpha$. If possible, let $\mu(x_0, t') > \alpha$ and $\nu(x_0, t') < 1 - \alpha$, then by the continuity of $\mu(x_0, \cdot)$ and $\nu(x_0, \cdot)$ at t', there exists $t'' < t'$ such that $\mu(x_0, t'') > \alpha$ and $\nu(x_0, t'') < 1 - \alpha$ which is impossible, since $t' = \wedge\{s : \mu(x_0, s) \geq \alpha,\ \nu(x_0, s) \leq 1 - \alpha\}$. Thus $\mu(x_0, t') \leq \alpha$ and $\nu(x_0, t') \geq 1 - \alpha$. Hence we get $\mu(x_0, t') = \alpha$ and $\nu(x_0, t') = 1 - \alpha$.

Next if $\mu(x_0, t') = \alpha$ and $\nu(x_0, t') = 1 - \alpha$, $\alpha \in (0,1)$, then from the definition $||x_0||_\alpha = \wedge\{t : \mu(x_0, t) \geq \alpha, \nu(x_0, t) \leq 1 - \alpha\} = t'$ (Since $\mu(x_0, \cdot)$ is strictly increasing on $\{t > 0 : 0 < \mu(x, t) < 1\}$ and ν is strictly decreasing on $\{t > 0 : 0 < \nu(x, t) < 1\}$). This completes the proof. □

Theorem 7. *Let (X, μ, ν) be an IF-normed linear spaces satisfying (★) and (★★) and $\{||\cdot||_\alpha : \alpha \in (0,1)\}$ be the family of corresponding α-norms of (μ, ν) on X. Then for any increasing (decreasing) sequence $(\alpha_n)_n$ in $(0,1)$, $\alpha_n \to \alpha$ implies $||x||_{\alpha_n} \to ||x||_\alpha$ for all $x \in X$.*

Proof. For $x = 0$, clearly $\alpha_n \to \alpha$ implies $||x||_{\alpha_n} \to ||x||_\alpha$.

Suppose $x \neq 0$. From Lemma 4, for $x \neq 0$, $\alpha \in (0,1)$ and $t' > 0$ we have $||x||_\alpha = t'$ iff $\mu(x, t') = \alpha$ and $\nu(x, t') = 1 - \alpha$. Let $(\alpha_n)_n$ be an increasing sequence in $(0,1)$ such that $\alpha_n \to \alpha \in (0,1)$. Let $||x||_{\alpha_n} \to t_n$ and $\mu(x, t) = \alpha$ and $\nu(x, t) = 1 - \alpha$. Since $\{||\cdot||_\alpha : \alpha \in (0,1)\}$ is an increasing family of norms, $(t_n)_n$ is an increasing sequence of real numbers and it is bounded above by t (since $||x||_{\alpha_n} \leq ||x||_\alpha$ for all $n \in \mathbb{N}$). Hence $(t_n)_n$ is convergent. Thus $\lim_{n\to\infty} \mu(x, t_n) = \lim_{n\to\infty} \alpha_n$ and $\lim_{n\to\infty} \nu(x, t_n) = \lim_{n\to\infty}(1 - \alpha_n)$ which implies $\mu(x, \lim_{n\to\infty} t_n) = \alpha$ and $\nu(x, \lim_{n\to\infty}, t_n) = 1 - \alpha$. Thus we have $\mu(x, \lim_{n\to\infty} t_n) = \mu(x, t)$ and $\nu(x, \lim_{n\to\infty}, t_n) = \nu(x, t)$, which implies $\lim_{n\to\infty} t_n = t$ by (★★). Therefore $\lim_{n\to\infty} ||x||_{\alpha_n} = ||x||_\alpha$. Similarly, if $(\alpha_n)_n$ is a decreasing sequence in $(0,1)$ and $\alpha_n \to \alpha \in (0,1)$, then it can be shown that $||x||_{\alpha_n} \to ||x||_\alpha$ for all $x \in X$. □

Theorem 8. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces satisfying (★) and (★★) and $T : X \to Y$ be a linear operator . Then T is weakly IF-bounded iff T is bounded w.r.t. α-norms of (μ_1, ν_1) and (μ_2, ν_2), $\alpha \in (0,1)$.*

Proof. First we suppose that T is weakly fuzzy bounded. Thus for all $\alpha \in (0,1)$, there exists $h_\alpha > 0$ such that for all $x \in X$, for all $t \in \mathbb{R}$ we have

$$\mu_1(h_\alpha x, t) \geq \alpha \text{ and } \nu_1(h_\alpha x, t) \leq 1 - \alpha \Rightarrow \mu_2(T(x), t) \geq \alpha \text{ and } \nu_2(T(x), t) \leq 1 - \alpha.$$

I.e.

$$\begin{aligned} \vee\{\beta &\in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha \Rightarrow \\ \vee\{\beta &\in (0,1) : ||T(x)||^2_\beta \leq t\} \geq \alpha. \end{aligned} \tag{i}$$

Now we show that,

$$\vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha \Leftrightarrow ||h_\alpha x||^1_\alpha \leq t.$$

If $x = 0$ then the relation is obvious. Suppose $x \neq 0$. Now if

(ii) $$\vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} > \alpha \text{ then } ||h_\alpha x||^1_\beta \leq t.$$

If $\vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} = \alpha$, then there exists an increasing sequence $(\alpha_n)_n$ in $(0,1)$ such that $\alpha_n \uparrow \alpha$ and $||h_\alpha x||^1_{\alpha_n} \leq t$. Then by Theorem 7 we have

(iii) $$||h_\alpha x||^1_\alpha \leq t.$$

Thus from (ii) and (iii) we get,

(iv) $$\vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha \Rightarrow ||h_\alpha x||^1_\alpha \leq t.$$

Next we suppose that $||h_\alpha x||^1_\alpha \leq t$.

If $||h_\alpha x||^1_\alpha < t$ then $\mu_1(h_\alpha x, t) \geq \alpha$ and $\nu_1(h_\alpha x, t) \leq 1 - \alpha$. I.e.

(v) $$\vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha.$$

If $||h_\alpha x||^1_\alpha = t$ i.e. $\wedge\{s : \mu_1(h_\alpha x, s) \geq \alpha \text{ and } \nu_1(h_\alpha x, s) \leq 1 - \alpha\} = t$, then there exists a decreasing sequence $(s_n)_n$ in $\mathbb{R}$ such that $s_n \downarrow t$ and $\mu_1(h_\alpha x, s_n) \geq \alpha$ and $\nu_1(h_\alpha x, s_n) \leq 1 - \alpha$

$\Rightarrow \lim_{n\to\infty} \mu_1(h_\alpha x, s_n) \geq \alpha$ and $\lim_{n\to\infty} \nu_1(h_\alpha x, s_n) \leq 1 - \alpha$

$\Rightarrow \mu_1(h_\alpha x, \lim_{n\to\infty} s_n) \geq \alpha$ and $\nu_1(h_\alpha x, \lim_{n\to\infty} s_n) \leq 1 - \alpha$ by ($\bigstar\bigstar$).

$\Rightarrow \mu_1(h_\alpha x, t) \geq \alpha$ and $\nu_1(h_\alpha x, t) \leq 1 - \alpha$.

(vi) $$\Rightarrow \vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha.$$

From (v) and (vi) it follows that,

(vii) $$||h_\alpha x||^1_\alpha \leq t \Rightarrow \vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha.$$

Hence from (iv) and (vii) we have,

(viii) $$\vee\{\beta \in (0,1) : ||h_\alpha x||^1_\beta \leq t\} \geq \alpha \Leftrightarrow ||h_\alpha x||^1_\alpha \leq t.$$

In a similar way we can show that,

(ix) $$\vee\{\beta \in (0,1) : ||T(x)||^2_\beta \leq t\} \geq \alpha \Leftrightarrow ||T(x)||^2_\alpha \leq t.$$

Therefore from (viii) and (ix) we have

$\mu_1(h_\alpha x, t) \geq \alpha$ and $\nu_1(h_\alpha x, t) \leq 1 - \alpha \Rightarrow \mu_2(T(x), t) \geq \alpha$ and $\nu_2(T(x), t) \leq 1 - \alpha$

then $||h_\alpha x||^1_\alpha \leq t \Rightarrow ||T(x)||^2_\alpha \leq t$. This implies that $||T(x)||^2_\alpha \leq h_\alpha ||x||^1_\alpha$ for all $\alpha \in (0,1)$.

Conversely suppose that for all $\alpha \in (0,1)$, there exists $h_\alpha > 0$ such that $||T(x)||^2_\alpha \leq h_\alpha ||x||^1_\alpha$ for all $x \in X$.

Then for $x \neq 0$, $||h_\alpha x||^1_\alpha \leq t \Rightarrow ||T(x)||^2_\alpha \leq t$, for all $t > 0$, i.e.,

$$\begin{aligned} \wedge\{s \;&:\; \mu_1(h_\alpha x, s) \geq \alpha \text{ and } \nu_1(h_\alpha x, s) \leq 1 - \alpha\} \leq t \Rightarrow \\ \wedge\{s \;&:\; \mu_2(T(x), s) \geq \alpha \text{ and } \nu_2(T(x), s) \leq 1 - \alpha\} \leq t. \end{aligned}$$

In a similar way as above we can show that

$$\begin{aligned} \wedge\{s \;&:\; \mu_1(h_\alpha x, s) \geq \alpha \text{ and } \nu_1(h_\alpha x, s) \leq 1 - \alpha\} \leq t \\ &\Leftrightarrow \mu_1(h_\alpha x, s) \geq \alpha \text{ and } \nu_1(h_\alpha x, s) \leq 1 - \alpha \end{aligned}$$

and

$$\begin{aligned} \wedge\{s \;&:\; \mu_2(T(x), s) \geq \alpha \text{ and } \nu_2(T(x), s) \leq 1 - \alpha\} \leq t \\ &\Leftrightarrow \mu_2(T(x), t) \geq \alpha \text{ and } \nu_2(T(x), t) \leq 1 - \alpha. \end{aligned}$$

Thus we have

$\mu_1(h_\alpha x, t) \geq \alpha$ and $\nu_1(h_\alpha x, t) \leq 1-\alpha \Rightarrow \mu_2(T(x), t) \geq \alpha$ and $\nu_2(T(x), t) \leq 1-\alpha$ for all $x \in X$.

If $x \neq 0$, $t \leq 0$ and $x = 0$, $t > 0$ then the above relation is obvious. Hence the theorem follows. □

4. *IF*-Compact Operators

Definition 10. *A subset A of an IF-normed linear space (X, μ, ν) is said to be IF-bounded iff there exist $t > 0$ and $0 < r < 1$ such that $\mu(x, t) > 1 - r$ and $\nu(x, r) < r$ for all $x \in A$.*

Definition 11. *A subset A of an IF-normed linear space (X, μ, ν) is said to be IF-compact if any sequence $(x_n)_n$ in A has a subsequence converging to an element of A.*

Definition 12. *The IF-closure of a subset B of an IF-normed linear space (X, μ, ν) is denoted by $\overline{B}$ and defined by the set of all $x \in X$ such that there is a sequence $(x_n)_n$ of elements of B with $x_n \to x$. We say that B is IF-closed if $B = \overline{B}$.*

Definition 13. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces. A linear operator $T : X \to Y$ is called IF-compact operator if for every IF-bounded subset M of X the subset of $T(M)$ of Y is relatively compact, that is the IF-closure of $T(M)$ is a IF-compact set.*

Example 3. *Let $(X, ||\cdot||_1)$ and $(Y, ||\cdot||_2)$ be two ordinary normed linear spaces, and $T : X \to Y$ be a compact operator. Then it is easy to see that $T : (X, \mu_1, \nu_1) \to (Y, \mu_2, \nu_2)$ is a IF-compact operator, where (μ_1, ν_1) and (μ_2, ν_2) are the standard IF-norms induced by ordinary norms $||\cdot||_1$ and $||\cdot||_2$, respectively, i.e.,*

$$\mu_i(x,t) = \begin{cases} \frac{t}{t+||x||_i} & if \quad t > 0,\ t \in \mathbb{R}, \\ 0 & if \quad t \leq 0 \end{cases},$$

and

$$\nu_i(x,t) = \begin{cases} \frac{||x||_i}{t+||x||_i} & if \quad t > 0,\ t \in \mathbb{R}, \\ 0 & if \quad t \leq 0 \end{cases}$$

for $i = 1, 2$.

Example 4. *Let $C[0,1]$ be the set of all real valued continuous functions on $[0,1]$ with the IF-norm*

$$\mu(\varphi(x), t) = \frac{t}{t + \sup_{x\in[0,1]} |\varphi(x)|} \text{ and } \nu(\varphi(x), t) = \frac{\sup_{x\in[0,1]} |\varphi(x)|}{t + \sup_{x\in[0,1]} |\varphi(x)|},$$

where $\varphi(x) \in C[0,1]$ and $t > 0$. If $k(x,y)$ with $x, y \in [0,1]$ is a real valued continuous function, then the operator $T : C[0,1] \to C[0,1]$ defined by

$$(T\varphi)(x) = \int_0^1 k(x,y)\varphi(y)dy,$$

where $\varphi \in C[0,1]$ is an IF-compact operator.

Theorem 9. *Let $T : (X, \mu_1, \nu_1) \to (Y, \mu_2, \nu_2)$ be a linear operator. Then T is IF-compact iff it maps every IF-bounded sequence $(x_n)_n$ in X onto a sequence $(T(x_n))_n$ in Y which has an IF-convergent subsequence.*

Proof. Suppose that T be a IF-compact operator and $(x_n)_n$ be an IF-bounded sequence in (X, μ_1, ν_1). The IF-closure of $\{T(x_n) : n \in \mathbb{N}\}$ is an IF-compact set. So $(T(x_n))_n$ has an IF-convergent subsequence by definition. Conversely, let A be a IF-bounded subset of (X, μ_1, ν_1). We show that the IF-closure of $T(A)$ is IF-compact. Let $(x_n)_n$ be a sequence in the closure of $T(A)$. For given $\varepsilon > 0$, $n \in \mathbb{N}$ and $t > 0$, there exists $(y_n)_n$ in $T(A)$ such that $\mu_2(x_n - y_n, \frac{t}{2}) > 1 - \varepsilon$ and $\nu_2(x_n - y_n, \frac{t}{2}) < \varepsilon$. Let $y_n = T(z_n)$, where $z_n \in A$. Since A is IF-bounded set, so is $\{z_n : n \in \mathbb{N}\}$. On the other hand, because T is IF-compact operator, $T(z_n)$ has an IF-convergent subsequence $(y_{n_k})_k = (T(z_{n_k}))_k$. Let $y_{n_k} \to y$ for some $y \in Y$. Hence $\mu_2\left(y_{n_k} - y, \frac{t}{2}\right) > 1 - \varepsilon$ and $\nu_2\left(y_{n_k} - y, \frac{t}{2}\right) < \varepsilon$ for all $n_k > n_0$. We have

$$\begin{aligned}
\mu_2\left(x_{n_k} - y, t\right) &\geq \min\left\{\mu_2\left(x_{n_k} - y_{n_k}, \frac{t}{2}\right), \mu_2\left(y_{n_k} - y, \frac{t}{2}\right)\right\} > 1 - \varepsilon, \\
\nu_2\left(x_{n_k} - y, t\right) &\leq \max\left\{\nu_2\left(x_{n_k} - y_{n_k}, \frac{t}{2}\right), \nu_2\left(y_{n_k} - y, \frac{t}{2}\right)\right\} < \varepsilon
\end{aligned}$$

for all $n_k > n_0$. Hence $(x_{n_k})_k$ is an IF-convergent subsequence of $(x_n)_n$. Thus the IF-closure of $T(A)$ is an IF-compact set. □

Definition 14. *Let (X, μ, ν) be an IF-normed linear space. We define the following subset of X:*

$$B_\alpha[x, r] = \{y \in X : \mu\left(x - y, t\right) \geq \alpha \text{ and } \nu\left(x - y, t\right) \leq 1 - \alpha\},$$

where $x \in X$, $\alpha \in (0, 1)$ and $r > 0$.

Theorem 10. *Let (X, μ, ν) be an IF-normed linear space satisfying (★)and, $\mu\left(x, \cdot\right)$ and $\nu\left(x, \cdot\right)$ are continuous functions on $\mathbb{R}$. Then X is finite dimensional iff $B_\alpha[x, r]$ is an IF-compact set in X, for each $\alpha \in (0, 1)$ and $r > 0$.*

Proof. Let $A_\alpha[x, r] = \{y \in X : ||x - y||_\alpha \leq r\}$, where $\alpha \in (0, 1)$ and $r > 0$. We first show that $B_\alpha[x, r] = A_\alpha[x, r]$. If $y \in B_\alpha[x, r]$, then $\mu\left(x - y, t\right) \geq \alpha$ and $\nu\left(x - y, t\right) \leq 1 - \alpha$. Since $||x - y||_\alpha \leq r$, then $y \in A_\alpha[x, r]$. Now if $y \in A_\alpha[x, r]$, then $||x - y||_\alpha \leq r$, or $\wedge\{t > 0 : \mu\left(x - y, t\right) \geq \alpha$ and $\nu\left(x - y, t\right) \leq 1 - \alpha\} \leq r$. If $\wedge\{t > 0 : \mu\left(x - y, t\right) \geq \alpha$ and $\nu\left(x - y, t\right) \leq 1 - \alpha\} < r$, then $\mu\left(x - y, t\right) \geq \alpha$ and $\nu\left(x - y, t\right) \leq 1 - \alpha$. Thus $y \in B_\alpha[x, r]$. If $\wedge\{t > 0 : \mu\left(x - y, t\right) \geq \alpha$ and $\nu\left(x - y, t\right) \leq 1 - \alpha\} = r$, there is $\left(t_n\right)_n$ such that $t_n \to r$, and $\mu\left(x - y, t\right) \geq \alpha$ and $\nu\left(x - y, t\right) \leq 1 - \alpha$. By continuity of $\mu\left(x, \cdot\right)$ and $\nu\left(x, \cdot\right)$ we obtain $\mu\left(x - y, r\right) = \lim_{n\to\infty} \mu\left(x - y, t_n\right) \geq \alpha$ and $\nu\left(x - y, r\right) = \lim_{n\to\infty} \nu\left(x - y, t_n\right) \leq 1 - \alpha$. Hence $y \in B_\alpha[x, r]$. Consequently $B_\alpha[x, r] = A_\alpha[x, r]$.

Suppose now that $\dim X < \infty$, $x \in X$, and $r > 0$. Choose the sequence $(x_n)_n$ in $B_\alpha[x, r]$. It is clear that $A_\alpha[x, r]$ is a compact subset of $(X, ||\cdot||_\alpha)$. Hence there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and $v \in A_\alpha[x, r]$ such that $x_{n_k} \stackrel{||\cdot||_\alpha}{\to} v$. Because in finite dimensional spaces all norms are equivalent, $x_{n_k} \stackrel{||\cdot||_\beta}{\to} v$, for all $\beta \in (0, 1)$. Thus by Lemma 2 we obtain $x_{n_k} \stackrel{(\mu,\nu)}{\to} v$. Since $B_\alpha[x, r] = A_\alpha[x, r]$, we have $v \in B_\alpha[x, r]$.

Conversely, let $B_\alpha[x, r]$ be IF-compact. To show that X is finite dimensional, it suffices to prove that $A_\alpha[x, r]$ is compact with respect to α-norm. Choose a sequence $(x_n)_n$ of $A_\alpha[x, r]$. Since $B_\alpha[x, r]$ is IF-compact, it has a IF-convergent subsequence $(x_{n_k})_k$. Lemma 2 implies that $(x_{n_k})_k$ is convergent under $||\cdot||_\alpha$. Thus $A_\alpha[x, r]$ is compact in normed linear space $(X, ||\cdot||_\alpha)$. This shows that X is finite dimensional. □

Lemma 5. *Let (X,μ_1,ν_1) and (Y,μ_2,ν_2) be two IF-normed linear spaces satisfying ($\bigstar$) and $T : X \to Y$ be an IF-fuzzy compact operator. Then $T : (X, ||\cdot||^1_\alpha) \to (Y, ||\cdot||^2_\alpha)$ is an ordinary compact operator for all $\alpha \in (0,1)$.*

Proof. We show that for each bounded sequence $(x_n)_n$ in $(X, ||\cdot||^1_\alpha)$, the sequence $(T(x_n))_n$ has a convergent subsequence in $(Y, ||\cdot||^2_\alpha)$. Let $(x_n)_n$ be a bounded sequence in $(X, ||\cdot||^1_\alpha)$. There exists $M > 0$ such that $||x_n||^1_\alpha < M$ for all $n \in \mathbb{N}$. Hence $\mu_1(x_n, M) \geq \alpha$ and $\nu_1(x_n, M) \leq 1-\alpha$, for all n, that is $(x_n)_n$ is IF-bounded. Thus $(T(x_n))_n$ has an IF-convergent subsequence $(T(x_{n_k}))_k$. By Lemma 2, $(T(x_{n_k}))_k$ is convergent under $||\cdot||^2_\alpha$. $\square$

Theorem 11. *Let (X,μ_1,ν_1) and (Y,μ_2,ν_2) be two IF-normed linear spaces satisfying ($\bigstar$) and ($\bigstar\bigstar$). Then*

(a) Every IF-compact linear operator $T : X \to Y$ is weakly IF-continuous.
(b) If $\dim X = \infty$ then the identity operator $I : X \to X$ is not an IF-compact operator.

Proof. (a) Choose $\alpha \in (0,1)$. Let $||\cdot||^1_\alpha$ and $||\cdot||^2_\alpha$ are α-norms on X and Y corresponding to the IF-norms (μ_1,ν_1) and (μ_2,ν_2), respectively. By Lemma 5, $T : (X, ||\cdot||^1_\alpha) \to (Y, ||\cdot||^2_\alpha)$ is a compact operator. Since compact operator is bounded, there exists $M_\alpha > 0$ such that $||T(x)||^2_\alpha \leq M_\alpha ||x||^1_\alpha$. Hence T is weakly IF-bounded by Theorem 8. Now Theorem 6 implies that T is weakly IF-continuous.

(b) The identity operator I maps $B_\alpha[0,1]$ to itself. Suppose on the contrary that I is a IF-compact operator. Then $\overline{B}_\alpha[0,1]$ is IF-compact for all $\alpha \in (0,1)$. Now $\overline{B}_\alpha[0,1] \subseteq A_\alpha[0,1] = B_\alpha[0,1]$ implies that $B_\alpha[0,1]$ is closed and so IF-compact. Thus X is finite dimensional by Theorem 10, which is a contradiction. $\square$

Theorem 12. *Let (X,μ_1,ν_1) and (Y,μ_2,ν_2) be two IF-normed linear spaces. Then the set of all IF-compact linear operators from X into Y is a linear subspace of $IF'(X,Y)$.*

Proof. Suppose that T_1 and T_2 are IF-compact linear operators from X into Y. Let $(x_n)_n$ be any IF-bounded sequence in X. The sequence $(T_1(x_n))_n$ has a IF-convergent subsequence $(T_1(x_{n_k}))_k$. The sequence $(T_2(x_{n_k}))_k$ also has an IF-convergent subsequence $(T_2(z_n))_n$. Hence $(T_1(z_n))_n$ and $(T_2(z_n))_n$ are IF-convergent sequences. Let $T_1(z_n) \to u$, and $T_2(z_n) \to v$. If $t > 0$, we have

$$\lim_{n\to\infty} \mu_2((T_1+T_2)(z_n)-u-v,t) \geq \lim_{n\to\infty} \min\left\{\mu_2\left(T_1(z_n)-u, \frac{t}{2}\right), \mu_2\left(T_2(z_n)-v, \frac{t}{2}\right)\right\},$$

and

$$\lim_{n\to\infty} \nu_2((T_1+T_2)(z_n)-u-v,t) \leq \lim_{n\to\infty} \max\left\{\nu_2\left(T_1(z_n)-u, \frac{t}{2}\right), \nu_2\left(T_2(z_n)-v, \frac{t}{2}\right)\right\}.$$

Thus, $\lim_{n\to\infty} \mu_2((T_1+T_2)(z_n)-u-v,t) = 1$ and $\lim_{n\to\infty} \nu_2((T_1+T_2)(z_n)-u-v,t) = 0$, for all $t > 0$. This implies T_1+T_2 is an IF-compact operator. Now if $T_1(x_{n_k}) \to y$, then

$$\begin{aligned}
\lim_{n\to\infty} \mu_2\left(\alpha T_1(x_{n_k}) - \alpha y, t\right) &= \lim_{n\to\infty} \mu_2\left(T_1(x_{n_k}) - y, \frac{t}{|\alpha|}\right) = 1 \text{ and} \\
\lim_{n\to\infty} \nu_2\left(\alpha T_1(x_{n_k}) - \alpha y, t\right) &= \lim_{n\to\infty} \nu_2\left(T_1(x_{n_k}) - y, \frac{t}{|\alpha|}\right) = 0
\end{aligned}$$

for all $\alpha \in \mathbb{R}\backslash\{0\}$, and $t > 0$. Hence αT_1 is also an IF-compact operator. $\square$

Theorem 13. *Let (X, μ, ν) be an IF-normed linear space, $T : X \to X$ be an IF-compact linear operator, and $S : X \to X$ be a strongly IF-continuous linear operator. Then ST and TS are IF-compact operators.*

Proof. Let $(x_n)_n$ be any IF-bounded sequence in X. Then $(T(x_n))_n$ has an IF-convegent subsequence $(T(x_{n_k}))_k$. Let $\lim_{n\to\infty} T(x_{n_k}) = y$. Since S is strongly IF-continuous, by Theorem 3 we have $S(T(x_{n_k})) \to S(y)$. Hence $ST(x_n)$ has an IF-convergent subsequence. This proves ST is IF-compact. Now to show that TS is IF-compact, choose any IF-bounded sequence $(x_n)_n$. There exist $t_0 > 0$ and $r_0 \in (0,1)$ such that $\mu_1(x_n, t_0) > 1 - r_0$ and $\nu_1(x_n, t_0) < r_0$ for all $n \geq 1$. By Theorem 6 we conclude that the operator S is a strongly IF-bounded linear operator. Thus there exists $M > 0$ such that $\mu_1(S(x_n), t_0M) > 1 - r_0$ and $\nu_1(S(x_n), t_0M) < r_0$, for all n. It follows that $(S(x_n))_n$ is IF-bounded sequence in $S(X)$. Because T is fuzzy compact, $(TS(x_n))_n$ has an IF-convergent subsequence. This completes the proof. □

Lemma 6. *Let (X, μ, ν) be an IF-normed linear space satisfying (★), $\mu(x, \cdot)$ and $\nu(x, \cdot)$ are continuous functions on $\mathbb{R}$ and $\dim X < \infty$. Then each IF-bounded sequence $(x_n)_n$ in (X, μ, ν) has an IF-convergent subsequence.*

Proof. Let $(x_n)_n$ be an IF-bounded sequence in (X, μ, ν). There are $t_0 > 0$ and $r_0 \in (0,1)$ such that $\mu(x_n, t_0) > 1 - r_0$ and $\nu(x_n, t_0) < r_0$, for all $n \in \mathbb{N}$. Hence $(x_n)_n \in B_{1-r_0}[0, t_0]$. By Theorem 10, $B_{1-r_0}[0, t_0]$ is an IF-compact set, so $(x_n)_n$ has an IF-convergent subsequence. □

Theorem 14. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces satisfying (★) and (★★). If $T : X \to Y$ is a linear operator where $\dim X < \infty$, then T is weakly IF-continuous.*

Proof. Since (X, μ_1, ν_1) and (Y, μ_2, ν_2) satisfy (★), we may suppose that $||\cdot||^1_\alpha$ and $||\cdot||^2_\alpha$ are α-norms on X and Y corresponding to the IF-norms (μ_1, ν_1) and (μ_2, ν_2), respectively. Since T is of finite dimension, thus $T : (X, ||\cdot||^1_\alpha) \to (X, ||\cdot||^2_\alpha)$ is a bounded linear operator for each $\alpha \in (0,1)$. Thus by Theorem 8, it follows that T is weakly IF-bounded. □

Theorem 15. *Let (X, μ_1, ν_1) and (Y, μ_2, ν_2) be two IF-normed linear spaces satisfying (★) and $\mu_2(x, \cdot)$ and $\nu_2(x, \cdot)$ are continuous function on $\mathbb{R}$, and $T : X \to Y$ a linear operator. Then the following hold:*

(a) If T is weakly IF-bounded and $\dim T(X) < \infty$, then T is an IF-compact operator.
(b) In addition if (X, μ_1, ν_1) and (Y, μ_2, ν_2) satisfying (★★) and $\dim T(X) < \infty$, then T is an IF-compact operator.

Proof. (a) Let $(x_n)_n$ be an IF-bounded sequence of X. There are $t_0 > 0$ and $r_0 \in (0,1)$ such that $\mu(x_n, t_0) > 1 - r_0$ and $\nu(x_n, t_0) < r_0$, for all $n \in \mathbb{N}$. Since T is weakly IF-bounded, there is $M_{1-r_0} > 0$ such that for all n,

$$\begin{aligned} \mu_1(x_n, t_0) &\geq 1 - r_0 \text{ and } \nu_1(x_n, t_0) \leq r_0 \Rightarrow \\ \mu_2\left(T(x_n), \frac{t_0}{M_{1-r_0}}\right) &\geq 1 - r_0 \text{ and } \nu_2\left(T(x_n), \frac{t_0}{M_{1-r_0}}\right) \leq r_0. \end{aligned}$$

It follows that $(T(x_n))_n$ is an IF-bounded sequence in $T(X)$. Since $\dim T(X) < \infty$, the sequence $(T(x_n))_n$ has a convergent subsequence by Lemma 6. Hence T is IF-compact.

(b) Theorem 14 implies that T is weakly IF-continuous. We also imply by Theorem 6 that T is weakly IF-bounded. Since $\dim T(X) < \infty$, by the (a) we conclude that T is an IF-compact operator. $\square$

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,75-81,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

$\mathcal{I}$-CONVERGENCE OF POSITIVE LINEAR OPERATORS ON Lp WEIGHTED SPACES

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ABSTRACT. In this paper, using the concept of $\mathcal{I}$-convergence we prove a Korovkin type approximation by means of positive linear operators defined on the weighted space $L_{p,\omega}(\mathbb{R})$. Also we state its n-dimensional analogue for the weighted space $L_{p,\Omega}(\mathbb{R}^n)$. Also we display an example such that our method of convergence is stronger than the usual convergence in the weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence by Steinhaus [7] and Fast [2]. If $K \subseteq \mathbb{N}$, the set of natural numbers, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $d(K) := \lim_n \frac{1}{n}|K_n|$, if limit exists. A sequence $x = (x_k)$ of real numbers is statistically convergent to L if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has the natural density zero.

A generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of $\mathbb{N}$ is given by Kostyrko, Mačaj, Šalát and Sleziak [6].

A non-void class $\mathcal{I} \subset P(\mathbb{N})$ is called the ideal if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$, $B \subset A \Rightarrow B \in \mathcal{I}$).

Throughout in this paper we consider admissible ideals, i.e. those which are different from $P(\mathbb{N})$ and contain all singletons. It is easy to check that $\mathcal{I} = \{K \subset \mathbb{N} : d(K) = 0\}$ forms an admissible ideal. A sequence $x = (x_k)$ of real numbers is $\mathcal{I}$-convergent to L if $\{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. In this case we write $\mathcal{I} - \lim x = L$. It is known that any convergent sequence is $\mathcal{I}$-convergent, but not conversely. Some examples and properties of $\mathcal{I}$-convergence may be found in [6]. [1 2]

2. THE MAIN RESULTS

For a sequence (L_n) of positive linear operators on $C(a,b)$, which is the space of continuous functions on $[a,b]$ and bounded on real axis $\mathbb{R}$, Korovkin [5] solved a problem which based on the existence of the limit $\lim_n L_n(f;x) = f(x)$. Also, Curtis [1] has extended this theorem for the functions in $L_p(-\pi,\pi)$. Gadjiev [3] stated and proved weighted Korovkin type theorems in the space of locally integrable functions in $\mathbb{R}$. Recently, Gadjiev and Aral [4] have investigated Korovkin type approximation theorems in the weighted spaces $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$.

The purpose of the present paper is to study a Korovkin type approximation theorem via $\mathcal{I}$-convergence in the weighted space $L_{p,\omega}(\mathbb{R})$ and $L_{p,\Omega}(\mathbb{R}^n)$.

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[1]2000 AMS *Subject Classifications:* Primary 41A10, 41A25, 41A36; Secondary 40A05, 40A30.
Key words and phrases. $\mathcal{I}$-convergence, positive linear operator, Korovkin theorem, weighted space.
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For a fixed $p \in [1,\infty)$, let ω be a positive continuous function on $\mathbb{R}$ satisfying the condition

$$\int\limits_{\mathbb{R}} t^{2p}\omega(t)\,dt < \infty. \tag{2.1}$$

By $L_{p,\,\omega}(\mathbb{R})$ $(1 \leq p < \infty)$ we denote the linear space of all functions f which are measurable, $p-$absolutely integrable on $\mathbb{R}$ with respect to the weight function ω, that is,

$$L_{p,\,\omega}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \|f\|_{p,\,\omega} := \left(\int\limits_{\mathbb{R}} |f(t)|^{p}\,\omega(t)\,dt \right)^{\frac{1}{p}} < \infty \right\}.$$

The minimum and maximum values of the function ω on finite intervals will be denoted by ω_{min} and ω_{max} respectively.

Theorem 1. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Let $(L_n)_{n\in N}$ be the sequence of positive linear operators $L_n : L_{p,\,\omega}(\mathbb{R}) \to L_{p,\,\omega}(\mathbb{R})$ and let the sequence $\{\|L_n\|\}$ be a uniformly bounded. If*

$$\mathcal{I} - \lim \left\| L_n\left(t^{i},\ x\right) - x^{i} \right\|_{p,\ \omega} = 0,\ i = 0,\ 1,\ 2, \tag{2.2}$$

then for any function $f \in L_{p,\,\omega}(\mathbb{R})$, *we have*

$$\mathcal{I} - \lim \|L_n f - f\|_{p,\ \omega} = 0.$$

Proof. We follow the proof of Theorem 1 in [4] up to a certain stage.

We can choose a such large number A such that for every $\varepsilon > 0$

$$\left\| f\chi_2^{A} \right\|_{p,\,\omega} < \varepsilon, \tag{2.3}$$

where χ_1^{A} be characteristic function of the interval $[-A,\,A]$ and $\chi_2^{A}(t) = 1 - \chi_1^{A}(t)$.

Since L_n is linear operator, we have

$$\begin{aligned} \|L_n f - f\|_{p,\,\omega} &= \left\| L_n\left(\chi_1^{A} + \chi_2^{A}\right) f - \left(\chi_1^{A} + \chi_2^{A}\right) f \right\|_{p,\,\omega} \\ &\leq \left\| L_n\left(\chi_1^{A} f\right) - \chi_1^{A} f \right\|_{p,\,\omega} + \left\| L_n\left(\chi_2^{A} f\right) - \chi_2^{A} f \right\|_{p,\,\omega} \\ &= I_n^{'} + I_n^{''}. \end{aligned} \tag{2.4}$$

Firstly, we compute $I_n^{''}$. Since $\{L_n\}$ is a uniformly bounded sequence, there exists a constant $K > 0$ such that

$$\|L_n\|_{p,\,\omega} \leq K. \tag{2.5}$$

Hence, from (2.3), we have

$$\begin{aligned} I_n^{''} &\leq \left\| L_n\left(\chi_2^{A} f\right) \right\|_{p,\,\omega} + \left\| \chi_2^{A} f \right\|_{p,\,\omega} \\ &\leq (K+1) \left\| \chi_2^{A} f \right\|_{p,\,\omega} \\ &< (K+1)\,\varepsilon. \end{aligned} \tag{2.6}$$

Furthermore, for every function $f \in L_{p,\omega}(\mathbb{R})$ the inequality

$$\|\chi_1^{A} f\|_{p} \leq \omega_{min}^{-1/p} \|f\|_{p,\omega}$$

implies $L_{p,\omega}(\mathbb{R}) \subset L_p(-A, A)$. Since the space of continuous functions is dense in $L_p(-A, A)$, given $f \in L_{p,\omega}(\mathbb{R})$, for each $\varepsilon' > 0$, there exists a continuous function φ on $[-A,\, A]$ satisfying the condition $\varphi(x) = 0$ for $|x| > A$ such that

$$\left\| (f-\varphi)\, \chi_1^A \right\|_p < \frac{\varepsilon'}{(K+1)\omega_{max}^{1/p}}.$$

We can write from these inequalities and (2.5)

$$\begin{aligned} I_n' &= \left\| L_n\left(\chi_1^A f\right) - \chi_1^A f \right\|_{p,\,\omega} \qquad (2.7)\\ &\leq \left\| L_n\left(f-\varphi\right)\chi_1^A \right\|_{p,\,\omega} + \left\| L_n\left(\varphi\chi_1^A\right) - \varphi\chi_1^A \right\|_{p,\,\omega} + \left\| \left(f-\varphi\right)\chi_1^A \right\|_{p,\,\omega} \\ &\leq \left\| L_n\left(\varphi\chi_1^A\right) - \varphi\chi_1^A \right\|_{p,\,\omega} + \varepsilon'. \end{aligned}$$

Since $\chi_2^{A_1}\chi_1^A\varphi = 0$ for some $A_1 > A$, we obtain the following inequality for the first term of (2.7)

$$\begin{aligned} \left\| L_n\left(\varphi\chi_1^A\right) - \varphi\chi_1^A \right\|_{p,\,\omega} &= \left\| \left(\chi_1^{A_1} + \chi_2^{A_1}\right) L_n\left(\varphi\chi_1^A\right) - \left(\chi_1^{A_1} + \chi_2^{A_1}\right)\varphi\chi_1^A \right\|_{p,\,\omega} \qquad (2.8)\\ &\leq \left\| \left[L_n\left(\varphi\chi_1^A\right) - \varphi\chi_1^A\right]\chi_1^{A_1} \right\|_{p,\,\omega} + \left\| \chi_2^{A_1} L_n\left(\varphi\chi_1^A\right) \right\|_{p,\,\omega}. \end{aligned}$$

Now, by denoting $M_\varphi = \max\limits_{t\in\mathbb{R}} \left|\varphi\left(t\right)\right| \chi_1^A\left(t\right)$, we get

$$\begin{aligned} \left\| \chi_2^{A_1} L_n\left(\varphi\chi_1^A\right) \right\|_{p,\,\omega} &= \left(\int\limits_{|t|>A_1} \left| L_n\left(\varphi\chi_1^A;\, t\right) \right|^p \omega\left(t\right) dt \right)^{\frac{1}{p}} \\ &\leq M_\varphi \left(\int\limits_{|t|>A_1} \left| L_n\left(1;\, t\right) - 1 \right|^p \omega\left(t\right) dt \right)^{\frac{1}{p}} \\ &\quad + M_\varphi \left(\int\limits_{\mathbb{R}} \chi_2^{A_1}\left(t\right)\omega\left(t\right) dt \right)^{\frac{1}{p}}. \end{aligned}$$

According to the hypotheses of the theorem, since $\omega \in L_1(\mathbb{R})$ we can choose the number A_1 such that

$$\left(\int\limits_{\mathbb{R}} \chi_2^{A_1}(t)\omega(t)dt \right)^{\frac{1}{p}} < \frac{\varepsilon'}{M_\varphi}.$$

So we have

$$\left\| \chi_2^{A_1} L_n\left(\varphi\chi_1^A\right) \right\|_{p,\,\omega} \leq M_\varphi \left\| L_n\left(1;\, x\right) - 1 \right\|_{p,\,\omega} + \varepsilon'. \qquad (2.9)$$

Substituting these inequalities into (2.7), we have

$$I_n' \leq 2\varepsilon' + M_\varphi \left\| L_n\left(1;\, x\right) - 1 \right\|_{p,\,\omega} + \left\| \left[L_n\left(\varphi\chi_1^A\right) - \varphi\chi_1^A\right]\chi_1^A \right\|_{p,\,\omega}. \qquad (2.10)$$

By the continuity of $\varphi\chi_1^A$ on $[-A,\ A]$, for given any $\varepsilon'>0$ there is a $\delta>0$ such that

$$\left|\varphi(t)\,\chi_1^A(t)-\varphi(x)\,\chi_1^A(x)\right|<\varepsilon'+2M_\varphi\frac{(t-x)^2}{\delta^2}.$$

Thus,

$$\begin{aligned}\left\|\left[L_n\left(\varphi\chi_1^A\right)-\varphi\chi_1^A\right]\chi_1^{A_1}\right\|_{p,\,\omega} &\leq \left\|\left[L_n\left(\left|\varphi(t)\,\chi_1^A(t)-\varphi(x)\,\chi_1^A(x)\right|;\,x\right)\right]\chi_1^{A_1}(x)\right\|_{p,\,\omega} \qquad (2.11)\\ &\quad+\left\|\varphi(x)\,\chi_1^{A_1}(x)\left(L_n(1;\,x)-1\right)\right\|_{p,\,\omega}\\ &\leq \left(\varepsilon'+\frac{2M_\varphi}{\delta^2}A^2+M_\varphi\right)\|L_n(1,x)-1\|_{p,\omega}\\ &\quad+\frac{4M_\varphi}{\delta^2}A\,\|L_n(t;\,x)-x\|_{p,\,\omega}+\frac{2M_\varphi}{\delta^2}\left\|L_n\left(t^2;\,x\right)-x^2\right\|.\end{aligned}$$

Substituting (2.11) into (2.10), we get

$$\begin{aligned}I_n' &\leq 2\varepsilon'+\left(\varepsilon'+\frac{2M_\varphi}{\delta^2}A^2+M_\varphi\right)\|L_n(1,x)-1\|_{p,\omega}+\frac{4M_\varphi}{\delta^2}A\left(\|L_n(t;\,x)-x\|_{p,\,\omega}\right.\\ &\quad\left.+\frac{2M_\varphi}{\delta^2}\left\|L_n\left(t^2;\,x\right)-x^2\right\|_{p,\,\omega}\right..\end{aligned}$$

Then, the inequality (2.4) is obtained the following,

$$\begin{aligned}\|L_nf-f\|_{p,\,\omega} &\leq 2\varepsilon'+(K+1)\,\varepsilon+\left(\varepsilon'+\frac{2M_\varphi}{\delta^2}A^2+M_\varphi\right)\|L_n(1,x)-1\|_{p,\omega}\\ &\quad+\frac{4M_\varphi}{\delta^2}A\,\|L_n(t;\,x)-x\|_{p,\,\omega}+\frac{2M_\varphi}{\delta^2}\left\|L_n\left(t^2;\,x\right)-x^2\right\|_{p,\,\omega}.\end{aligned}$$

We conclude that

$$\begin{aligned}\|L_nf-f\|_{p,\,\omega} &\leq 2\varepsilon'+(K+1)\,\varepsilon+B\left\{\|L_n(1,x)-1\|_{p,\omega}+\|L_n(t;\,x)-x\|_{p,\,\omega}\right.\\ &\quad\left.+\left\|L_n\left(t^2;\,x\right)-x^2\right\|_{p,\,\omega}\right\}\end{aligned}$$

where $B:=\max\left\{\varepsilon'+\frac{2M_\varphi}{\delta^2}A^2+M_\varphi,\frac{4M_\varphi}{\delta^2},\frac{2M_\varphi}{\delta^2}\right\}$. The last inequality shows that, for any $\varepsilon''>0$,

$$\begin{aligned}\left\{n:\|L_nf-f\|_{p,\,\omega}\geq\varepsilon''\right\} &\subseteq \left\{n:2\varepsilon'+(K+1)\,\varepsilon+B\left\{\|L_n(1,x)-1\|_{p,\omega}\right.\right. \qquad (2.12)\\ &\quad+\|L_n(t;\,x)-x\|_{p,\,\omega}\\ &\quad\left.\left.+\left\|L_n\left(t^2;\,x\right)-x^2\right\|_{p,\,\omega}\geq\varepsilon''\right\}\right\}.\end{aligned}$$

Now we write

$$\begin{aligned} D \quad &: \quad = \left\{ n : \|L_n f - f\|_{p,\,\omega} \geq \varepsilon^{''} \right\}, \\ D_1 \quad &: \quad = \left\{ n : \|L_n\,(1;\,x) - 1\|_{p,\,\omega} \geq \frac{\varepsilon^{''} - (2\varepsilon' + (K+1)\,\varepsilon)}{3B} \right\}, \\ D_2 \quad &: \quad = \left\{ n : \|L_n(t;x) - x\|_{p,\omega} \geq \frac{\varepsilon^{''} - (2\varepsilon' + (K+1)\,\varepsilon)}{3B} \right\}, \\ D_3 \quad &: \quad = \left\{ n : \left\|L_n(t^2;x) - x^2\right\|_{p,\omega} \geq \frac{\varepsilon^{''} - (2\varepsilon' + (K+1)\,\varepsilon)}{3B} \right\}. \end{aligned}$$

Then, it follows from (2.12) that $D \subseteq D_1 \cup D_2 \cup D_3$. By (2.2), $D_i \in \mathcal{I}$ for each $i = 1, 2, 3$. So, by the definition of ideal, $D_1 \cup D_2 \cup D_3 \in \mathcal{I}$, which yields $D \in \mathcal{I}$. So we have

$$\left\{ n : \|L_n f - f\|_{p,\,\omega} \geq \varepsilon^{''} \right\} \in \mathcal{I}$$

whence the result. ■

Now we give an example of a sequence of positive linear operators such that this operator satisfies the conditions of Theorem 1 but does not satisfy the conditions of classical Korovkin theorem in weighted space $L_{p,\,\omega}\,(\mathbb{R})$.

Example 2. *We choose* $\omega\,(x) = \left(\frac{1}{1+x^{6m}}\right)^p$, $p \geq 1$. *By* $L_{p,\,m}\,(\mathbb{R})$ *we denote the space of* $L_{p,\,\omega}\,(\mathbb{R})$. *Note that this selection of* $\omega\,(x)$ *satisfies the condition* (2.1). *Also note that for* $1 \leq p < \infty$

$$L_{p,\,m}\,(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} : \left(1 + x^{6m}\right)^{-1} f\,(x) \in L_p\,(\mathbb{R}) \right\},$$

where m *is a positive integer.*

The Kantorovich variant of the Szasz-Mirakyan operators [8] *by replacing* $f\left(\frac{kb_n}{n}\right)$ *with with an integral mean of* $f\,(x)$ *over the interval* $[(k+1)\,b_n/n,\ kb_n/n]$ *as follows:*

$$K_n\,(f;x) := \frac{n}{b_n} \sum_{k=0}^{\infty} P_{n,k}\,(x) \int_{kb_n/n}^{(k+1)b_n/n} f\,(t)\,dt, \quad n \in \mathbb{N}, x \in [0, b_n) \tag{2.13}$$

where (b_n) *is a sequence of positive real numbers satisfying the condition*

$$\lim_{n\to\infty} \frac{b_n}{n} = 0 \text{ and } \lim_{n\to\infty} b_n = \infty.$$

and

$$P_{n,k}\,(x) := e^{-nx/b_n} \frac{(nx)^k}{k! b_n^k},\ k = 0, 1, 2, ...$$

This operator satisfies the conditions of the classical Korovkin theorem in [4]. *Now define* $T_n : L_{p,\,m}\,(\mathbb{R}^+) \to L_{p,\,m}\,(\mathbb{R}^+)$ *by* $T_n\,(f;x) = (1 + \alpha_n)\,K_n\,(f;x)$ *where* (K_n) *is defined by* (2.13) *and* (α_n) *is an* $\mathcal{I}$*-convergent null sequence but not convergent. So it is possible to construct such an* (α_n). *Without of generality we may assume that* (α_n) *is a non-negative. It is known that*

$$K_n\,(1;x) = 1,\ K_n\,(t;x) = x + \frac{b_n}{n} \text{ and } K_n\left(t^2;x\right) = x^2 + \frac{2b_n}{n}x + \frac{b_n^2}{3n^2}.$$

Hence, (T_n) satisfies all conditions of Theorem 1. So, for any function $f \in L_{p,\omega}(\mathbb{R})$, we have

$$\mathcal{I} - \lim \|L_n f - f\|_{p,\ \omega} = 0,$$

but (T_n) does not satisfy the classical Korovkin theorem.

Now we establish an analogue of Theorem 1 for the weight space $L_{p,\Omega}(\mathbb{R}^n)$.
Let Ω be a positive continuous function in $\mathbb{R}^n$, satisfying the condition

$$\int_{\mathbb{R}^n} |t|^{2p}\, \Omega(t)\, dt < \infty$$

and for $1 \le p < \infty$

$$L_{p,\Omega}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R};\ \|f\|_{p,\Omega} = \left(\int_{\mathbb{R}^n} |f(t)|^p\, \Omega(t)\, dt \right)^{1/p} < \infty \right\}.$$

Theorem 3. *Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. Let $(L_n)_{n\in N}$ be the sequence of positive linear operators $L_n : L_{p,\Omega}(\mathbb{R}^n) \to L_{p,\Omega}(\mathbb{R}^n)$ and let the sequence $\{\|L_n\|\}$ be a uniformly bounded. If*

$$\begin{aligned}
\mathcal{I} - \lim_{n\to\infty} \|L_n(1;\ x) - 1\|_{p,\Omega} &= 0, \\
\mathcal{I} - \lim_{n\to\infty} \|L_n(t_i;\ x) - x_i\|_{p,\Omega} &= 0, \quad i = 1, 2, .., n, \\
\mathcal{I} - \lim_{n\to\infty} \left\| L_n\left(|t|^2;\ x\right) - |x|^2 \right\|_{p,\Omega} &= 0,
\end{aligned}$$

then for any function $f \in L_{p,\Omega}(\mathbb{R}^n)$,we have

$$\mathcal{I} - \lim_{n\to\infty} \|L_n f - f\|_{p,\Omega} = 0.$$

Proof. Using a similar idea, the wanted result is obtained. ■

Conclusion 4. *If we define $\mathcal{I}_f = \{K \subset \mathbb{N} : |K| < \infty\}$, $\mathcal{I}_d = \{K \subset \mathbb{N} : d(K) = 0\}$, $\mathcal{I}_{d_A} = \{K \subset \mathbb{N} : d_A(K) = 0\}$ and $\mathcal{I}_\mu = \{K \subset \mathbb{N} : \mu(K) = 0\}$, then we get the definitions of usual convergence, statistical convergence, A-statistical convergence and μ-statistical convergence, respectively. Details may be found in* [5]. *So, Theorem 1 and Theorem 3 are valid in this cases.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,83-100,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Identification of planar screens at low frequencies in thermoelasticity

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Abstract

In this paper the problem of determining a screen in an isotropic and homogeneous thermoelastic medium at low frequencies is considered. We formulate the direct problem for the planar screen in the thermoelastic medium and present an equivalent model for the problem under consideration at low-frequencies based on an non - homogeneous formulation via appropriate Dirac measures. We prove that the corresponding inverse problem of reconstructing the planar screen for two important cases: the thermal stress dislocation and the thermal displacement dislocation from boundary measurements has a unique solution. Finally, we present a reconstruction method for the above cases based on a proper use of certain vector test functions and the application of the two-sided Laplace transform.

1 Introduction

Scattering theory has played a central role in the scientific area of mathematical physics. The inverse scattering problem is a basic one in many applications, so, it concentrates the most interest and is in the foreground of the mathematical research. An excellent presentation of theoretical results and methods exploiting the inverse problem can be found in [10]. Problems connected with the scattering of waves by a very thin obstacle became very important, finding applications especially in non-destructive tests. For the

identification of cracks by boundary measurements significant results among others have been obtained by Friedman and Vogelius [14], Alessandrini *et al* [1] and Kress *et al* [9]. Amari *et al* in a series of papers, between them [2, 3], have proposed a new method for solving inverse problems using low-frequency waves. More precisely in [2] a computational attractive method is introduced for the identification of planar cracks located deep inside a heterogeneous conducting body based on low-frequency asymptotic analysis of Maxwell's equations. In [4] complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion are presented while in [15] identification of elastic inclusions and elastic moment tensors by boundary measurements is investigated.

In this work we examine an inverse scattering problem for a screen in thermoelasticity. Thermoelasticity combines the theory of elastodynamics with the theory of heat conduction. The governing equations form a coupled system of three hyperbolic and one parabolic equation. The hyperbolic equations have a source term which is proportional to the temperature gradient, while the parabolic equation has a source term which is proportional to the divergence of the velocity. In [16] can be found an excellent introduction of the theory of thermoelasticity. In [11] the scattering process, the scattering amplitudes and cross sections in thermoelasticity are presented. The theory of thermoelastic waves in the low-frequency region has been developed in [11]. Results for specific applications at low frequencies can be found in [8, 12]. In [7] a complete analysis of the three-dimensional thermoelastic scattering problems from screens is presented. In a recent work [5] the reciprocity gap principle is exploited in order to identify planar cracks in which are all located in the same plane in a homogeneous and isotropic thermoelastic medium. Their method estimates the plane by an explicit formula.

In Section 2 we formulate the direct scattering problem for the planar screen in the thermoelastic medium. Two types of screens are considered, thermal stress dislocations and displacements dislocations. The governing equations, the boundary conditions for each case and the radiation conditions for the well-posedness of the problem are presented. In Section 3 an equivalent model for the problem under consideration at low-frequencies is presented. Based on results for the elastic case given in [6] we prove that the direct problem for the screen can be replaced by a model prescribed by a non-homogeneous equation having a force term corresponding to a thermoelastic sources distribution on the planar screen. In Section 4 we face the inverse problem to determine the shape of the screen and the estimation of the cor-

responding Burgers vectors which describe mainly the physical properties of the dislocation [6]. The reconstruction method for the thermal stress dislocation as well as the thermal displacement dislocation is presented. Firstly, the method gives the plane of the screen and secondly provides a reconstruction of the screen by an inversion of a two-sided Laplace transform applied on data produced by Green's formula.

2 Formulation of the direct scattering problem

We consider the scattering process in a homogeneous and isotropic thermoelastic medium of Biot type in $\mathbb{R}^3$. The Biot medium is characterized by the real Lamé constants λ, μ where $\mu > 0$, $\lambda + 2\mu > 0$ and mass density ρ, the coefficient of thermal diffusity κ and the coupling constants γ and η.

We assume that inside the thermoelastic medium we have a planar dislocation Σ which is a bounded, simply connected, orientable smooth surface with a smooth non-self intersecting boundary. We denote the two sides of Σ as Σ_+ and Σ_- and we will use the superscripts + and - to indicate that the corresponding quantities are measured at neighboring points on Σ_+ and Σ_- respectively. The unit normal vector is denoted as $\widehat{\boldsymbol{\nu}}$ and, due to the fact that we consider a planar dislocation, it is the same on each point of Σ. We assume that $\widehat{\boldsymbol{\nu}} = \widehat{\boldsymbol{\nu}}_- = -\widehat{\boldsymbol{\nu}}_+$.

If we consider only time harmonic fields then the Biot system assumes the following spectral form

$$\begin{aligned} \left(\Delta^* + \rho\,\omega^2\right)\mathbf{u}\left(\mathbf{x}\right) &= \gamma\ \nabla\theta\left(\mathbf{x}\right), \quad \mathbf{x} \in \mathbb{R}^3 \backslash \Sigma && (1) \\ \left(\Delta + \frac{i\,\omega}{\kappa}\right)\theta\left(\mathbf{x}\right) &= i\,\omega\ \eta\ \nabla\cdot\mathbf{u}\left(\mathbf{x}\right), \quad \mathbf{x} \in \mathbb{R}^3 \backslash \Sigma && (2) \end{aligned}$$

where $\mathbf{u}\left(\mathbf{x}\right)$ is the elastic displacement, $\theta\left(\mathbf{x}\right)$ denotes the thermal variation field, Δ^* is the Lamé operator and ω is the frequency. We define as $q = \frac{i\omega}{\kappa}$ the spectral thermal constant. Note that whenever $\gamma \to 0+$, $\eta \to 0+$ the Biot system decouples into the Navier equation of dynamic elasticity and the heat conduction equation. Introducing a four dimensional vector notation as in [11], we have

$$\mathbf{U}\left(\mathbf{x}\right) := \begin{pmatrix} \mathbf{u}\left(\mathbf{x}\right) \\ \theta\left(\mathbf{x}\right) \end{pmatrix} \qquad (3)$$

and the Biot system (1, 2) is simply written as

$$L\mathbf{U}(\mathbf{x}) = \mathbf{0}, \quad , \mathbf{x} \in \mathbb{R}^3 \backslash \Sigma \tag{4}$$

where L is the differential operator $L := \begin{bmatrix} (\Delta^* + \rho\,\omega^2)\, I_3 & -\gamma\ \nabla \\ q\ \kappa\ \eta\nabla\cdot & \Delta + q \end{bmatrix}$. The thermoelastic operator L is a 4x4 matrix elliptic differential operator. It is not self-adjoint and the adjoint operator L^* may be obtained by L by replacing γ with $i\,\omega\ \eta$ and vice versa.

A unified field $\mathbf{U}(\mathbf{x})$ which satisfies (4) admits a decomposition into three vector fields

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}^1(\mathbf{x}) + \mathbf{U}^2(\mathbf{x}) + \mathbf{U}^s(\mathbf{x}) \tag{5}$$

with

$$\mathbf{U}^1(\mathbf{x}) = \left(\mathbf{u}^1(\mathbf{x}), \theta^1(\mathbf{x})\right), \ \mathbf{U}^2(\mathbf{x}) = \left(\mathbf{u}^2(\mathbf{x}), \theta^2(\mathbf{x})\right), \mathbf{U}^s(\mathbf{x}) = (\mathbf{u}^s(\mathbf{x}), 0) \tag{6}$$

The displacement fields $\mathbf{u}^1(\mathbf{x}), \mathbf{u}^2(\mathbf{x})$ and $\mathbf{u}^s(\mathbf{x})$ satisfy the following vectorial Helmholtz equations

$$\left(\Delta + k_1^2\right)\mathbf{u}^1(\mathbf{x}) = \mathbf{0}, \ \left(\Delta + k_2^2\right)\mathbf{u}^2(\mathbf{x}) = \mathbf{0}, \ \left(\Delta + k_s^2\right)\mathbf{u}^s(\mathbf{x}) = \mathbf{0} \tag{7}$$

The temperature fields satisfy the scalar Helmholtz equations

$$\left(\Delta + k_1^2\right)\theta^1(\mathbf{x}) = 0, \ \left(\Delta + k_2^2\right)\theta^2(\mathbf{x}) = 0 \tag{8}$$

The dispersion relations characterizing equation (4) are written as

$$k_1^2 + k_2^2 = q(1+\epsilon) + k_p^2, \ k_1^2 k_2^2 = q\ k_p^2, \ \mu\ k_s^2 = \rho\ \omega^2 \tag{9}$$

where k_1 and k_2 are the complex wavenumbers of the elastothermal and thermoelastic waves respectively and are given by

$$k_j = \omega/v_j + id_j, \ v_j > 0, \ d_j > 0, \ j = 1, 2$$

v_j are the phase velocities and d_j determine the corresponding dissipation coefficients, $k_s = \omega\sqrt{\rho/\mu}$ is the wavenumber of the uncoupled transverse wave; $k_p = \omega\sqrt{\rho/(\lambda + 2\mu)}$ is the wavenumber of the longitudinal wave in the absence of thermal interactions and $\epsilon = \gamma\ \eta\ \kappa/(\lambda + 2\mu)$ is the dimensionless thermoelastic coupling constant. From the dispersion relations it comes out

that the transverse field is not affected by the existence of the thermal field. The longitudinal field gives birth to two curl free fields, $\mathbf{u}^1(\mathbf{x})$ and $\mathbf{u}^2(\mathbf{x})$, which are characterized as the elastothermal and thermoelastic field respectively and in view of (9) are involved in the thermoelastic propagation of waves.

On the surface of the dislocation we assume that one of the following boundary conditions holds

$$B_j(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}})\,\mathbf{U}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \Sigma, \ j = 1, 2, 3, 4 \tag{10}$$

where the unified thermoelastic total field $\mathbf{U}(\mathbf{x})$ is given by

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}_{in}(\mathbf{x}) + \mathbf{U}_{sc}(\mathbf{x}) \tag{11}$$

$\mathbf{U}_{in}(\mathbf{x})$ is the incident field, which is an entire solution of (4), and $\mathbf{U}_{sc}(\mathbf{x})$ is the scattered field. If the incident field is a plane which propagates in the $\widehat{\mathbf{d}}$ direction then it admits the following form

$$\mathbf{U}_{in}(\mathbf{x}) = A^1 \begin{pmatrix} \widehat{\mathbf{d}} \\ \beta_1 \end{pmatrix} e^{i\ k_1 \widehat{\mathbf{d}}\cdot\mathbf{x}} + A^2 \begin{pmatrix} \beta_2 \widehat{\mathbf{d}} \\ 1 \end{pmatrix} e^{i\ k_2 \widehat{\mathbf{d}}\cdot\mathbf{x}} + A^3 \begin{pmatrix} \widehat{\mathbf{b}} \\ 0 \end{pmatrix} e^{i\ k_s \widehat{\mathbf{d}}\cdot\mathbf{x}} \tag{12}$$

where $\beta_1 = \frac{i\ k_1 q \kappa \eta}{k_1^2 - q}$, $\beta_1 = \frac{i k_2 \gamma}{\left(k_1^2 - k_2^2\right)} .The four boundary differential operators \mathrm{B}_j$ are given by

$$B_1(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) = \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

$$B_2(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) = R(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) = \begin{bmatrix} T(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) & -\gamma\ \widehat{\boldsymbol{\nu}} \\ 0 & \partial_{\mathbf{v}} \end{bmatrix} \tag{14}$$

$$B_3(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) = \begin{bmatrix} \mathbf{I}_3 & 0 \\ 0 & \partial_{\mathbf{v}} \end{bmatrix} \tag{15}$$

$$B_4(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) = \begin{bmatrix} T(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) & -\gamma\ \widehat{\boldsymbol{\nu}} \\ 0 & 1 \end{bmatrix} \tag{16}$$

$\mathbf{I}_3$ is 3x3 unit matrix and the operator $T(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}})$ is the surface traction operator of elasticity and is given by

$$T(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}}) = 2\mu\ \widehat{\boldsymbol{\nu}} \cdot \nabla + \lambda\ \widehat{\boldsymbol{\nu}}\,\nabla \cdot + \mu\ \ \widehat{\boldsymbol{\nu}} \times \nabla \times \tag{17}$$

The first thermoelastic problem corresponds to a rigid screen at constant temperature, it is of Dirichlet type, the second to a screen with a Neumann type condition in thermal insulation, the third to a Dirichlet condition in thermal insulation and the fourth to a Neumann type problem at a constant temperature.

For the well posedness of the exterior boundary value problems, the scattered field $\mathbf{U}_{sc}(\mathbf{x})$ must satisfy the Kupradze radiation conditions [16] as $r = |\mathbf{x}| \to \infty$ for $i = 1, 2, 3$ and $j = 1, 2$

$$\begin{aligned} \mathbf{u}^j(\mathbf{x}) &= o\left(\frac{1}{r}\right), \partial_{x_i}\mathbf{u}^j(\mathbf{x}) = O\left(\frac{1}{r^2}\right), \\ \theta^j(\mathbf{x}) &= o\left(\frac{1}{r}\right), \partial_{x_i}\theta^j(\mathbf{x}) = O\left(\frac{1}{r^2}\right) \\ \mathbf{u}^s(\mathbf{x}) &= O\left(\frac{1}{r}\right), \, r\left(\partial_{x_i}\mathbf{u}^s(\mathbf{x}) - i\,k_s\mathbf{u}^s(\mathbf{x})\right) = O\left(\frac{1}{r}\right) \end{aligned} \tag{18}$$

Uniqueness and existence theorems for thermoelastic screens in $\mathbb{R}^3$ have been proved by Cakoni [7].

3 The equivalent model at low frequencies

In what follows, we will consider the first two boundary value problems corresponding to the boundary operators (13) and (14). For the remaining boundary conditions, corresponding to the boundary operators (15) and (16), the consideration is similar and all the remaining analysis is straightforward.

Let us assume that the unified thermoelastic vector $\mathbf{U}(\mathbf{x})$ and the surface traction - flux vector $R(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}})\mathbf{U}(\mathbf{x})$ be discontinuous across Σ. We define as usually with the bracket notation the discontinuities of the thermal displacement field

$$[\mathbf{U}(\mathbf{x})] := \mathbf{U}_+(\mathbf{x}) - \mathbf{U}_-(\mathbf{x}) \tag{19}$$

and the thermal stress field

$$[R(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}})\mathbf{U}(\mathbf{x})] := -R(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}})\mathbf{U}_+(\mathbf{x}) - R(\partial_{\mathbf{x}}, \widehat{\boldsymbol{\nu}})\mathbf{U}_-(\mathbf{x}) \tag{20}$$

We consider a sufficiently large domain Ω containing the dislocation Σ. Using

the thermoelastic Green's integral formula [16] in Ω, we have

$$\int_{\Omega\backslash\Sigma} (\mathbf{W}\cdot L\ \mathbf{U} - \mathbf{U}\cdot L^*\ \mathbf{W})\, dx = -\int_{\Sigma_+} \mathbf{W}\cdot [R\ \mathbf{U}]\, ds$$
$$-\int_{\Sigma_+} [\mathbf{U}]\cdot R^*\mathbf{W} ds + \int_{\partial\Omega} (\mathbf{W}\cdot R\ \mathbf{U} - \mathbf{U}\cdot R^*\mathbf{W})\, ds \tag{21}$$

where R^* is the adjoint differential operator to R and can be applied by interchanging γ with $i\,\omega\ \eta$. The unified thermoelastic fundamental solution $\tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}')$ satisfies the equation

$$L\tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}') = -\mathbf{4}\ \boldsymbol{\pi}\ \delta(\mathbf{x}-\mathbf{x}')\,\tilde{\mathbf{I}}_4 \tag{22}$$

and its explicit form can be found in [7]. Applying Green's formula (21) for the fundamental solution $\tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}')$ and the field $\mathbf{U}$ and taking into account (22) we conclude to

$$\begin{aligned}\mathbf{U}(\mathbf{x}) \;=\; & -\int_{\Sigma_+} \tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}')\cdot[R\ \mathbf{U}(\mathbf{x}')]\, ds(\mathbf{x}') \\ & -\int_{\Sigma_+} [\mathbf{U}(\mathbf{x}')]\cdot R^*\ \tilde{\mathbf{E}}^T(\mathbf{x},\mathbf{x}')\, ds(\mathbf{x}') \\ & +\int_{\partial\Omega} \left(\tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}')\cdot R\mathbf{U}(\mathbf{x}') - \mathbf{U}(\mathbf{x}')\cdot R^*\ \tilde{\mathbf{E}}^T\ (\mathbf{x},\mathbf{x}')\right) ds(\mathbf{x}')\end{aligned} \tag{23}$$

From the properties of the delta function, equation (23) yields

$$\begin{aligned}\mathbf{U}(\mathbf{x}) \;=\; & \int_{\Omega} \tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}')\cdot\Bigg(\ -\int_{\Sigma_+} \delta(\mathbf{x}'-\mathbf{y})\ \ [R\ \mathbf{U}(\mathbf{y})]\, ds(\mathbf{y}) \\ & -\int_{\Sigma_+} [\mathbf{U}(\mathbf{y})]\cdot(R^*\ \delta(\mathbf{x}'-\mathbf{y}))\, ds(\mathbf{y})\ \Bigg)\, d\mathbf{x}' \\ & +\int_{\partial\Omega} \left(\tilde{\mathbf{E}}(\mathbf{x},\mathbf{x}')\cdot R\mathbf{U}(\mathbf{x}') - \mathbf{U}(\mathbf{x}')\cdot R^*\ \ \tilde{\mathbf{E}}^T(\mathbf{x},\mathbf{x}')\right) ds(\mathbf{x}')\end{aligned} \tag{24}$$

From equation (24) we infer that any boundary value problem concerning a dislocation problem on Σ, can be considered as an equivalent nonhomogeneous problem

$$L\mathbf{U}(\mathbf{x}) = \ -\delta_\Sigma\ \ [R\ \mathbf{U}(\mathbf{x}')] - [\mathbf{U}(\mathbf{x}')]\cdot(R^*\ \delta_\Sigma)\,, \mathbf{x} \in\mathbb{R}^3 \tag{25}$$

where δ_Σ is the Dirac measure on Σ.

Now, if we have the first boundary value problem, the field $\mathbf{U}(\mathbf{x}')$ vanishes on Σ which means that $[\mathbf{U}(\mathbf{x}')] = 0$ on Σ. So, for this case only the discontinuity of the unified thermal stress field remains in (25). Similarly, for the second boundary value problem, only the second term remains in the right hand side of (25) .

In what follows we will examine the problem in the frame of low frequencies. We can consider problems in the low frequency regime in cases where the characteristic dimension of the scatterer is much less than the wavelength of the incident field [12].

Let us now consider the thermal stress dislocation. In the right hand side of (25) only the term concerning the discontinuity of the stress field appears because the other term vanishes due to the continuity of the displacement field on Σ.

The low frequency properties of the Biot system of thermoelasticity is presented in [12]. In the low frequency limit we have $\omega \to 0$ and this means that all wavenumbers tend to 0. We define the constants

$$k = k_s,\ \tau_p = \frac{k_p}{k_s},\ \tau_q = \frac{\sqrt{q}}{k_s},\ \tau_m = \frac{k_m}{k_s},\ m = 1, 2$$

Using this notation, it is shown in [12] that the scattered field $\mathbf{U}(\mathbf{x})$ can be expanded in the form

$$\mathbf{U}(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{(i\,k)^n}{n!} \mathbf{U}_n(\mathbf{x})$$

for certain vector functions $\mathbf{U}_n(\mathbf{x})$.

The incident plane wave of the form (12) assumes the following expansion:

$$\begin{aligned} \mathbf{U}_{in}(\mathbf{x}) &= \sum_{n=0}^{\infty} \frac{(i\,k)^n}{n!} \left[A^1 \tau_1^n \begin{pmatrix} \widehat{\mathbf{d}} \left(\widehat{\mathbf{d}} \cdot \widehat{\mathbf{x}}\right)^n \\ n\bar{\beta}_1 \left(\widehat{\mathbf{d}} \cdot \widehat{\mathbf{x}}\right)^{n-1} \end{pmatrix} + A^2 \tau_2^n \begin{pmatrix} \bar{\beta}_2 \widehat{\mathbf{d}} \left(\widehat{\mathbf{d}} \cdot \widehat{\mathbf{x}}\right)^n \\ n \left(\widehat{\mathbf{d}} \cdot \widehat{\mathbf{x}}\right)^{n-1} \end{pmatrix} \right. \\ &\quad \left. + A^s \begin{pmatrix} \widehat{\mathbf{d}} \left(\widehat{\mathbf{d}} \cdot \widehat{\mathbf{x}}\right)^n \\ 0 \end{pmatrix} \right] \end{aligned} \tag{26}$$

We observe that the first term in the expansion is only of elastic character, that is $\mathbf{U}_{in}(\mathbf{x}) = \mathbf{U}_0(\mathbf{x}) + O(k)$ where

$$\mathbf{U}_{in,0}(\mathbf{x}) = A^1 \tau_1^0 \begin{pmatrix} \widehat{\mathbf{d}} \\ 0 \end{pmatrix} + A^2 \tau_2^0 \begin{pmatrix} \bar{\beta}_2 \widehat{\mathbf{d}} \\ 0 \end{pmatrix} + A^s \begin{pmatrix} \widehat{\mathbf{d}} \\ 0 \end{pmatrix}$$

Since we are interested to consider scattering problems in the low frequency region, we assume that the incident field is approximately constant. The same result holds also for the stress field, since the stress tensor has first order derivatives, the first order terms are deduced from $\mathbf{U}_{in,1}(\mathbf{x})$ where the application of the derivatives on $\left(\widehat{\mathbf{d}}\cdot\widehat{\mathbf{x}}\right)$ produce only constant terms. From this argument we can assume that the non - homogeneous term in (25) for the thermal stress dislocation is approximately the same at all points on the screen. So, supposing that

$$[R\ \mathbf{U}(\mathbf{x}')] = \mathbf{B} \tag{27}$$

where $\mathbf{B} = \begin{pmatrix} \mathbf{b} \\ \theta_b \end{pmatrix}$ is a constant vector, equation (25) is written as

$$L\mathbf{U}(\mathbf{x}) = -\delta_\Sigma\ \mathbf{B}\ ,\quad \mathbf{x}\in\mathbb{R}^3 \tag{28}$$

For the case of the thermal displacement dislocation, assuming again that at low frequencies the displacement field is constant at all points on the screen, equation (25) yields the form

$$L\mathbf{U}(\mathbf{x}) = \ -\mathbf{A}\cdot(R^*\ \delta_\Sigma)\ ,\quad \mathbf{x}\in\mathbb{R}^3 \tag{29}$$

where $\mathbf{A} = \begin{pmatrix} \mathbf{a} \\ \theta_a \end{pmatrix}$ is an unknown constant vector.

We next derive an identity which is crucial to the reconstruction method. For any C^2 vector $\mathbf{W}$ using the corresponding Green's formula [16] and equation (25), we conclude that

$$\begin{aligned} &\frac{1}{|\Sigma|}\int_\Sigma \mathbf{W}(\mathbf{x}')\cdot[R\ \mathbf{U}(\mathbf{x}')]\,ds(\mathbf{x}') + \int_{\Omega\backslash\Sigma}\mathbf{U}\cdot L^*\mathbf{W}dx \\ = &\int_{\partial\Omega}\left(\mathbf{U}(\mathbf{x}')\cdot R^*\mathbf{W}(\mathbf{x}') - \mathbf{W}(\mathbf{x}')\cdot R\ \mathbf{U}(\mathbf{x}')\right)ds(\mathbf{x}') \end{aligned} \tag{30}$$

where $|\Sigma|$ is the area of Σ. Assuming that $\mathbf{W} := \begin{pmatrix} \mathbf{w} \\ \theta_w \end{pmatrix}$ equation (30) can be written as

$$\begin{aligned} &\frac{1}{|\Sigma|}\int_\Sigma(\mathbf{w}\cdot\mathbf{b} + \theta_w\ \theta_b)\,ds + \int_{\Omega\backslash\Sigma}\mathbf{U}\cdot L^*\mathbf{W}dx \\ = &-\int_{\partial\Omega}\left(\mathbf{w}\cdot(T\mathbf{u} - qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta) + \theta_w\ \ \widehat{\boldsymbol{\nu}}\cdot\ \nabla\theta\right)ds \\ &+\int_{\partial\Omega}\left(\mathbf{u}\cdot(T\mathbf{w} - \gamma\ \widehat{\boldsymbol{\nu}}\ \theta_w\ ) + \theta\ \widehat{\boldsymbol{\nu}}\cdot\ \nabla\theta_w\ \right)ds \end{aligned} \tag{31}$$

4 Uniqueness of the inverse thermoelastic problem for a screen

This uniqueness of the inverse scattering problem is strongly related with the unique continuation principle. It is well known that for elliptic equations the unique continuation principle is equivalent with the uniqueness of the Cauchy problem. In our case, the uniqueness for the Cauchy problem is satisfied since we have homogeneous and isotropic thermoelastic medium, the coefficients are analytic and Holmgren's theorem can be applied [17].

The inverse problem we face is the determination of the shape of the planar screen Σ and the vector $\mathbf{B}$ if we have a thermal stress dislocation or the vector $\mathbf{A}$ if we have displacement dislocation. These vectors characterize the physical properties of the dislocation. We assume that the available information is the knowledge of the field $\mathbf{U}$ on the surface of a sphere Ω totally enclosing the planar screen.

We assume that we know a priori the type of the unknown screen that is we know that we have a thermal stress dislocation or a displacement dislocation. We prove the following uniqueness theorem for this inverse thermal stress dislocation problem:

Theorem 1 *Let Σ_1 and Σ_2 be two planar screens and $\mathbf{B}_1$, $\mathbf{B}_2$ be two constant vectors. Assume that $\mathbf{U}_j$, $j = 1, 2$ are solutions of the scattering problem:*

$$L\mathbf{U}_j(\mathbf{x}) = -\delta_{\Sigma_j}\mathbf{B}_j,\ j = 1, 2,\ \mathbf{x} \in \mathbb{R}^3 \tag{32}$$

and the fields are satisfying the radiation conditions (18). If $\mathbf{U}_1 = \mathbf{U}_2$ on $\partial\Omega$ then $\Sigma_1 = \Sigma_2$ and $\mathbf{B}_1 = \mathbf{B}_2$.

Proof. Suppose $\Sigma_1 \neq \Sigma_2$ and we have two fields $\mathbf{U}_1$ and $\mathbf{U}_2$ such that $\mathbf{U}_1 = \mathbf{U}_2$ on $\partial\Omega$. Then there exists a point $\mathbf{x}_0 \in \Sigma_1$ such that $\mathbf{x}_0 \notin \bar{\Sigma}_1$. It is well known that solutions of the thermoelasticity for a homogeneous and isotropic material admit the unique continuation property [17]. So, from equality of the vector fields $\mathbf{U}_1 = \mathbf{U}_2$ on $\partial\Omega$ and unique continuation we infer that there exists a constant ρ_0 such that

$$\mathbf{U}_1(\mathbf{x}_0 + \rho\widehat{\boldsymbol{\nu}}) = \mathbf{U}_2(\mathbf{x}_0 + \rho\widehat{\boldsymbol{\nu}}),\ 0 < \rho < \rho_0 \tag{33}$$

From the singular behavior of the Green's dyadic as $\mathbf{x}$ approaches $\mathbf{x}_0$ we know that its measure is of order

$$O\left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|}\right)$$

and consequently the same behavior has the field $\mathbf{U}_1$ in the vicinity of the point $\mathbf{x}_0$. So, $\mathbf{U}_2$ remains bounded as $\rho \to 0$ but $|\mathbf{U}_1| \underset{\rho\to 0}{\to} \infty$ which contradicts equality in the vicinity of $\mathbf{x}_0$ stated in (33). Therefore, $\Sigma_1 = \Sigma_2$.

The equality of the vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ follows directly from the application of the definition of the Dirac measure: We take the inner product in (32) with any vector distribution $\mathbf{\Psi}$ and integrate over Ω, that is $\int_\Omega L\mathbf{U}_j(\mathbf{x}') \cdot \mathbf{\Psi}(\mathbf{x}')\, dv = -|\Sigma_j|\, \mathbf{B}_j \cdot \mathbf{\Psi}(\mathbf{x})$, $j = 1, 2$. But we have proved that $\mathbf{U}_1 = \mathbf{U}_2$ on Ω, $\Sigma_1 = \Sigma_2$ and obviously $|\Sigma_1| = |\Sigma_2|$ so, $\mathbf{B}_1 \cdot \mathbf{\Psi}(\mathbf{x}) = \mathbf{B}_2 \cdot \mathbf{\Psi}(\mathbf{x})$ for any distribution on $\Sigma = \Sigma_1 = \Sigma_2$. which means that $\mathbf{B}_1 = \mathbf{B}_2$. ■

With some slight modifications we prove that uniqueness holds for the inverse problem from a thermal displacement dislocation:

Theorem 2 *Let Σ_1 and Σ_2 be two planar screens and $\mathbf{A}_1$, $\mathbf{A}_2$ be two constant vectors. Assume that $\mathbf{U}_j$, $j = 1, 2$ are solutions of the scattering problem:*

$$L\mathbf{U}_j(\mathbf{x}) = -\mathbf{A}_j \cdot \left(R^* \ \delta_{\Sigma_j}\right), \quad j = 1, 2, \mathbf{x} \in \mathbb{R}^3 \tag{34}$$

and the fields are satisfying the radiation conditions (18). If $\mathbf{U}_1 = \mathbf{U}_2$ on $\partial\Omega$ then $\Sigma_1 = \Sigma_2$ and $\mathbf{A}_1 = \mathbf{A}_2$.

Proof. Using the same argument as before we prove that there exists a point $\mathbf{x}_0 \in \Sigma_1$ such that $\mathbf{x}_0 \notin \bar{\Sigma}_1$ and

$$\mathbf{U}_1(\mathbf{x}_0 + \rho\widehat{\boldsymbol{\nu}}) = \mathbf{U}_2(\mathbf{x}_0 + \rho\widehat{\boldsymbol{\nu}}),\ 0 < \rho < \rho_0 \tag{35}$$

and consequently the same behavior has the field $\mathbf{U}_1$ in the vicinity of the point $\mathbf{x}_0$. So, $\mathbf{U}_2$ remains bounded as $\rho \to 0$ but $|\mathbf{U}_1| \underset{\rho\to 0}{\to} \infty$ [6] which contradicts equality in the vicinity of $\mathbf{x}_0$ stated in (33). Therefore, $\Sigma_1 = \Sigma_2$. The equality of the vectors $\mathbf{A}_1$ and $\mathbf{A}_2$ follows from the application of the definition of the Dirac measure: We take the inner product in (32) with any vector distribution $\mathbf{\Psi}$ and integrate over Ω, that is

$$\int_\Omega L\mathbf{U}_j(\mathbf{x}') \cdot \mathbf{\Psi}(\mathbf{x}')\, dv = -|\Sigma_j|\ \mathbf{A}_j \cdot R^*\mathbf{\Psi}(\mathbf{x}),\ j = 1, 2$$

But we have proved that $\mathbf{U}_1 = \mathbf{U}_2$ on Ω, $\Sigma_1 = \Sigma_2$ and obviously $|\Sigma_1| = |\Sigma_2|$ so, $\mathbf{A}_1 \cdot R^*\mathbf{\Psi}(\mathbf{x}) = \mathbf{A}_2 \cdot R^*\mathbf{\Psi}(\mathbf{x})$ for any distribution on $\Sigma = \Sigma_1 = \Sigma_2$. which means that $\mathbf{A}_1 = \mathbf{A}_2$. ■

5 Reconstruction method for thermal stress dislocation and thermal displacement dislocation

The purpose of our work is to extract geometrical and physical information about a screen in thermoelasticity at low frequencies. The reconstruction of the shape of the screen Σ and the estimation of its physical characteristics as the vector $\mathbf{b}$ and the θ_b is of great importance in applications. Following the ideas presented in [2] we examine first the case of the thermal stress dislocation. We assume that the fields $\mathbf{u}$, θ, $T\mathbf{u}$ and $\nabla\theta$ are known on $\partial\Omega$.

Suppose now that we want to simplify the form of (31) in low frequencies in such a way that we can use it for the inverse problem. If $\mathbf{U}$ is a solution of the basic differential operator then the term which is not possible to compute is $\int_{\Omega\setminus\Sigma}\mathbf{U}\cdot L^{*}\mathbf{W}dx$ since in the inverse problem Σ is unknown. The integral over $\partial\Omega$ for arbitrary test function is computable because measurements are performed there. The operator L^{*} can be written as $L^{*}=L_0^{*}+L_1^{*}$ where $L_0^{*}=\begin{bmatrix}\Delta^{*} & 0\\ \gamma\ \ \nabla\cdot & \Delta\end{bmatrix}$ and $L_1^{*}=\begin{bmatrix}\rho\,\omega^2 I_3 & 0\\ i\ \omega\ \eta\ \nabla\cdot & i\ \omega/\kappa\end{bmatrix}$. We observe that if we choose a function $\mathbf{W}=\begin{pmatrix}\mathbf{w}\\ \theta_w\end{pmatrix}$ which is a solution of the static adjoint version of the basic equation, then it holds that the application of L on $\mathbf{W}$ is approximately of order $O(\omega)$, that is $L^{*}\mathbf{W}=O(\omega^2)$. Static solutions of the adjoint operator have to satisfy the system of equations

$$\left\{\begin{array}{c}\Delta^{*}\mathbf{w}=\mathbf{0}\\ \Delta\theta_w=\gamma\ \ \nabla\cdot\mathbf{w}\end{array}\right.$$

In all cases we will use only elastic displacement fields satisfying $\nabla\cdot\mathbf{w}=0$ and simple harmonic functions θ_w.

In the sequel, we will make simple choices of functions $\mathbf{w}$ and θ_w in (31) in order to extract information about the screen. We first take as $\begin{pmatrix}\mathbf{w}\\ \theta_w\end{pmatrix}$ the constant test functions $\begin{pmatrix}\mathbf{e}_j\\ 0\end{pmatrix}$, $j=1,2,3$ where $\mathbf{e}_j$ are the unit vectors of the coordinate system. Then equation (31) gives the stress dislocation vector $\mathbf{b}=b\,\widehat{\mathbf{b}}=b\,(b_1,b_2,b_3)$. We have that

$$\mathbf{e}_j\cdot\mathbf{b} = b\ b_j = \int_{\partial\Omega}\left\{\mathbf{e}_j\cdot\left(T\mathbf{u} - qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right)\right\}ds + O\left(\omega\right),\ \ j = 1,2,3 \tag{36}$$

and summing

$$b\ = \sqrt{\sum_{j=1}^{3}\int_{\partial\Omega}\left|\mathbf{e}_j\cdot\left(T\mathbf{u} - qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right)\right|^2 ds + O\left(\sqrt{\omega}\right)}$$

If we define as $b^{(1)}$ the approximate magnitude of the vectors $\mathbf{b}$, that is $b^{(1)}\ = \sqrt{\sum_{j=1}^{3}\int_{\partial\Omega}\left|\mathbf{e}_j\cdot\left(T\mathbf{u} - qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right)\right|^2 ds}$ then the components of the vector $\mathbf{b}$, can be approximated from (36) by

$$b_j^{(1)} = \frac{1}{b^{(1)}}\int_{\partial\Omega}\left\{\mathbf{e}_j\cdot\left(T\mathbf{u} - qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right)\right\}ds$$

in the sense that $b_j = b_j^{(1)} + O\left(\sqrt{\omega}\right)$.

The thermal term θ_b can be derived similarly using the test vector $\begin{pmatrix}\mathbf{0}\\ 1\end{pmatrix}$, with this choice equation (31) is written as

$$\theta_b = \int_{\partial\Omega}\left\{\ \widehat{\boldsymbol{\nu}}\ \nabla\theta + qk\eta\ \mathbf{u}\cdot\ \widehat{\boldsymbol{\nu}}\right\}ds + O\left(\omega\right) \tag{37}$$

We will face, now, the most important problem, which is the determination of the shape of the screen. Suppose that the equation of the plane where the screen is located is given by $x_3 = A\ x_1 + B\ x_2 + C$. We will compute the constants A, B and C which a priori are not known. Define as

$$\begin{aligned} f\left(\mathbf{w},\theta_w\right) \ &= \ \int_{\partial\Omega}\left(\mathbf{u}\cdot\left(T\mathbf{w} - \gamma\ \widehat{\boldsymbol{\nu}}\ \theta_w\ \right) + \theta\ \widehat{\boldsymbol{\nu}}\cdot\ \ \nabla\theta_w\ \right)ds \\ &\quad -\int_{\partial\Omega}\left(\mathbf{w}\cdot\left(T\mathbf{u} - qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right) + \theta_w\ \ \widehat{\boldsymbol{\nu}}\cdot\ \ \nabla\theta\right)ds \end{aligned} \tag{38}$$

Then using the test functions of the form $\begin{pmatrix} x_i\mathbf{e}_j \\ 0\end{pmatrix}$, $i,j = 1,2,3$ we have

$$\frac{1}{|\Sigma|}\int_{\Sigma}x_i ds\ b_j = f\left(x_i\mathbf{e}_j,0\right)$$

and consequently we derive the linear system

$$A\,\frac{f\left(x_1\mathbf{e}_j,0\right)}{b_1}+B\,\frac{f\left(x_2\mathbf{e}_j,0\right)}{b_2}+C=\frac{f\left(x_3\mathbf{e}_j,0\right)}{b_3},\ j=1,2,3 \tag{39}$$

from which the equation of the plane is always determined. Of course, since we have in a preceding step computed approximately the constants b_j, denoted by $b_j^{(1)}$ the computation of the constants A, B, C is again up to an error.

The next important problem is to estimate the shape of the planar dislocation. For simplicity we assume that $x_3=0$ and in this case the unit normal is $\mathbf{e}_3$.

We consider the harmonic function $\phi(\mathbf{x})=e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}$, where $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=0$. We construct the function $\mathbf{w}:=\nabla\phi(\mathbf{x})=i\ \boldsymbol{\xi}\ e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}$ and set $\mathbf{W}=\begin{pmatrix}\nabla\phi(\mathbf{x})\\ 0\end{pmatrix}$. Then from (31) we infer that

$$\begin{aligned}&\frac{1}{|\Sigma|}\int_\Sigma\left(ie^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\right)\ \boldsymbol{\xi}\ \cdot\mathbf{b}\ ds\\ =\ &-\int_{\partial\Omega}\left\{i\ e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\ \boldsymbol{\xi}\cdot\left(T\mathbf{u}-qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right)-\mathbf{u}\cdot T\left(i\ \boldsymbol{\xi}\ e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\right)\right\}ds\end{aligned} \tag{40}$$

In this case we want to estimate approximately the shape of the planar dislocation $\Sigma^{(1)}$ writing (40) in the form

$$\frac{1}{\left|\Sigma^{(1)}\right|}\int_{\Sigma^{(1)}}e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\ \ ds=g\left(\ \boldsymbol{\xi}\ \right) \tag{41}$$

where

$$g\left(\ \boldsymbol{\xi}\ \right)=\frac{-1}{\boldsymbol{\xi}\ \cdot\mathbf{b}}\int_{\partial\Omega}\left\{\ e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\ \boldsymbol{\xi}\cdot\left(T\mathbf{u}-qk\eta\ \widehat{\boldsymbol{\nu}}\ \theta\right)-\mathbf{u}\cdot T\left(\boldsymbol{\xi}\ e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\right)\right\}ds$$

The integral $\frac{1}{\left|\Sigma^{(1)}\right|}\int_{\Sigma^{(1)}}e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}}\ \ ds$ can be written as $\frac{1}{\left|\Sigma^{(1)}\right|}\int_{\mathbb{R}^3}e^{i\ \boldsymbol{\xi}\cdot\mathbf{x}'}h_{\Sigma^{(1)}}\left(\mathbf{x}'\right)\ \ d\mathbf{x}'$ where $h_{\Sigma^{(1)}}\left(\mathbf{x}'\right)$ is the characteristic function of $\Sigma^{(1)}$. Writing $\mathbf{x}'=(x_1,x_2,x_3)$ and $\boldsymbol{\xi}\ =\left(\boldsymbol{\xi}'\ ,\sqrt{-\boldsymbol{\xi}'\ \cdot\boldsymbol{\xi}'}\right),\ \boldsymbol{\xi}\ '=(\xi_{\ 1},\xi_{\ 2})$ then equation (41) yields

$$\frac{1}{\left|\Sigma^{(1)}\right|}\int_{\mathbb{R}^3}e^{i\ \boldsymbol{\xi}'\cdot\mathbf{x}'}h_{\Sigma^{(1)}}\left(\mathbf{x}'\right)\ \ d\mathbf{x}'=e^{-i\ \sqrt{-\boldsymbol{\xi}'\ \cdot\boldsymbol{\xi}'}\ \ x_3}g\left(\ \boldsymbol{\xi}\ \right),\ x_3=0 \tag{42}$$

From the above equation we see that $e^{-i\sqrt{-\boldsymbol{\xi}'\cdot\boldsymbol{\xi}'}\,x_3}h(\boldsymbol{\xi})$ is the Laplace transform of $\frac{1}{|\Sigma^{(1)}|}h_{\Sigma^{(1)}}(\mathbf{x}')$ with compact support. Therefore

$$e^{-i\sqrt{-\boldsymbol{\xi}'\cdot\boldsymbol{\xi}'}\,x_3}h(\boldsymbol{\xi})$$

can be uniquely determined by the inverse Laplace transform since this function can be uniquely extended as an analytic function of the complex variable $\boldsymbol{\xi}'$. The inverse Laplace is

$$\Sigma^{(1)} = \sup p \int_{ip-\infty}^{ip+\infty} e^{-i\sqrt{-\boldsymbol{\xi}'\cdot\boldsymbol{\xi}'}\,x_3} g\left(\boldsymbol{\xi}', \sqrt{-\boldsymbol{\xi}'\cdot\boldsymbol{\xi}'}\right) d\boldsymbol{\xi}' \tag{43}$$

For the reconstruction of the displacement dislocation we start with the equation (21)

$$\begin{aligned} &\frac{1}{|\Sigma|}\int_{\Sigma} R^*\mathbf{W}(\mathbf{x}')\cdot[\mathbf{U}(\mathbf{x}')]\ ds(\mathbf{x}') + \int_{\Omega\backslash\Sigma}\mathbf{U}\cdot L^*\mathbf{W}dx \\ = &\int_{\partial\Omega}(\mathbf{U}(\mathbf{x}')\cdot R^*\mathbf{W}(\mathbf{x}') - \mathbf{W}(\mathbf{x}')\cdot R\,\mathbf{U}(\mathbf{x}'))\,ds(\mathbf{x}') \end{aligned} \tag{44}$$

and setting approximately $[\mathbf{U}(\mathbf{x}')] = \mathbf{A} = \begin{pmatrix} a\,\widehat{\mathbf{a}} \\ \theta_a \end{pmatrix}$ where $\mathbf{A}$ is a constant vector, we derive the relation

$$\begin{aligned} &\frac{1}{|\Sigma|}\int_{\Sigma}[a\,\widehat{\mathbf{a}}\cdot(T\mathbf{w}-\gamma\,\widehat{\boldsymbol{\nu}}\,\theta_w) + \theta_a\widehat{\boldsymbol{\nu}}\cdot\nabla\theta_w]\,ds + \int_{\Omega\backslash\Sigma}\mathbf{U}\cdot L^*\mathbf{W}dx \\ = &-\int_{\partial\Omega}(\mathbf{w}\cdot(T\mathbf{u}-qk\eta\,\widehat{\boldsymbol{\nu}}\,\theta) + \theta_w\ \widehat{\boldsymbol{\nu}}\cdot\nabla\theta)\,ds \\ &\int_{\partial\Omega}(\mathbf{u}\cdot(T\mathbf{w}-\gamma\,\widehat{\boldsymbol{\nu}}\,\theta_w) + \theta\,\widehat{\boldsymbol{\nu}}\cdot\nabla\theta_w)\,ds \end{aligned} \tag{45}$$

This case is more complicated although the idea is the same: compute the unknown parameters using test functions which have constant terms in $T\mathbf{w} - \gamma\,\widehat{\boldsymbol{\nu}}\,\theta_w$ and $\widehat{\boldsymbol{\nu}}\cdot\nabla\theta_w$. For simplicity we assume that $\widehat{\mathbf{a}} = \widehat{\boldsymbol{\nu}}$ which simulates a "slit" situation. Then using only the thermal constant test function $\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$ we have

$$\frac{-\gamma a}{|\Sigma|}\int_{\Sigma} ds = -\int_{\partial\Omega}(\widehat{\boldsymbol{\nu}}\cdot\nabla\theta + \gamma\,\widehat{\boldsymbol{\nu}})\ ds + O(\omega)$$

from which we compute

$$a = \frac{1}{\gamma} \int_{\partial\Omega} \left(\widehat{\boldsymbol{\nu}} \cdot \nabla\theta + \gamma \, \widehat{\boldsymbol{\nu}} \right) \, ds + O(\omega)$$

Again for simplicity that we know that the plane where the screen is located is the plane $x_3 = c$. Then $v_1 = v_2 = 0$ and $v_3^2 = 1$. Using the test functions $\begin{pmatrix} \mathbf{w}_4 \\ 0 \end{pmatrix}$, where $\mathbf{w}_4 = (2x_1^2, -x_2^2, -x_3^2)$, we infer that .

$$2\mu a \frac{1}{|\Sigma|} \int_{\Sigma} \left(2v_1^2 x_1 - v_2^2 x_2 - v_3^2 x_3 \right) ds = g(\mathbf{w}_4) \tag{46}$$

or

$$-2\mu a \; c = g(\mathbf{w}_4) \Longleftrightarrow c = \frac{1}{-2\mu a} g(\mathbf{w}_4) \tag{47}$$

If we do not have a plane parallel to the $x_3 = 0$ then we have to use more functions like $(-x_1^2, 2x_2^2, -x_3^2)$ and $(-x_1^2, -x_2^2, 2x_3^2)$.

To estimate the shape of the planar dislocation we can use the same procedure. Assume that we have the plane $x_3 = 0$. We use the same function $\phi(\mathbf{x}) = e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}}$, where $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0$ and construct the test function $\mathbf{W} = \begin{pmatrix} \nabla\phi(\mathbf{x}) \\ 0 \end{pmatrix}$. Then from (44) we infer that

$$\begin{aligned} & \frac{-2\mu}{|\Sigma|} \int_{\Sigma} \left(i e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}} \right) \; (\boldsymbol{\xi} \cdot \widehat{\mathbf{n}})(\boldsymbol{\xi} \cdot \widehat{\mathbf{a}}) \; ds \\ = & \int_{\partial\Omega} \left\{ -2\mu \, i \; (\mathbf{u} \cdot \boldsymbol{\xi})(\widehat{\boldsymbol{\nu}} \cdot \boldsymbol{\xi}) \; e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}} - i \, e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}} \, \boldsymbol{\xi} \cdot (T\mathbf{u} - qk\eta \, \widehat{\boldsymbol{\nu}} \, \theta) \right\} ds \end{aligned} \tag{48}$$

In this case we want to estimate approximately the shape of the planar dislocation $\Sigma^{(1)}$ writing (40) in the form

$$\frac{1}{|\Sigma^{(1)}|} \int_{\Sigma^{(1)}} e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}} \; ds = g(\boldsymbol{\xi}) \tag{49}$$

where

$$\begin{aligned} g(\boldsymbol{\xi}) \; = \; & \frac{-1}{2\mu \; (\boldsymbol{\xi} \cdot \widehat{\mathbf{n}})(\boldsymbol{\xi} \cdot \widehat{\mathbf{a}})} \int_{\partial\Omega} \left[-2\mu \, i \; (\mathbf{u} \cdot \boldsymbol{\xi})(\widehat{\boldsymbol{\nu}} \cdot \boldsymbol{\xi}) \; e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}} \right. \\ & \left. - i \, e^{i\,\boldsymbol{\xi}\cdot\mathbf{x}} \, \boldsymbol{\xi} \cdot (T\mathbf{u} - qk\eta \, \widehat{\boldsymbol{\nu}} \, \theta) \right] ds \end{aligned} \tag{50}$$

Using the same inversion procedure we can reconstruct the shape $\Sigma^{(1)}$ of the screen.

We mention here that this work can be directly applied to a problem in elasticity by vanishing all quantities describing thermal effects.

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$(\bar{\lambda}, \sigma)$ - DOUBLE SEQUENCE SPACES VIA ORLICZ FUNCTION

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ABSTRACT. In this paper we define and study two concepts which arise from the notions of invariant means and de la Valle-Poussin mean namely: strongly double $(\bar{\lambda}, \sigma)$- convergence defined by Orlicz function and uniform $(\bar{\lambda}, \sigma)$-statistical convergence and establish natural characterization for the underline sequence spaces.

1. Introduction and Background

Let l_∞ be the Banach space of bounded $x = (x_k)$ with the usual norm $||x|| = \sup_n |x_n|$. A sequence $x \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [5] proved that

$$\hat{c} = \{x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + \cdots + x_{m+n}}{m+1}.$$

The following space of strongly almost convergent sequence was introduced by Maddox in [7]

$$[\hat{c}] = \{x \in l_\infty : \lim_m t_{m,n}(|x - Le|) \text{ exists uniformly in } n \text{ for some } L \in c\}$$

where $e = (1, 1, \ldots)$.

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean provided that

i $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all k,

ii $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and

iii $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in l_\infty$.

For certain class of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the bounded sequence of all whose σ-mean are equal. If $x = (x_k)$, set $Tx = (Tx_k) = (x_{\sigma(n)})$, it can be shown that

$$V_\sigma = \left\{x \in l_\infty : \lim_m t_{m,n}(x) = L \text{ uniformly in } n,\ L = \sigma - \lim x\right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^m(n)}}{m+1},\ t_{-1,n}(x) = 0,$$

Date: February 3, 2006.

2000 *Mathematics Subject Classification.* Primary 40A99; Secondary 40A05.

Key words and phrases. Double sequence spaces, Orlicz function, de la Valle-Poussin means, double statistical convergent .

(see, Schaefer [21]) . We say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$. The space $[V_\sigma]$ is of strongly σ-convergent sequence was introduced by Mursaleen [13] as follows: A sequence $x = (x_k)$ is said to be strongly σ-convergent if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k}|x_{\sigma^i(m)} - L| \to 0 \tag{1.1}$$

as $k \to \infty$ uniformly in m. We will denote $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When (1.1) holds we write $[V_\sigma] - \lim x = L$. If we let $\sigma(m) = m + 1$, then $[V_\sigma] = [\hat{c}]$. Thus strong σ-convergence generalizes the concept of strong almost convergence sequence space.

Recall in [3] that an Orlicz function $M : [0, \infty) \to [0, \infty)$ is continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently the notion of Orlicz function was used to define sequence spaces by Parashar and B.Choudhary [18] and other authors. An Orlicz function M can be represented in the following integral form: $M(x) = \int_0^x p(t)dt$ where p is the known kernel of M, right differential for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

If the convexity property of Orlicz function M is replaced with $M(x + y) \leq M(x) + M(y)$ then this function is called Modulus function, which was presented and discussed by Ruckle [15] and Maddox [6]. Let ω'' denote the set of all double sequences of real numbers. In 1900 Pringsheim presented the following definition for the convergence of double sequences.

Definition 1.1 (Pringsheim, [14])**.** *A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as "P-convergent".*

We shall denote the space of all P-convergent sequences by c''. By a bounded double sequence we shall mean there exists a positive number M such that $|x_{k,l}| < M$ for all (k, l), and denote such bounded by $||x||_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty$. We shall also denote the set of all bounded double sequences by l''_∞. We also note in contrast to the case for single sequence, a P-convergent double sequence need not be bounded. Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0.$$

The generalized de la Valee-Poussin mean is defined by

$$t_i(x) = \frac{1}{\lambda_i}\sum_{k\in I_i} x_k$$

where $I_i = [i - \lambda_i + 1, i]$. A sequence $x = (x_n)$ is said to be (V, λ)-summable to a number L, if $t_i(x) \to L$ as $i \to \infty$ (see [4]) .

Definition 1.2. *Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers both of which tends to ∞ as i and j approach ∞, respectively.*

Also let $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0$ *and* $\mu_{j+1} \leq \mu_j + 1, \mu_1 = 0$. *We write the generalized double de la Valee-Poussin mean by*

$$t_{i,j}(x) = \frac{1}{\lambda_i \mu_j} \sum_{(k,l) \in \bar{I}_{i,j}} x_{k,l},$$

where $I_j = [j - \mu_j + 1, j]$.

A sequence $x = (x_{k,l})$ is said to be $(V^{''}, \lambda, \mu)$-summable to a number L, if $t_{ij}(x) \to L$ as $i, j \to \infty$ in the Pringsheim sense. Throughout this paper we shall denote $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$, $(\sigma^k(m), \sigma^l(n))$ by $\sigma^{k,l}(m,n)$, and $(k \in I_i, l \in I_j)$ by $(k,l) \in \bar{I}_{i,j}$.

We shall now generalize Definition 1.2 via double σ-convergence as follows:

Definition 1.3. *A bounded double sequence* $x = (x_{k,l})$ *of real number is said to be* $(\bar{\lambda}, \sigma)$*-convergent to L provided that*

$$P - \lim_{i,j} T^{i,j}_{m,n} = L \text{ uniformly in } (m,n),$$

where

$$T^{i,j}_{m,n} = \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l) \in \bar{I}_{i,j}} x_{\sigma^{k,l}(m,n)}.$$

In this case we write $(\bar{\lambda}, \sigma) - \lim x = L$. We shall also denote the set of all $(\bar{\lambda}, \sigma)$-convergent sequences by $V^{''}_{(\bar{\lambda},\sigma)}$. Clearly $V^{''}_{(\bar{\lambda},\sigma)} \subset l^{''}_{\infty}$. One can see that in contrast to the case for single sequences, a P-convergent sequence need not be $(\bar{\lambda}, \sigma)$-convergent. But, it is easy to see that every bounded P-convergent double sequence is $(\bar{\lambda}, \sigma)$-convergent. In addition, if we let $\sigma(m) = m + 1$, $\sigma(n) = n + 1$, and $\bar{\lambda}_{i,j} = ij$ in the above definition then $(\bar{\lambda}, \sigma)$-convergence reduces to the almost P-convergence which was defined by Móricz and Rhoades in [8]. It is quite natural to expect that the sets of sequences that are strongly double $(\bar{\lambda}, \sigma)$-summable to zero, strongly double $(\bar{\lambda}, \sigma)$-summable and strongly double $(\bar{\lambda}, \sigma)$-bounded by de la Valee-Poussin mean method can be defined by combining the concepts of Orlicz function, λ-method, and σ-mean. Such a combination would be a multidimensional analogues of the definition presented by E. Savas and R. Savas in [20]. We now ready to present the multidimensional sequence spaces.

Definition 1.4. *Let* M *be an Orlicz function and* $p = (p_{k,l})$ *be any factorable double sequence of strictly positive real numbers. Let* $\lambda = (\lambda_i)$ *and* $\mu = (\mu_j)$ *be the same as in above.*

$$[V^{''}_{\sigma}, \bar{\lambda}, M, p] = \{x \in w^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l) \in \bar{I}_{i,j}} \left[M \left(\frac{x_{\sigma^{k,l}(m,n)} - l}{\rho} \right) \right]^{p_{k,l}} = 0$$

$$\text{uniformly in } (m,n), \text{ for some } \rho > 0 \text{ and some } l > 0\},$$

$$[V^{''}_{\sigma}, \bar{\lambda}, M, p]_0 = \{x \in w^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l) \in \bar{I}_{i,j}} \left[M \left(\frac{x_{\sigma^{k,l}(m,n)}}{\rho} \right) \right]^{p_{k,l}} = 0$$

$$\text{uniformly in } (m,n), \text{ for some } \rho > 0\},$$

and

$$[V_\sigma^{''}, \bar{\lambda}, M, p]_\infty = \{x \in w^{''} : \sup_{i,j,m,n} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{x_{\sigma^{k,l}(m,n)}}{\rho} \right) \right]^{p_{k,l}} < \infty$$

$$\textit{for some } \rho > 0\}.$$

If we consider various assignment of M, $\bar{\lambda}$, and p in the above sequence spaces we are granted the following:

(1) If $M(x) = x$, $\bar{\lambda}_{i,j} = ij$, and $p_{k,l} = 1$ for all (k,l) then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [V_\sigma^{''}]$, $[V_\sigma^{''} M, \bar{\lambda}, p]_0 = [V_\sigma^{''}]_0$, and $[V_\sigma^{''} M, \bar{\lambda}, p]_\infty = [V_\sigma^{''}]_\infty$.

(2) If $p_{k,l} = 1$ for all (k,l) then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [V_\sigma^{''}, M, \bar{\lambda}]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [V_\sigma^{''}, M, \bar{\lambda}]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [V_\sigma^{''}, M, \bar{\lambda}]_\infty$.

(3) If $p_{k,l} = 1$ for all (k,l) and $\bar{\lambda}_{i,j} = ij$ then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [V_\sigma^{''}, M]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [V_\sigma^{''}, M]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [V_\sigma^{''}, M]_\infty$.

(4) If $M(x) = x$ and $p_{k,l} = p$ for all (k,l) then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [V_\sigma^{''}, \bar{\lambda}]^p$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [V_\sigma^{''}, \bar{\lambda}]_0^p$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [V_\sigma^{''}, \bar{\lambda}]_\infty^p$.

(5) If $\bar{\lambda}_{i,j} = ij$ then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [V_\sigma^{''}, M, p]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [V_\sigma^{''}, M, p]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [V_\sigma^{''}, M, p]_\infty$ which was studied by Savas and Patterson in [19].

(6) If $M(x) = x$, $\bar{\lambda}_{i,j} = ij$, $\sigma(m) = m+1$, and $\sigma(n) = n+1$, then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [\hat{c}^{''}, p]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [\hat{c}^{''}, p]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [\hat{c}^{''}, p]_\infty$.

(7) If $\sigma(m) = m+1$ and $\sigma(n) = n+1$ then $[V_\sigma^{''}, M, \bar{\lambda}, p] = [\hat{c}^{''}, M, \bar{\lambda}]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [\hat{c}^{''}, M, \bar{\lambda}]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [\hat{c}^{''}, M, \bar{\lambda}]_\infty$.

(8) If $M(x) = x$, $\bar{\lambda}_{i,j} = ij$, $\sigma(m) = m+1$, $\sigma(n) = n+1$, and $p_{k,l} = 1$ for all (k,l) $[V_\sigma^{''}, M, \bar{\lambda}, p] = [\hat{c}^{''}]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0 = [\hat{c}^{''}]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty = [\hat{c}^{''}]_\infty$, $[\hat{c}^{''}]$ was studied in [8].

The following inequalities will be used throughout this paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \le \sup_{k,l} p_{k,l} = H$ and $D = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have as in Maddox [7]

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \le D(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}).$$

2. Main Results

We begin the characterization of the above sequence spaces by presenting the following theorems:

Theorem 2.1. *Let the sequence $(p_{k,l})$ be bounded then $[V_\sigma^{''}, M, \bar{\lambda}, p]$, $[V_\sigma^{''}, M, \bar{\lambda}, p]_0$, and $[V_\sigma^{''}, M, \bar{\lambda}, p]_\infty$ are linear spaces over the set of complex numbers.*

Proof. We shall only prove that $[V_\sigma^{''}, M, \bar{\lambda}, p]_0$ is linear. The proof of the others follow in a similar manner. If $\alpha, \beta \in C$ then there exist two positive numbers ρ_1 and ρ_2 such that

$$P - \lim_{i,j\to\infty} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{x_{\sigma^{k,l}(m,n)}}{\rho_1} \right) \right]^{p_{k,l}} = 0$$

uniformly in (m,n) and

$$P-\lim_{i,j\to\infty}\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{y_{\sigma^{k,l}(m,n)}}{\rho_2}\right)\right]^{p_{k,l}}=0$$

uniformly in (m,n). Let $\rho_3=\max\{2|\alpha|\rho_1,2|\beta|\rho_2\}$. Since M is non-decreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{|\alpha x_{\sigma^{k,l}(m,n)}+\beta y_{\sigma^{k,l}(m,n)}|}{\rho_3}\right)\right]^{p_{k,l}}\\
= &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{|\alpha x_{\sigma^{k,l}(m,n)}|}{\rho_3}+\frac{|\beta y_{\sigma^{k,l}(m,n)}|}{\rho_3}\right)\right]^{p_{k,l}}\\
\leq &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\frac{1}{2^{p_{k,l}}}\left[M\left(\frac{|x_{\sigma^{k,l}(m,n)}|}{\rho_1}\right)+M\left(\frac{|y_{\sigma^{k,l}(m,n)}|}{\rho_2}\right)\right]^{p_{k,l}}\\
\leq &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{|x_{\sigma^{k,l}(m,n)}|}{\rho_1}\right)+M\left(\frac{|y_{\sigma^{k,l}(m,n)}|}{\rho_2}\right)\right]^{p_{k,l}}\\
\leq &D\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{|x_{\sigma^{k,l}(m,n)}|}{\rho_1}\right)\right]^{p_{k,l}}+D\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{|y_{\sigma^{k,l}(m,n)}|}{\rho_2}\right)\right]^{p_{k,l}}.
\end{aligned}$$

Now since each statement of the last inequality tends to zero as (i,j) approaches in the Pringsheim sense, uniformly in (m,n), $[V_\sigma^{''},M,\bar{\lambda},p]_0$ is linear. $\square$

Definition 2.1. *An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K>0$ such that $M(2u)\leq KM(u)$ for all $u\geq 0$. The Δ_2-condition is equivalent to the inequality $M(lu)\leq K(l)M(u)$ for all values of u and for $l\geq 1$ being satisfied.*

Theorem 2.2. *Let M be an Orlicz function. If $\beta=\lim_{t\to\infty}\frac{M(t\backslash\rho)}{t}\geq 1$, then $[V_\sigma^{''},M,\bar{\lambda}]=[V_\sigma^{''},\bar{\lambda}]$.*

Proof. Let $x\in[V_\sigma^{''},\bar{\lambda}]$, then

$$S_{m,n}^{i,j}=P-\lim_{i,j}\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}|x_{\sigma^{k,l}(m,n)}-L|=0,\ \text{uniformly in } (m,n).$$

Let $\epsilon>0$ be given and choose $0<\delta<1$ such that $M(u)<\epsilon$ for every $0\leq u\leq\delta$. We can write for each (m,n)

$$\begin{aligned}
\sum_{(k,l)\in\bar{I}_{i,j}}M\left(\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right) &= \sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^k(m),\sigma^l(n)}-L|\leq\delta}M\left(\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right)\\
&+\sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^{k,l}(m,n)}-L|>\delta}M\left(\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right).
\end{aligned}$$

It is clear that:

$$\sum_{(k,l)\in \bar{I}_{i,j} \& |x_{\sigma^{k,l}(m,n)}-L|\le\delta} M\left(\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right) < \epsilon(\bar{\lambda}_{i,j}).$$

On the other hand, we use the fact that

$$|x_{\sigma^{k,l}(m,n)}-L| < 1 + \left[\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right]$$

where $[h]$ denotes the integer part of h.

Since M is an Orlicz function we have:

$$M\left(\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right) \ge M(1).$$

Now, let us consider the second part where the sum is taken over $|x_{\sigma^{k,l}(m,n)}-L| > \delta$. Thus

$$\begin{aligned}\sum_{(k,l)\in \bar{I}_{i,j} \& |x_{\sigma^{k,l}(m,n)}-L|>\delta} M\left(\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right) &\le \sum_{(k,l)\in \bar{I}_{i,j}} M\left(1+\left[\frac{|x_{\sigma^{k,l}(m,n)}-L|}{\rho}\right]\right)\\ &\le 2M(1)\frac{1}{\delta}(\bar{\lambda}_{i,j})T^{i,j}_{m,n}.\end{aligned}$$

Therefore

$$\sum_{(k,l)\in \bar{I}_{i,j}} M\left(|x_{\sigma^{k,l}(m,n)}-L|\right) \le \epsilon(\bar{\lambda}_{i,j}) + 2M(1)\frac{1}{\delta}(\bar{\lambda}_{i,j})T^{i,j}_{m,n}$$

for every (m,n). Hence $x \in [V_\sigma^{''}, M, \bar{\lambda}]$. Observe that in this part of the proof we did not need $\beta \ge 1$. Let $\beta \ge 1$ and $x \in [V_\sigma^{''}, M, \bar{\lambda}]$. Since $\beta \ge 1$ we have $M(t) \ge \beta(t)$ for all $t \ge 0$. It follows that $x_{k,l} \to L[V_\sigma^{''}, M, \bar{\lambda}]$ implies $x_{k,l} \to L[V_\sigma^{''}, \bar{\lambda}]$. This implies $[V_\sigma^{''}, M, \bar{\lambda}] = [V_\sigma^{''}, \bar{\lambda}]$. □

3. Double statistical convergence

A real number sequence x is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}|\{k < n : |x_k - L| \ge \epsilon\}| = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, ..., n$ and the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_1-\lim x = L$ or $x_k \to L(st_1)$. Statistical convergence is a generalization of the usual notion of convergence for real valued sequences that parallels the usual theory of convergence. The idea of statistical convergence was first introduce by Fast [1], but the rapid development were started after the papers of Fridy [2] and Šalát, [16]. Now statistical convergence has become one of the most active area of research in the field of Summability theory.

Before we present the new definitions and the main theorems, we shall state a few known results. The following definition was presented by Mursaleen in [12]. A sequence x is said to be λ-statistically convergent or S_λ-convergent to L, if for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n}|\{k \in I_n : |x_k - L \ge \epsilon|\}| = 0,$$

where the vertical bars indicate the numbers of elements in the enclosed set. In this case we write $S_\lambda - \lim x = L$ or $x_k \to L(S_\lambda)$. Savas [17] presented and studied the concepts of uniformly λ-statistical convergence as follows: A sequence x is said to be uniformly λ-statistically convergent or $\hat{S}_\lambda$-convergent to L, if for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \max_m |\{k \in I_n : |x_{k+m} - L \geq \epsilon|\}| = 0.$$

In this case we write $\hat{S}_\lambda - \lim x = L$ or $x_k \to L(\hat{S}_\lambda)$.

Recently Mursaleen and Edely [10] presented the notion statistical convergence for double sequence $x = (x_{k,l})$ as follows: A real double sequence $x = (x_{k,l})$ is said to be statistically convergent to L, provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} |\{(k,l) : k \leq m \text{ and } l \leq n, |x_{k,l} - L| \geq \epsilon\}| = 0$$

. In this section of this paper our goal is to define and study $(\bar{\lambda}, \sigma)$-statistical analogue of convergence for double sequence. We now present $(\bar{\lambda}, \sigma)$- statistical analogues for double sequence $x = (x_{k,l})$ as follows:

Definition 3.1. *A double sequence $x = (x_{k,l})$ is said to be uniformly $S''_{(\bar{\lambda},\sigma)}$-convergent or uniformly $(\bar{\lambda}, \sigma)$-statistical convergent to L, provided that for every $\epsilon > 0$*

$$P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \max_{m,n} |\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L| > \epsilon\}| = 0.$$

In this case we write $S''_{(\bar{\lambda},\sigma)} - \lim x = L$ or $x_{k,l} \to L(S''_{(\bar{\lambda},\sigma)})$ and $S''_{(\bar{\lambda},\sigma)} = \{x : \exists L \in \mathbb{R}, S''_{(\bar{\lambda},\sigma)} - \lim x = L\}$. If we take $\bar{\lambda}_{i,j} = ij$ the above definition reduce to the following which was defined in [19]:

Definition 3.2. *A double sequence $x = (x_{k,l})$ of real numbers is said to be uniformly double (σ'')-statistically convergent to* 0*, if*

$$P - \lim_{i,j} \frac{1}{ij} \max_{m,n \geq 0} |\{k < i \text{ and } l < j : |x_{\sigma^{k,l}(m,n)}| \geq \epsilon\}| = 0.$$

We denote by (S''_σ), the set of sequences $x = (x_{k,l})$ which are uniformly double σ-statistically convergent to L. In particular, if σ is the translation in both dimension $(S''_\sigma)_0$ reduces to the set of sequences $x = (x_{k,l})$ which are uniformly double almost statistically convergent to 0, which was defined in [9] as follows: A double sequence $x = (x_{k,l})$ of real numbers is said to be uniformly almost statistically convergent to 0 if

$$P - \lim_{i,j} \frac{1}{ij} \max_{m,n \geq 0} |\{k < i \text{and} l < j : |x_{k+m,l+n}| \geq \epsilon\}| = 0.$$

The set of double almost statistically convergent sequences shall be denoted by $\hat{S}''$.

Theorem 3.1. *Let $\bar{\lambda} = (\lambda_{i,j})$ be the same as in above, and let $0 < p < \infty$, then*

(1) *$x_{k,l} \to L[V''_\sigma, \bar{\lambda}]^p$ implies $x_{k,l} \to L(S''_{(\bar{\lambda},\sigma)})$;*

(2) *if $x \in l''_\infty$ and $x_{k,l} \to L(S''_{(\bar{\lambda},\sigma)})$ then $x_{k,l} \to L[V''_\sigma, \bar{\lambda}]^p$;*

(3) *$S''_{(\bar{\lambda},\sigma)} \cap l''_\infty = [V''_\sigma, \bar{\lambda}]^p \cap l''_\infty$.*

Proof.

Let $\epsilon > 0$ and $x_{k,l} \to L[V_\sigma^{''}, \bar{\lambda}]^p$ and since

$$\begin{aligned}\sum_{(k,l)\in \bar{I}_{i,j}} |x_{\sigma^{k,l}(m,n)} - L|^p &\geq \sum_{(k,l)\in \bar{I}_{i,j}} |x_{\sigma^{k,l}(m,n)} - L|^p \text{ for } |x_{\sigma^{k,l}(m,n)}| \geq \epsilon \\ &\geq \epsilon^p |\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L|\}| \geq \epsilon.\end{aligned}$$

We have $x_{k,l} \to L(S^{''}_{(\bar{\lambda},\sigma)})$.

Suppose that $x_{k,l} \to L(S^{''}_{(\bar{\lambda},\sigma)})$ and $x \in l_\infty^{''}$ such that $|x_{\sigma^{k,l}(m,n)}| \leq K$ for all (k,l) and (m,n). Let $\epsilon > 0$ be given and N_ϵ such that

$$\frac{1}{\bar{\lambda}_{i,j}} |\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L| \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p}}\}| \leq \frac{\epsilon}{2Kp}$$

for all $i, j \geq N_\epsilon$. Also let

$$L_{i,j} = \{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)}| \geq \left(\frac{\epsilon}{2}\right)^{\frac{1}{p}}\}.$$

Now for all $i, j > N_\epsilon$ we are granted

$$\begin{aligned}\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} |x_{\sigma^{k,l}(m,n)} - L|^p &= \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in L_{i,j}} |x_{\sigma^{k,l}(m,n)} - L|^p \\ &+ \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\notin L_{i,j}} |x_{\sigma^{k,l}(m,n)} - L|^p \\ &\leq \frac{1}{\bar{\lambda}_{i,j}} \frac{\bar{\lambda}_{i,j}\epsilon}{2K^p} K^p \\ &+ \frac{1}{\bar{\lambda}_{i,j}} \bar{\lambda}_{i,j} \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence $x_{k,l} \to L[V_\sigma^{''}, \bar{\lambda}]^p$. The third part follows immediately from (1) and (2). □

If we let $\bar{\lambda}_{i,j} = ij$ and $p = 1$ in Theorem 3.1. then we have the following corollary:

Corollary 3.1. (1) $x_{k,l} \to L[V_\sigma^{''}]$ *implies* $x_{k,l} \to L(S_\sigma^{''})$
(2) *If* $x \in l_\infty^{''}$ *and* $x_{k,l} \to L(S_\sigma^{''})$ *then* $x_{k,l} \to L[V_\sigma^{''}]$
(3) $S_\sigma^{''} \cap l_\infty^{''} = [V_\sigma^{''}] \cap l_\infty^{''}$.

Theorem 3.2. $S_\sigma^{''} - \lim x = L$ *implies* $S^{''}_{(\bar{\lambda},\sigma)} - \lim x = L$ *if and only if*

$$P - \liminf_{i,j} \frac{\bar{\lambda}_{i,j}}{ij} > 0. \tag{3.1}$$

Proof. For given $\epsilon > 0$ we have $\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon\} \subset \{k \leq i \& l \leq j : |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon\}$. Therefore

$$\begin{aligned}\frac{1}{ij} |\{k \leq i, l \leq j : |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon\}| &\geq \frac{1}{ij} |\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon\}| \\ &\geq \frac{\bar{\lambda}_{i,j}}{ij} \frac{1}{\bar{\lambda}_{i,j}} |\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon\}|.\end{aligned}$$

Taking the limit as $(i,j) \to \infty$ in the Pringsheim sense and the hypothesis, we have the following: $x_{k,l} \to L[S_\sigma^{''}]$ implies $x_{k,l} \to L[S^{''}_{(\bar{\lambda},\sigma)}]$.

Conversely, suppose that $P-\liminf_{i,j}\frac{\bar{\lambda}_{i,j}}{ij}=0$ and $x\in S''_{\sigma}$. Then as in [12] we can choose subsequences $\{i(p)\}_{p=1}^{\infty}$ and $\{j(q)\}_{q=1}^{\infty}$ such that $\frac{\lambda_i(p)}{i(p)}<\frac{1}{p}$ and $\frac{\mu_j(q)}{j(q)}<\frac{1}{q}$. Let us consider the following double sequence:

$$x_{k,l}:=\begin{cases}1, & \text{if } k\in I_i(p) \text{ and } l\in I_j(q), (p,q=1,2,3,\ldots)\\ 0, & \text{otherwise .}\end{cases}$$

Then $x\in[V''_{\sigma}]$ and hence by Corollary 3.1 $x\in S''_{\sigma}$. But on the other hand $x\notin[V''_{(\bar{\lambda},\sigma)}]^p$ and thus Theorem 3.1 implies $x\notin(S''_{(\bar{\lambda}\sigma)})$; a contradiction and hence (3.1) holds. This completes the proof. □

Theorem 3.3. *If M be an Orlicz function and $0<h=\inf p_k\le p_k\le\sup_k p_k=H<\infty$ then $[V''_{\sigma},M,\bar{\lambda},p]\subset S''_{(\bar{\lambda},\sigma)}$*

Proof. Let $x\in[V''_{\sigma},M,\bar{\lambda},p]$. Then there exists $\rho>0$ such that

$$\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{x_{\sigma^{k,l}(m,n)}-L}{\rho}\right)\right]^{p_{k,l}}\to 0$$

as $(i,j)\to\infty$ in the Pringsheim sense uniformly in (m,n). If $\epsilon>0$ and let $\epsilon_1=\frac{\epsilon}{\rho}$, then we obtain the following:

$$\begin{aligned}
&\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}}\left[M\left(\frac{x_{\sigma^{k,l}(m,n)}-L}{\rho}\right)\right]^{p_{k,l}}\\
=\ &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^{k,l}(m,n)}|\ge\epsilon}\left[M\left(\frac{x_{\sigma^{k,l}(m,n)}-L}{\rho}\right)\right]^{p_{k,l}}\\
+\ &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^{k,l}(m,n)}|<\epsilon}\left[M\left(\frac{x_{\sigma^{k,l}(m,n)}-L}{\rho}\right)\right]^{p_{k,l}}\\
\ge\ &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^{k,l}(m,n)}|\ge\epsilon}\left[M\left(\frac{x_{\sigma^{k,l}(m,n)}-L}{\rho}\right)\right]^{p_{k,l}}\\
\ge\ &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^{k,l}(m,n)}|\ge\epsilon}[M(\epsilon_1)]^{p_{k,l}}\\
\ge\ &\frac{1}{\bar{\lambda}_{i,j}}\sum_{(k,l)\in\bar{I}_{i,j}\&|x_{\sigma^{k,l}(m,n)}|\ge\epsilon}\min\{[M(\epsilon_1)]^{\inf p_{k,l}},[M(\epsilon_1)]^H\}\\
\ge\ &\frac{1}{\bar{\lambda}_{i,j}}|\{(k,l)\in\bar{I}_{i,j}:|x_{\sigma^{k,l}(m,n)}-L|\ge\epsilon\}|\min\{[M(\epsilon_1)]^h,[M(\epsilon_1)]^H\}.
\end{aligned}$$

Hence $x\in S''_{(\bar{\lambda},\sigma)}$.

□

Theorem 3.4. *Let M be a bounded Orlicz function and $0<h=\inf p_k\le p_k\le\sup_k p_k=H<\infty$. Then $S''_{(\bar{\lambda},\sigma)}\subset[V''_{\sigma},M,\bar{\lambda},p]$.*

Proof. Suppose that M is bounded. Then there exists an integer K such that $M(x) < K$ for $x > 0$. Thus

$$\begin{aligned}
&\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{|x_{\sigma^{k,l}(m,n)} - L|}{\rho} \right) \right]^{p_{k,l}} \\
= &\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \& |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon} \left[M\left(\frac{|x_{\sigma^{k,l}(m,n)} - L|}{\rho} \right) \right]^{p_{k,l}} \\
+ &\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \& |x_{\sigma^{k,l}(m,n)} - L| < \epsilon} \left[M\left(\frac{|x_{\sigma^{k,l}(m,n)} - L|}{\rho} \right) \right]^{p_{k,l}} \\
\leq &\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \& |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon} \max\{K^h, K^H\} \\
+ &\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \& |x_{\sigma^{k,l}(m,n)} - L| < \epsilon} \left[M\left(\frac{\epsilon}{\rho} \right) \right]^{p_{k,l}} \\
\leq &\max\{K^h, K^H\} \frac{1}{\bar{\lambda}_{i,j}} |\{(k,l) \in \bar{I}_{i,j} : |x_{\sigma^{k,l}(m,n)} - L| \geq \epsilon\}| \\
+ &\max\left\{ \left[M\left(\frac{\epsilon}{\rho} \right) \right]^h, \left[M\left(\frac{\epsilon}{\rho} \right) \right]^H \right\}.
\end{aligned}$$

Hence $x \in [V_\sigma^{''}, M, \bar{\lambda}, p]$. This completes the proof. □

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.1,113-125,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, motivated by the result of Aoyama et al., we first introduce a new iterative scheme for an accretive operator in a Banach space and obtain some strong convergence theorems in a Banach space under appropriate conditions on parameters.

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Keywords: Strong convergence; accretive operators; fixed point; uniformly smooth and uniformly convex Banach spaces.

2000 AMS Mathematics Subject Classifications: 47H05, 47H10.

1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, C be a nonempty closed convex subset of H and A be a monotone operator of C into H.

The variational inequality problem is formulated as finding a point $x^* \in C$ such that

$$\langle x - x^*, Ax^* \rangle \geq 0 \tag{1}$$

for all $x \in C$. Such a point $x^* \in C$ is called a *solution* of the problem (1). The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. In the case when $C = H$, $\mathrm{VI}(H, A) = A^{-1}0$ holds, where

$$A^{-1}0 = \{x^* \in H : Ax^* = 0\}.$$

An element of $A^{-1}0$ is called a *zero point* of A. An operator A of C into H is said to be *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \tag{2}$$

for all $x, y \in C$ (see [1]-[3]). It is known that, if T is a nonexpansive mapping of C into itself, then $A = (I - T)$ is $\frac{1}{2}$-inverse strongly monotone and $F(T) = VI(C, A)$, where $F(T)$ is the set of fixed points of T.

Let $P : H \to C$ be a mapping and $x \in H$. Then there exists a unique point $Px \in C$ such that

$$\|x - Px\| = d(x, C).$$

Such a mapping P of H onto C is called the *metric projection* onto C. It is well-known that $\|x - y\| = d(x, C)$ if and only if $\langle x - y, y - z \rangle \geq 0$ for all $z \in C$.

In the case of $C = H = R^N$, to find a zero point of an inverse strongly monotone operator, in 1979, Golśhtein and Tretýakov [4] proved the following theorem:

Theorem GT. *Let R^N be the N-dimensional Euclidean space and A be an α-inverse strongly monotone operator of R into itself with $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_1 = x \in R^N, \\ x_{n+1} = x_n - \lambda_n A x_n \end{cases} \tag{3}$$

for all $n = 1, 2, \cdots,$ *where* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *If* $\{\lambda_n\}$ *is chosen so that* $\lambda_n \in [a, b]$ *for some* a, b *with* $0 < a < b < 2\alpha$, *then* $\{x_n\}$ *defined by* (3) *converges to some element of* $A^{-1}0$.

To find a solution of the variational inequality for an inverse strongly monotone operator, Iiduka et al. [5] proved the following weak convergence theorem:

Theorem I. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* A *be an* α*-inverse strongly monotone operator of* C *into* H *with* $VI(C, A) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined as follows:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n) P_C(x_n - \lambda_n A x_n)) \end{cases} \tag{4}$$

for all $n = 1, 2, \cdots,$ *where* P_C *is the metric projection from* H *onto* C, $\{\alpha_n\}$ *is a sequence in* $[-1, 1]$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2\alpha]$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\alpha_n \in [a, b]$ *for some* a, b *with* $-1 < a < b < 1$ *and* $\lambda_n \in [c, d]$ *for some* c, d *with* $0 < c < d < 2(1 + a)\alpha$, *then* $\{x_n\}$ *defined by* (4) *converges weakly to some element of* $VI(C, A)$.

Let X be a Banach space, X^* be the dual space of X and $\langle \cdot, \cdot \rangle$ denote the pairing between X and X^*. For $q > 1$, the *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in X$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and, usually, write $J_2 = J$. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$.
(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0, \infty)$.
(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{5}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (5) is attained uniformly for $x, y \in U$. The norm of X is said to be Fréchet differentiable if, for any $x \in U$, the limit (5) is attained uniformly for all $y \in U$. The *modulus of smoothness* of X is defined by

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0, \infty) \to [0, \infty)$ is a function. It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho(\tau) \leq c\tau^q$$

for all $\tau > 0$.

Note that

(1) X is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of X.

(2) All Hilbert spaces, L_p (or l_p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L_p (or l_p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.

Let C be a nonempty closed convex subset of a Banach space X. An operator A of C into X is said to be *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0$$

for all $x, y \in C$. And an operator A of C into X is said to be *α-inverse strongly accretive* if, for any $\alpha > 0$,

$$\langle Ax - Ay, J(x-y)\rangle \geq \alpha\|Ax - Ay\|^2$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D. We know the following lemma concerning sunny nonexpansive retraction:

Lemma 1.1. ([10]) *Let C be a closed convex subset of a smooth Banach space X, D be a nonempty subset of C and Q be a retraction from C onto D. Then Q is sunny and nonexpansive if and only if*

$$\langle u - Pu, j(y - Pu)\rangle \leq 0$$

for all $u \in C$ and $y \in D$.

Remark 1.1. (1) If X is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

(2) If C is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space X and T is a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$, then the set $F(T)$ is a sunny nonexpansive retract of C.

Recently, Aoyama et al. [6] first considered the following generalized variational inequality problem in a Banach space:

Problem A. *Let X be a smooth Banach space and C be a nonempty closed convex subset of X. Let A be an accretive operator of C into X. Find a point $x^* \in C$ such that, for some $j(x - x^*) \in J(x - y)$,*

$$\langle Ax^*, j(x - x^*)\rangle \geq 0 \tag{6}$$

for all $x \in C$.

Problem A is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [7].

In order to find a solution of Problem A, Aoyama et al. [6] introduced the following iterative scheme for an accretive operator A in a Banach space X:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \end{cases} \tag{7}$$

for all $n = 1, 2, \cdots$, where Q_C is a sunny nonexpansive retraction from X onto C. Then they proved a weak convergence theorem in a Banach space which is generalized simultaneously theorems of [1] and [4] as follows:

Theorem A. *Let X be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of X. Let Q_C be a sunny nonexpansive retraction from X onto C, $\alpha > 0$ and A be an α-inverse strongly accretive operator of C into X with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0,\ x \in C\}.$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by (6) *converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of X.*

In this paper, motivated by Theorem A, we first introduce the following iterative algorithm $\{x_n\}$ for an accretive operator A in a Banach space X: for any fixed $u \in C$,

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n A x_n) \end{cases} \tag{8}$$

for all $n = 1, 2, \cdots$, where Q_C is a sunny nonexpansive retraction from X onto C, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ and $\{\lambda_n\}$ is a sequence of real numbers, and obtain some strong convergence theorems in a Banach space under appropriate conditions on parameters.

2. Preliminaries

We need the following lemmas for proof of our main results:

Lemma 2.1. ([9]) *Let q be a given real number with $1 < q \leq 2$ and X be a q-uniformly smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$$

for all $x, y \in X$, where K is the q-uniformly smoothness constant of X.

The following lemma is characterized the set of solutions of Problem A by using sunny nonexpansive retractions.

Lemma 2.2. ([6]) *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C and let A be an accretive operator of C into X. Then, for all $\lambda > 0$,*

$$S(C, A) = F(Q(I - \lambda A)).$$

Lemma 2.3. ([11]) *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$, then x is a fixed point of T.*

Lemma 2.4. ([12]) *Let $\{x_n\}$, $\{z_n\}$ be bounded sequences in a Banach space X and $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

Suppose that

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) z_n$$

for all $n = 0, 1, 3, \cdots$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.5. ([10]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n$$

for all $n = 0, 1, 3, \cdots$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. The Main Results

In this section, we obtain some strong convergence theorems for finding a solution of Problem A for an inverse strongly accretive operator in a uniformly convex and 2-uniformly smooth Banach space.

Theorem 3.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of X. Let Q_C be a sunny nonexpansive retraction from X onto C, $\alpha > 0$ and A be an α-inverse strongly accretive operator of C into X with $S(C,A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are*

three sequences in $(0,1)$ *and* $\{\lambda_n\}$ *is a real number sequence in* $[a, \frac{\alpha}{K^2}]$ *for some* $a > 0$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n = 0, 1, 3, \cdots$*;*
(ii) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq 1$*;*
(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$*.*

Then the sequence $\{x_n\}$ *defined by* (7) *converges strongly to* $Q'u$*, where* Q' *is a sunny nonexpansive retraction of* C *onto* $S(C, A)$*.*

Proof. First, we observe that $I - \lambda_n A$ is nonexpansive. Indeed, for all $x, y \in C$ and $\lambda_n \in \left(0, \frac{\alpha}{K^2}\right]$, from Lemma 2.1, we have

$$\begin{aligned} &\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\ &= \|(x - y) - \lambda_n (Ax - Ay)\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_n \langle Ax - Ay, J(x - y)\rangle \\ &\quad + 2K^2 \lambda_n^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda_n \alpha \|Ax - Ay\|^2 \\ &\quad + 2K^2 \lambda_n^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + 2\lambda_n (K^2 \lambda_n - \alpha)\|Ax - Ay\|^2. \end{aligned} \tag{9}$$

So, if $0 < \lambda_n \leq \frac{\alpha}{K^2}$, then $I - \lambda_n A$ is a nonexpansive mapping.

Letting $p \in S(C, A)$, it follows from Lemma 2.2 that $p = Q_C(p - \lambda_n Ap)$. Putting $y_n = Q_C(x_n - \lambda_n Ax_n)$, it follows from (9) that

$$\begin{aligned} \|y_n - p\| &= \|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\| \\ &\leq \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{10}$$

Thus we have, from (7) and (10),

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n (u - p) + \beta_n (x_n - p) + \gamma_n (y_n - p)\| \\ &\leq \alpha_n \|u - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \max\{\|u - p\|, \|x_0 - p\|\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded and so is $\{y_n\}$.

Define a sequence $\{x_n\}$ in C by

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$$

for all $n = 0, 1, 2, \cdots$. Then it follows that

$$\begin{aligned} z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1} u + \gamma_{n+1} y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n y_n}{1-\beta_n} \\ &= \Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big) u + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (y_{n+1} - y_n) \\ &\quad + \Big(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\Big) y_n. \end{aligned} \tag{11}$$

Note that

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ &= \|Q_C(x_{n+1} - \lambda_{n+1} A x_{n+1}) - Q_C(x_n - \lambda_n A x_n)\| \\ &\le \|(x_{n+1} - \lambda_{n+1} A x_{n+1}) - (x_n - \lambda_n A x_n)\| \\ &= \|(x_{n+1} - \lambda_{n+1} A x_{n+1}) - (x_n - \lambda_{n+1} A x_n) + (\lambda_n - \lambda_{n+1}) A x_n\| \\ &\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\|. \end{aligned} \tag{12}$$

From (11) and (12), we have

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big| \|u\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|y_{n+1} - y_n\| \\ &\quad + \Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\Big| \|y_n\| \\ &\le \Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big| (\|u\| + \|y_n\|) \\ &\quad + \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\|, \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, from Lemma 2.4, we obtain $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Next, we show that

$$\limsup_{n\to\infty}\langle u - Q'u, j(x_n - Q'u)\rangle \leq 0. \tag{13}$$

To show (13), we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to z such that

$$\limsup_{n\to\infty}\langle u - Q'u, j(x_n - Q'u)\rangle = \limsup_{i\to\infty}\langle u - Q'u, j(x_{n_i} - Q'u)\rangle. \tag{14}$$

We first prove $z \in S(C, A)$. Since λ_n is in $[a, \frac{\alpha}{K^2}]$ for some $a > 0$, it follows that $\{\lambda_{n_i}\}$ is bounded and so there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ which converges to $\lambda_0 \in [a, \frac{\alpha}{K^2}]$. We may assume, without loss of generality, that $\lambda_{n_i} \to \lambda_0$. Since Q_C is nonexpansive, it follows from $y_{n_i} = Q_C(x_{n_i} - \lambda_{n_i} A x_{n_i})$ that

$$\begin{aligned}
&\|Q_C(x_{n_i} - \lambda_0 A x_{n_i}) - x_{n_i}\| \\
&\leq \|Q_C(x_{n_i} - \lambda_0 A x_{n_i}) - y_{n_i}\| + \|y_{n_i} - x_{n_i}\| \\
&\leq \|(x_{n_i} - \lambda_0 A x_{n_i}) - (x_{n_i} - \lambda_{n_i} A x_{n_i})\| + \|y_{n_i} - x_{n_i}\| \\
&\leq |\lambda_{n_i} - \lambda_0| \|A x_{n_i}\| + \|y_{n_i} - x_{n_i}\|,
\end{aligned}$$

which implies that

$$\lim_{i\to\infty}\|Q_C(I - \lambda_0 A)x_{n_i} - x_{n_i}\| = 0. \tag{15}$$

By Lemma 2.3 and (15), since we have $z \in F(Q_C(I - \lambda_0 A))$, it follows from Lemma 2.2 that $z \in S(C, A)$.

Now, from (14) and Lemma 1.1, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle u - Q'u, j(x_n - Q'u)\rangle &= \limsup_{i\to\infty}\langle u - Q'u, j(x_{n_i} - Q'u)\rangle \\
&= \langle u - Q'u, j(z - Q'u)\rangle \\
&\leq 0.
\end{aligned}$$

Therefore, we have (noting that (10))

$$
\begin{aligned}
&\|x_{n+1} - Q'u\|^2 \\
&= \langle \alpha_n u + \beta_n x_n + \gamma_n y_n - Q'u, j(x_{n+1} - Q'u)\rangle \\
&= \alpha_n \langle u - Q'u, j(x_{n+1} - Q'u)\rangle + \beta_n \langle x_n - Q'u, j(x_{n+1} \\
&\quad - Q'u)\rangle + \gamma_n \langle y_n - Q'u, j(x_{n+1} - Q'u)\rangle \\
&\le \alpha_n \langle u - Q'u, j(x_{n+1} - Q'u)\rangle + \frac{1}{2}\beta_n(\|x_n - Q'u\|^2 + \|x_{n+1} - Q'u\|^2) \\
&\quad + \frac{1}{2}\gamma_n(\|y_n - Q'u\|^2 + \|x_{n+1} - Q'u\|^2) \\
&\le \frac{1}{2}(1-\alpha_n)(\|x_n - Q'u\|^2 + \|x_{n+1} - Q'u\|^2) \\
&\quad + \alpha_n \langle u - Q'u, j(x_{n+1} - Q'u)\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|x_{n+1} - Q'u\|^2 \\
&\le (1-\alpha_n)\|x_n - Q'u\|^2 + 2\alpha_n \langle u - Q'u, j(x_{n+1} - Q'u)\rangle.
\end{aligned} \tag{16}
$$

Finally, by Lemma 2.5 and (16), we conclude that $\{x_n\}$ converges strongly to $Q'u$. This completes the proof.

Remark 3.1. (1) From (9), we know that $Q(I - \lambda_n A)$ is nonexpansive.

(2) If $S(C, A) \neq \emptyset$, it follows from Remark 1.1 and Lemma 2.2 that there exists a sunny nonexpansive retraction Q' of C onto $F(Q(I-\lambda_n A)) = S(C, A)$.

Let C be a subset of a smooth Banach space X and $\alpha > 0$. An operator A of C into X is said to be *α-strongly accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \ge \alpha \|x - y\|^2$$

for all $x, y \in C$. Let $L > 0$. An operator A of C into X is said to be *L-Lipschitz continuous* if

$$\|Ax - Ay\| \le L\|x - y\|$$

for all $x, y \in C$.

Let C be a nonempty closed convex subset of a Hilbert space H. One method of finding a point $x^* \in VI(C, A)$ is to use the projection algorithm $\{x_n\}$ defined by

$$
\begin{cases}
x_1 = x \in C, \\
x_{n+1} = P_C(x_n - \lambda A x_n)
\end{cases}
$$

for all $n = 1, 2, \cdots$, where P_C is the metric projection from H onto C, A is a monotone (accretive) operator of C into H and λ is a positive real number. It is well known that if A is an α-strongly accretive and L-Lipschitz continuous operator of C into H and $\lambda \in (0, \frac{2\alpha}{L^2})$, then the operator $P_C(I - \lambda A)$ is a contraction of C into itself.

Now, we prove a strong convergence theorem for strongly accretive operator.

Theorem 3.3. *Let X be a uniformly convex and 2-uniformly smooth Banach space and C be a nonempty closed convex subset of X. Let Q_C be a sunny nonexpansive retraction from X onto C, $\alpha > 0$ and A be an α-strongly accretive and L-Lipschitz continuous operator of C into X with $S(C, A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $(0, 1)$ and $\{\lambda_n\}$ is a real number sequence in $[a, \frac{\alpha}{K^2 L^2}]$ for some $a > 0$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n = 0, 1, 2, \cdots$;
(ii) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n \leq 1$;
(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

For fixed $u \in C$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n A x_n) \end{cases}$$

for all $n = 0, 1, 2, \cdots$. Then the sequence $\{x_n\}$ converges strongly to $Q'u$, where Q' is a sunny nonexpansive retraction of C onto $S(C, A)$.

Proof. Since A is an α-strongly accretive and L-Lipschitz continuous operator of C into X, we have

$$\langle Ax - Ay, J(x - y)\rangle \geq \alpha\|x - y\|^2 \leq \frac{\alpha}{L^2}\|Ax - Ay\|^2$$

for all $x, y \in C$. Therefore, A is $\frac{\alpha}{L^2}$-inverse strongly accretive. Using Theorem 3.1, we can obtain that $\{x_n\}$ converges strongly to $Q'u$. This completes the proof.

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TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL.10,NO.1,2008

Volume 10,Number 2 April 2008
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

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A quarterly international publication of Eudoxus Press, LLC

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A Sequential Probability Ratio Test

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Abstract

Sequential procedures are concerned with statistical analysis of data when the number of observations is not predetermined.

If the method of weight functions of Wald is not feasible and if Fraser sufficiency and invariance do not apply, then one should look for a procedure in which sample estimates replace the true values of the nuisance parameters.

The maximum likelihood SPRT's (SPRT= Sequential Probability Ratio Test) was proposed by Bartlett and by Cox.

The aim of this paper is to proof the theorem of Cox from [2], after we modified *Lemma 1* of Cramér and LeCam (see [2]).

AMS Subject Classification: 62-xx, 62Lxx, 62L12.

Keywords: maximum likelihood SPRT's, sequential likelihood ratio, sequential testing, statistics hypothesis, sequential procedures.

1 Introduction

In this problems we have to use the sequential testing of the statistics hypothesis.

Let $f(X;\ \gamma,\ \delta)$ be the probability density of a sequence of independent and identically distributed random variables $\{X_i\}_{i\geq 1}$, where $\gamma \in \Gamma$ and $\delta \in \Delta$ and let $\Theta = \Gamma \times \Delta$ be the parameter space.

We shall present Lemma 1 which is from [3] and which it was corrected of me in order to use it in proof of Theorem 1.

Lemma 1 *([3]) For each $(\gamma,\ \delta) \in \Theta$, assume that*

(i) $\partial \ln f/\partial\gamma$ *and* $\partial \ln f/\partial\delta$ *exist and are absolutely bounded by some functions that are integrable with respect to the density and*

(ii) $\partial^2 \ln f/\partial\gamma^2$, $\partial^2 \ln f/\partial\gamma\partial\delta$ *and* $\partial^2 \ln f/\partial\delta^2$ *exist, are absolutely bounded by some functions of* X *and their expectation exist.*

Then, we have

$$\begin{cases} n^{-1}\ \partial^2 \ln f(X_n;\ \gamma,\ \delta)/\partial\gamma^2 \to -I_{\gamma\gamma} \text{ in probability as } n \to \infty \\ n^{-1}\ \partial^2 \ln f(X_n;\ \gamma,\ \delta)/\partial\gamma\partial\delta \to -I_{\gamma\delta} \text{ in probability as } n \to \infty \\ n^{-1}\ \partial^2 \ln f(X_n;\ \gamma,\ \delta)/\partial\delta^2 \to -I_{\delta\delta} \text{ in probability as } n \to \infty \end{cases} \tag{1}$$

where $f(X_n;\ \gamma,\ \delta) = \prod_{i=1}^{n} f(X_i;\ \gamma,\ \delta)$,

$$\begin{cases} \mathrm{E}(\partial \ln f/\partial\gamma) = 0 = \mathrm{E}(\partial \ln f/\partial\delta), \\ \mathrm{Var}(\partial \ln f/\partial\gamma) = -\mathrm{E}(\partial^2 \ln f/\partial\gamma^2) = nI_{\gamma\gamma}, \\ \mathrm{Cov}(\partial \ln f/\partial\gamma,\ \partial \ln f/\partial\delta) = -\mathrm{E}(\partial^2 \ln f/\partial\gamma\partial\delta) = nI_{\gamma\delta}, \\ \mathrm{Var}(\partial \ln f/\partial\delta) = -\mathrm{E}(\partial^2 \ln f/\partial\delta^2) = nI_{\delta\delta}. \end{cases} \tag{2}$$

2 Maximum Likelihood SPRT's

We are interested in testing

$$\mathrm{H}_0 :\ \gamma = \gamma_0 \textit{ cu alternativa } \mathrm{H}_1 :\ \gamma = \gamma_1,$$

where δ is the nuisance parameter, no prior information about which is available.

One assume that

$$|\gamma_1 - \gamma| \textit{ and } |\gamma_0 - \gamma| \textit{ are of the order } n^{-1/2},$$

where γ denotes the true value.

Let $\hat{\gamma}_n$ and $\hat{\delta}_n$ denote the maximum likelihood estimators of γ and δ based on $(X_1, \ldots, X_n)$.

The consider the sequential procedure based on

$$Z_n' = \ln \frac{f(X_n;\ \gamma_1,\ \hat{\delta}_n)}{f(X_n;\ \gamma_0,\ \hat{\delta}_n)}, \tag{3}$$

which is called *the sequential likelihood ratio.*

We shall give Cox's procedure which is a slight modification of Bartlett's procedure.

The Taylor's expansions for $\ln f(X_n;\ \gamma_i,\ \hat{\delta}_n)$, for $i=0,\ 1$ about the true value $(\gamma,\ \delta)$ give

$$\ln f(X_n;\ \gamma_0,\ \hat{\delta}_n) \approx \ln f(X_n;\ \gamma,\ \delta) + (\gamma_0-\gamma)\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} +$$

$$+\frac{1}{2}(\gamma_0-\gamma)^2\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma^2} + (\gamma_0-\gamma)(\hat{\delta}_n-\delta)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta} \quad (4)$$

and

$$\ln f(X_n;\ \gamma_1,\ \hat{\delta}_n) \approx \ln f(X_n;\ \gamma,\ \delta) + (\gamma_1-\gamma)\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} +$$

$$+\frac{1}{2}(\gamma_1-\gamma)^2\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma^2} + (\gamma_1-\gamma)(\hat{\delta}_n-\delta)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta} \quad (5)$$

Substracting (4) from (5) we obtain

$$Z_n^{'} = (\gamma_1-\gamma_0)\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} + \frac{1}{2}(\gamma_1-\gamma_0)(\gamma_1+\gamma_0-2\gamma)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma^2} +$$

$$+(\gamma_1-\gamma_0)(\hat{\delta}_n-\delta)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta} + R_n(X_n), \quad (6)$$

where R_n contains the second order derivatives and it converges to zero in probability when $\gamma_i-\gamma,\ i=0,\ 1$ are sufficiently small and the second derivatives are uniform continue.

Expanding the functions $\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}$ and $\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}$ about $\hat{\gamma}_n$ and $\hat{\delta}_n$, which are the solutions of the equations

$$\frac{\partial \ln f(X_n;\ \hat{\gamma}_n,\ \hat{\delta}_n)}{\partial \hat{\gamma}_n} = 0 = \frac{\partial \ln f(X_n;\ \hat{\gamma}_n,\ \hat{\delta}_n)}{\partial \hat{\delta}_n} \quad (7)$$

we shall obtain

$$\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} = -(\hat{\gamma}_n-\gamma)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma^2} - (\hat{\delta}_n-\delta)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta} + R_n^{'}(X_n) \quad (8)$$

and

$$\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta} = -(\hat{\gamma}_n - \gamma)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta} - (\hat{\delta}_n - \delta)\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta^2} + R_n^{"}(X_n), \tag{9}$$

where $R_n^{'}$ and $R_n^{"}$ converge to zero in probability and therefore we can neglect them for sufficiently large n.

Theorem 1 *([3]) Under suitable regularity conditions, we have*

$$Z_n^{'} \approx (\gamma_1 - \gamma_0) I_{\gamma\gamma} n \left(\hat{\gamma}_n - \frac{1}{2}(\gamma_0 + \gamma_1) \right), \tag{10}$$

where $n\hat{\gamma}_n$ is asymptotically normal with mean $n\gamma$ and variance

$$\frac{n I_{\delta\delta}}{(I_{\gamma\gamma} I_{\delta\delta} - I_{\gamma\delta}^2)^2}.$$

Proof [I. Iatan]

Substituting (8) in (6) we shall have

$$Z_n^{'} = (\gamma_1 - \gamma_0) \left[-(\hat{\gamma}_n - \gamma) \frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma^2} - (\hat{\delta}_n - \delta) \frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta} \right] +$$

$$+ \frac{1}{2}(\gamma_1 - \gamma_0)(\gamma_1 + \gamma_0 - 2\gamma) \frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma^2} + (\gamma_1 - \gamma_0)(\hat{\delta}_n - \delta) \frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta};$$

and further,

$$Z_n^{'} \approx (\gamma_1 - \gamma_0) \left[(\hat{\gamma}_n - \gamma) n I_{\gamma\gamma} + (\hat{\delta}_n - \delta) n I_{\gamma\delta} \right] -$$

$$- \frac{1}{2}(\gamma_1 - \gamma_0)(\gamma_1 + \gamma_0 - 2\gamma) n I_{\gamma\gamma} - (\gamma_1 - \gamma_0)(\hat{\delta}_n - \delta) n I_{\gamma\delta}.$$

Finally, ignoring terms of order $O(1)$ we shall obtain

$$Z_n^{'} \approx (\gamma_1 - \gamma_0) n I_{\gamma\gamma} \left(\hat{\gamma}_n - \gamma - \frac{1}{2}\gamma_1 - \frac{1}{2}\gamma_0 + \gamma \right);$$

therefore

$$Z_n^{'} \approx (\gamma_1 - \gamma_0) I_{\gamma\gamma} n \left(\hat{\gamma}_n - \frac{1}{2}(\gamma_0 + \gamma_1) \right).$$

Using (1) in (8) and (9) it results

$$\begin{cases} n(\hat{\gamma}_n - \gamma)I_{\gamma\gamma} + n(\hat{\delta}_n - \delta)I_{\gamma\delta} = \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\gamma} \\ n(\hat{\gamma}_n - \gamma)I_{\gamma\delta} + n(\hat{\delta}_n - \delta)I_{\delta\delta} = \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\delta}. \end{cases} \tag{11}$$

If we multiply the first equation from (11) by $I_{\delta\delta}$ and the second by $-I_{\gamma\delta}$ and we sum the two equations, it obtains

$$n(\hat{\gamma}_n - \gamma)\left[I_{\gamma\gamma}I_{\delta\delta} - (I_{\gamma\delta})^2\right] = I_{\delta\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\gamma} - I_{\gamma\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\delta}.$$

Therefore,

$$n\hat{\gamma}_n = \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-1}\left[I_{\delta\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\gamma} - I_{\gamma\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\delta}\right] + n\gamma \tag{12}$$

We have

$$\mathrm{E}(n\hat{\gamma}_n) = \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-1}\left[I_{\delta\delta}E\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\gamma}\right) - I_{\gamma\delta}E\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\delta}\right)\right] + n\gamma \tag{13}$$

Using Lema 1, from (13) it results

$$\mathrm{E}(n\hat{\gamma}_n) = n\gamma.$$

We shall calculate

$$\mathrm{Var}(n\hat{\gamma}_n) = \mathrm{E}\left((n\hat{\gamma}_n - \mathrm{E}(n\hat{\gamma}_n))^2\right) = \mathrm{E}\left((n\hat{\gamma}_n - n\gamma)^2\right). \tag{14}$$

Taking into account (12), the relation (14) becomes

$$\mathrm{Var}(n\hat{\gamma}_n) = \mathrm{E}\left(\left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-2}\left[I_{\delta\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\gamma} - I_{\gamma\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\delta}\right]^2\right) =$$

$$= \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-2}\mathrm{E}\left(\left[I_{\delta\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\gamma} - I_{\gamma\delta}\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial\delta}\right]^2\right);$$

finally,

$$\mathrm{Var}(n\hat{\gamma}_n) = \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-2}\cdot[T - 2U + W], \tag{15}$$

where

$$T = I_{\delta\delta}^2 \mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right)^2\right),$$

$$U = I_{\delta\delta} I_{\gamma\delta} E\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} \cdot \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right),$$

$$W = I_{\gamma\delta}^2 \mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right)^2\right).$$

We know that:

$$\mathrm{Var}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right) = \mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right)^2\right) - \left[\mathrm{E}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right)\right]^2,$$

$$\mathrm{Cov}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma},\ \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) =$$

$$= \mathrm{E}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} \cdot \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) -$$

$$-\mathrm{E}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right) \cdot \mathrm{E}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right),$$

$$\mathrm{Var}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) = \mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right)^2\right) - \left[\mathrm{E}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right)\right]^2.$$

Using Lema 1 we shall obtain

$$\mathrm{Var}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right) = \mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right)^2\right),$$

$$\mathrm{Cov}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma},\ \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) =$$

$$= \mathrm{E}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma} \cdot \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right),$$

$$\mathrm{Var}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) = \mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right)^2\right),$$

namely,

$$\mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right)^2\right) = \mathrm{Var}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma}\right) = nI_{\gamma\gamma}, \tag{16}$$

$$\mathrm{E}\left(\frac{\partial^2 \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma \partial \delta}\right) = \mathrm{Cov}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \gamma},\ \frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) = nI_{\gamma\delta}, \tag{17}$$

$$\mathrm{E}\left(\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right)^2\right) = \mathrm{Var}\left(\frac{\partial \ln f(X_n;\ \gamma,\ \delta)}{\partial \delta}\right) = nI_{\delta\delta}. \tag{18}$$

Substituting (16)- (18) into (15) it results

$$\mathrm{Var}(n\hat{\gamma}_n) = \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-2} \cdot [I_{\delta\delta}^2 \cdot nI_{\gamma\gamma} - 2I_{\delta\delta}I_{\gamma\delta} \cdot nI_{\gamma\delta} + I_{\gamma\delta}^2 \cdot nI_{\delta\delta}] =$$

$$= \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-2} \cdot [I_{\delta\delta}^2 \cdot nI_{\gamma\gamma} - I_{\delta\delta}I_{\gamma\delta} \cdot nI_{\gamma\delta}] = \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-2} \cdot nI_{\delta\delta}(I_{\delta\delta}I_{\gamma\gamma} - I_{\gamma\delta}^2);$$

therefore

$$\mathrm{Var}(n\hat{\gamma}_n) = \frac{nI_{\delta\delta}}{(I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2)^2}.$$

From (12) we note that for large n, $n\hat{\gamma}_n$ is the sum of independent and identically distributed random variables $\{Y_i\}_{i=\overline{1,\ n}}$, where

$$Y_i - \gamma = \left[I_{\gamma\gamma}I_{\delta\delta} - I_{\gamma\delta}^2\right]^{-1}\left[I_{\delta\delta}\frac{\partial \ln f(X_n;\ \gamma;\ \delta)}{\partial \gamma} - I_{\gamma\delta}\frac{\partial \ln f(X_n;\ \gamma;\ \delta)}{\partial \delta}\right]. \tag{19}$$

We can use Wald's ([3]) approximations for the boundary values in terms of error probabilities.

Thus,

$$\begin{cases} B = \left(1 - I_{\gamma\delta}^2/I_{\gamma\gamma}I_{\delta\delta}\right)^{-1} \ln\left[\beta/(1-\alpha)\right] \approx c_n \ln\left[\beta/(1-\alpha)\right], \\ A = \left(1 - I_{\gamma\delta}^2/I_{\gamma\gamma}I_{\delta\delta}\right)^{-1} \ln\left[(1-\beta)/\alpha\right] \approx c_n \ln\left[(1-\beta)/\alpha\right], \end{cases} \tag{20}$$

where

$$c_n = \left(1 - I_{\hat{\gamma}_n\hat{\delta}_n}^2/I_{\hat{\gamma}_n\hat{\gamma}_n}I_{\hat{\delta}_n\hat{\delta}_n}\right)^{-1}.$$

3 Conclusions

Sequential procedures are concerned with statistical analysis of data when the number of observations is not predetermined.

Our goal is to use the sequential testing of the statistics hypothesis.

If the method of weight functions of Wald is not feasible and if Fraser sufficiency and invariance do not apply, then one should look for a procedure in which sample estimates replace the true values of the nuisance parameters. We confine ourselves to maximum likelihood estimators, which have desirable large sample properties.

The maximum likelihood SPRT's was proposed by Bartlett and by Cox.

In this paper we proof the theorem of Cox from [3], after we modified *Lemma 1.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.2,147-162,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

CONVERGENCE THEOREMS OF ITERATIVE SEQUENCES FOR GENERALIZED P-QUASICONTRACTIVE MAPPINGS IN P-CONVEX METRIC SPACES

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Abstract. We first introduce the concept of a p-convex metric space and a generalized p-quasicontractive mapping(or generalized quasicontractive mapping with respect to a weak distance p.) And also, we prove some convergence theorems of iterative sequences with errors for generalized p-quasicontractive mapping in p-convex metric spaces.

AMS Mathematics Subject Classification: 47H05, 47H10, 47H15, 54H25.

Key words and phrases : p-convex metric space, fixed point, Ishikawa type iterative sequence with errors, generalized p-quasicontractive mapping.

1. Introduction and preliminaries

Let K be a nonempty closed convex subset of a Banach space E and let $T : K \to K$ be a nonlinear pseudo-contractive mapping or accretive mapping. Recently concerning the problem of the Ishikawa iterative sequence $\{x_n\}$ [9]

defined by

$$\begin{cases} x_0 \in K, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad n \geq 0 \end{cases} \tag{1.1}$$

converging strongly to a fixed point of T or to a solution of the equation $Tx = f$ has been considered by many authors ([1,3-8,11-18,21,24,26), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying the certain conditions.

Rhoades [23] and Naimpally-Singh [22] suggested the following open question:

Can the Ishikawa iterative procedure be extended to nonlinear quasi-contractive mapping in a metric space?

This question is in fact solved in the affirmative (see Liu [19,20] and Xu [26]) for the Hilbert or Banach space setting.

Definition 1.1. [2] Let (E,d) be a metric space and $I = [0,1]$. For any positive integer $n \geq 2$, denote by $E^n = \underbrace{E \times E \times \cdots \times E}_{n}$, $I^n = \underbrace{I \times I \times \cdots \times I}_{n}$. A mapping $W : E^n \times I^n \to E$ is said to be a *convex structure* on E if it satisfies the following conditions: for any $u, x_1, x_2, \cdots, x_n \in E$ and for any α_1, α_2, $\cdots$, $\alpha_n \in I$ with $\sum_{i=1}^n \alpha_i = 1$

(1) $\quad W(x_1, x_2, \cdots, x_n; 0, 0, \cdots, \alpha_i, 0, \cdots, 0) = \alpha_i x_i = x_i, \quad i = 1, 2, \cdots, n;$

(2) $\quad d(u, W(x_1, x_2, \cdots, x_n; \alpha_1, \alpha_2, \cdots, \alpha_n)) \leq \displaystyle\sum_{i=1}^n \alpha_i d(u, x_i).$

E together with a metric d and a convex structure W is called a *convex metric space*, and denote it by (E,d,W).

In 1988, Ding [8] proved the following fixed point theorem using Ishikawa type iterative scheme: Let K be a nonempty closed convex subset of a complete convex metric space X with convex structure W and let $T : K \to K$ be a quasi-contractive mapping [7], i.e., there exists a constant $q \in [0,1)$ such that for all $x, y \in K$,

$$d(Tx,Ty) \leq q \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}. \tag{1.2}$$

Suppose that $\{x_n\}$ is a sequence defined by

$$\begin{cases} x_0 \in K, \\ x_{n+1} = W(Ty_n, x_n, \alpha_n), \\ y_n = W(Tx_n, x_n, \beta_n), \qquad n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \le \alpha_n, \beta_n \le 1$ and $\sum_{n=0}^{\infty} \alpha_n$ diverges. Then $\{x_n\}$ converges to a unique fixed point of T in K.

In 2003, Chang-Kim [2] introduced the concept of new type iterative sequence $\{x_n\}$ with errors in a metric space. Let (E, d) be a convex metric space with a convex structure $W : E^3 \times I^3 \to E$ and $T : E \to E$ be a generalized quasi contractive mapping. Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 \in E, \\ y_n = W(x_n, Tx_n, v_n; \xi_n, \eta_n, \delta_n), \\ x_{n+1} = W(x_n, Ty_n, u_n; \alpha_n, \beta_n, \gamma_n), \qquad n \ge 0 \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\xi_n + \eta_n + \delta_n = 1$, $n = 0, 1, 2, \cdots$ and $\{u_n\}$, $\{v_n\}$ are sequences in E, then $\{x_n\}$ is called the *Ishikawa type iterative sequence with errors* of T.

Especially, if $\eta_n = 0$ and $\delta_n = 0$ for all $n \ge 0$, it follows from the definition of the convex structure W that $y_n = x_n$. Hence from (1.4) we have

$$x_{n+1} = W(x_n, Tx_n, u_n; \alpha_n, \beta_n, \gamma_n). \tag{1.5}$$

The sequence defined by (1.5) is called the *Mann type iterative sequence with errors* of T. It should be pointed that if E is a linear normed space, then E is a convex metric space with a convex structure $W(x, y; 1-\lambda, \lambda) = (1-\lambda)x + \lambda y$, $\forall x, y \in E$, $\lambda \in I$. Therefore, the Ishikawa iterative sequence (1.1) is a special cases of (1.4) with $\gamma_n = 0$, $\delta_n = 0$ and $u_n = v_n = 0$, for all $n \ge 0$ and also, (1.4) with $\delta_n = \gamma_n = u_n = v_n = 0$ for all $n \ge 0$ reduces to (1.3).

And also Chang-Kim [2] proved the following theorem for the Ishikawa type iterative sequences with errors (1.4): Let (E, d, W) be a complete convex metric space, $W : E^3 \times I^3 \to E$ be the convex structure of E, T be a generalized quasi-contractive mapping defined by

$$\begin{aligned} &d(Tx, Ty) \\ &\le \Phi\Big(\max\{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\} \Big), \end{aligned} \tag{1.6}$$

for all $x, y \in E$, and $\{x_n\}$ be the Ishikawa type iterative sequence with errors of T defined by (1.4). Then the sequence $\{x_n\}$ converges to a unique fixed point z of T in E.

On the other hand, Kada-Suzuki-Takahashi [10] introduced the concept of ω-distance on a metric space as follows:

Let E be a metric space with a metric d. Then a function $p : E\times E \to [0,\infty)$ is called a ω-*distance* on E if the following satisfied:

(1) $p(x,z) \le p(x,y) + p(y,z)$ for any $x,y,z \in E$;
(2) for any $x \in E$, $p(x,\cdot) : E \to [0,\infty)$ is lower semi-continuous;
(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$.

Then we know that a metric d is a w-distance but the w-distance p cannot be a metric (see the examples in [10]).

Now, we introduce the concepts of a p-convex metric space and a p-convex structure W_p for the w-distance p.

Definition 1.2. Let (E,d) be a metric space with a ω-distance p and $I = [0,1]$. For any positive integer $n \ge 2$, denote by $E^n = \underbrace{E \times E \times \cdots \times E}_{n}$, $I^n = \underbrace{I \times I \times \cdots \times I}_{n}$. A mapping $W_p : E^n \times I^n \to E$ is said to a *p-convex structure* of E if it satisfies the following conditions: for any $u, x_1, x_2, \cdots, x_n \in E$ and for any $\alpha_1, \alpha_2, \cdots, \alpha_n \in I$ with $\sum_{i=1}^{n}\alpha_i = 1$,

$$
\begin{aligned}
&(1) \quad W_p(x_1,x_2,\cdots,x_n;0,0,\cdots,\alpha_i,0,\cdots,0) = \alpha_i x_i = x_i, \quad i = 1,2,\cdots,n;\\
&(2) \quad p(u,W_p(x_1,x_2,\cdots,x_n;\alpha_1,\alpha_2,\cdots,\alpha_n)) \le \sum_{i=1}^{n}\alpha_i p(u,x_i); \qquad (1.7)\\
&(3) \quad p(W_p(x_1,x_2,\cdots,x_n;\alpha_1,\alpha_2,\cdots,\alpha_n),u) \le \sum_{i=1}^{n}\alpha_i p(x_i,u).
\end{aligned}
$$

E together with a p-convex structure W_p is called a *p-convex metric space*, and denote it by (E,d,p,W_p).

A nonempty subset K of a p-convex metric space E with a p-convex structure W_p is said to be *p-convex* (cf. [25]) if

$$W_p(x_1,x_2,\cdots,x_i,\cdots,x_n;\alpha_1,\alpha_2,\cdots,\alpha_i,\cdots,\alpha_n) \in K,$$

for all $(x_1,x_2,\cdots,x_i,\cdots,x_n;\alpha_1,\alpha_2,\cdots,\alpha_i,\cdots,\alpha_n) \in K^n \times I^n$.

Since a metric d is w-distance, if we put $p = d$, then a convex metric space (E,d,W) is a p-convex metric space with a w-distance d.

The purpose of this paper is to prove some new convergence theorems for Ishikawa type iterative sequence with errors in p-convex metric spaces with a w-distance p and a p-convex structure W_p. The results of this paper not only extend and improve the well known results in Chang-Kim [2], Ćirić [7], Ding

[8], Kim et al. [11-18], Liu [19,20], Naimpally-Singh [22], Rhoades [23,24] and Xu [26] but also give an affirmative answer to the open question of Naimpally-Singh [22] in p-convex metric spaces.

Definition 1.3. (cf. [2]) A function $\Phi : [0, \infty) \to [0, \infty)$ is said to satisfy the *condition* (C_Φ), if it is nondecreasing, continuous from right, $\Phi(t) < t$, for all $t > 0$ and $\Phi(0) = 0$, and the graph of Φ intersects with the graph parallel moved toward the positive direction with respect to t-axis of the identity function I at only one point, that is,

$$\{(t, \Phi(t)) : t \geq 0\} \cap \{(t + a, I(t)) : t \geq 0, a \geq 0\} \text{ is a singleton.}$$

Definition 1.4. Let (E, d) be a metric space with a ω-distance p and $T : E \to E$ be a mapping. If there exists a function $\Phi : [0, \infty) \to [0, \infty)$ satisfying the condition (C_Φ) such that

$$\begin{aligned} &p(Tx, Ty) \\ &\leq \Phi\Bigg(\max\{p(x, x),\ p(x, y),\ p(x, Tx),\ p(y, Ty),\ p(x, Ty),\ p(y, Tx), \\ &\qquad\qquad p(y, y),\ p(y, x),\ p(Tx, x),\ p(Ty, y),\ p(Ty, x),\ p(Tx, y)\} \Bigg), \end{aligned}$$

for all $x, y \in E$ then T is said to be a *generalized p-quasicontractive* mapping (or *generalized quasicontractive* mapping *with respect to a weak distance* p).

If we put $p = d$, then a generalized p-quasicontractive mapping is generalized quasi-contractive mapping (1.6) and define $\Phi(t) = kt$, for all $t \in [0, 1)$ then the generalized quasi-contractive mapping (1.6) is quasi-contractive mapping (1.2).

2. Main results

We introduce the following lemma which plays a crucial role in the proofs of our main theorems.

Lemma 2.1. [10] *Let (E, d) be a metric space with a ω-distance p. Let $\{x_n\}$ be a sequence in E, let $\{a_n\}$ be a sequence in $[0, \infty)$ converging to zero, and let $x, y, z \in E$. Then the following hold*:

(i) *if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$,*
(ii) *if $p(x_n, x_m) \leq a_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

Using the concepts of w-distance p introduced by Kada-Suzuki-Takahashi [10] and a p-convex metric space, we have the following theorem which is a generalization of the result in [2].

Theorem 2.1. *Let K be a nonempty closed p-convex subset of a complete p-convex metric space E with a ω-distance p and a p-convex structure W_p. Let $T : K \to K$ be a generalized p-quasicontractive mapping*

$$\begin{aligned} &p(Tx, Ty) \\ &\leq \Phi\Big(\max\{p(x,x),\ p(x,y),\ p(x,Tx),\ p(y,Ty),\ p(x,Ty),\ p(y,Tx), \\ &\qquad p(y,y),\ p(y,x),\ p(Tx,x),\ p(Ty,y),\ p(Ty,x),\ p(Tx,y)\}\Big), \end{aligned} \tag{2.1}$$

for all $x, y \in K$ and some $\Phi : [0,\infty) \to [0,\infty)$ satisfying the condition (C_Φ) and such that

$$\inf\{p(x,y) + p(x,Tx) \mid x \in K\} > 0 \tag{2.2}$$

for every $y \in K$ with $y \neq Ty$. Suppose that $\{x_n\}$ is Ishikawa type iterative sequence with errors of T defined by

$$\begin{cases} x_0 \in K, \\ y_n = W_p(x_n, Tx_n, v_n; \xi_n, \eta_n, \delta_n), \\ x_{n+1} = W_p(x_n, Ty_n, u_n; \alpha_n, \beta_n, \gamma_n), \qquad n \geq 0 \end{cases} \tag{2.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, $\{\eta_n\}$ and $\{\delta_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\xi_n + \eta_n + \delta_n = 1$, $n = 0, 1, 2, \cdots$ and $\sum_{n=0}^{\infty} \beta_n = \infty$, and $\{u_n\}$, $\{v_n\}$ are sequences in E, and the following condition is satisfied:
For any nonnegative integers n, m with $0 \leq n < m$, if $\delta(A_{n,m}) > 0$, then

$$\begin{aligned} &\max_{n \leq i,j \leq m} \{p(x,y), p(y,x) \mid x \in \{u_i, v_i\}, y \in \{x_j, y_j, Tx_j, Ty_j, u_j, v_j\}\} \\ &< \delta(A_{n,m}), \end{aligned} \tag{2.4}$$

where

$$A_{n,m} = \{x_i, y_i, Tx_i, Ty_i, u_i, v_i : n \leq i \leq m\}$$

and

$$\delta(A_{n,m}) = \sup_{x,y \in A_{n,m}} p(x,y).$$

Then there is a unique fixed point z of T in K and the sequence $\{x_n\}$ converges to z.

Proof. Let $\mathbb{N}$ be the set of all nonnegative integers. Then, for any $n, m \in \mathbb{N}$, $0 \leq n < m$, we have

$$\delta(A_{n,m}) = \max\{D_1, D_1', D_2, D_3, D_3', D_4, D_4', D_5, D_6, D_6'\}, \tag{2.5}$$

where

$$
\begin{aligned}
D_1 &= \max\{p(x_n, Tx_i), p(x_n, Ty_i) \mid n \le i \le m\},\\
D_1' &= \max\{p(Tx_i, x_n), p(Ty_i, x_n) \mid n \le i \le m\},\\
D_2 &= \max\{p(Tx_i, Tx_j), p(Tx_i, Ty_j), p(Ty_i, Ty_j), p(Ty_j, Tx_i) \mid n \le i, j \le m\},\\
D_3 &= \max\{p(x_i, Tx_j), p(x_i, Ty_j) \mid n < i \le m, n \le j \le m\},\\
D_3' &= \max\{p(Tx_j, x_i), p(Ty_j, x_i) \mid n < i \le m, n \le j \le m\},
\end{aligned}
$$

$$
\begin{aligned}
D_4 &= \max\{p(y_i, Tx_j), p(y_i, Ty_j) \mid n \le i, j \le m\},\\
D_4' &= \max\{p(Tx_j, y_i), p(Ty_j, y_i) \mid n \le i, j \le m\},\\
D_5 &= \max\{p(x_i, x_j), p(x_i, y_j), p(y_i, y_j), p(y_j, x_i) \mid n \le i, j \le m\},\\
D_6 &= \max\{p(x, y) \mid x \in \{u_i, v_i\}, y \in \{x_j, y_j, Tx_j, Ty_j, u_j, v_j\} : n \le i, j \le m\},\\
D_6' &= \max\{p(y, x) \mid x \in \{u_i, v_i\}, y \in \{x_j, y_j, Tx_j, Ty_j, u_j, v_j\} : n \le i, j \le m\}.
\end{aligned}
$$

Now we prove that

$$\delta(A_{n,m}) = \max\{D_1, D_1'\}. \tag{2.6}$$

For the proof, we consider the following several steps.
(I) Since $T : K \to K$ is a generalized p-quasicontractive mappings,

$$D_2 \le \Phi(\delta(A_{n,m})). \tag{2.7}$$

(II) It follows from (2.3) and (1.7) that if $n < i \le m$, $n \le j \le m$, then we have

$$
\begin{aligned}
p(x_i, Tx_j) &= p(W_p(x_{i-1}, Ty_{i-1}, u_{i-1}; \alpha_{i-1}, \beta_{i-1}, \gamma_{i-1}), Tx_j)\\
&\le \alpha_{i-1} p(x_{i-1}, Tx_j) + \beta_{i-1} p(Ty_{i-1}, Tx_j) + \gamma_{i-1} p(u_{i-1}, Tx_j)\\
&\le \max\{p(x_{i-1}, Tx_j), \Phi(\delta(A_{n,m})), D_6\}.
\end{aligned}
$$

If $i - 1 > n$, then by the same way, we have

$$p(x_{i-1}, Tx_j) \le \max\{p(x_{i-2}, Tx_j), \Phi(\delta(A_{n,m})), D_6\}.$$

By induction, for $n < i \le m$, $n \le j \le m$, we can obtain

$$
\begin{aligned}
p(x_i, Tx_j) &\le \max\{p(x_{i-1}, Tx_j), \Phi(\delta(A_{n,m})), D_6\}\\
&\le \max\{p(x_{i-2}, Tx_j), \Phi(\delta(A_{n,m})), D_6\}\\
&\le \cdots\\
&\le \max\{p(x_n, Tx_j), \Phi(\delta(A_{n,m})), D_6\}.
\end{aligned}
$$

Similarly, for $n < i \le m$, $n \le j \le m$, we have

$$p(x_i, Ty_j) \le \max\{p(x_n, Ty_j), \Phi(\delta(A_{n,m})), D_6\}.$$

This implies that

$$\begin{aligned} D_3 &= \max\{p(x_i, Tx_j), p(x_i, Ty_j) \mid n < i \le m, n \le j \le m\} \\ &\le \max\{p(x_n, Tx_j), p(x_n, Ty_j), \Phi(\delta(A_{n,m})), D_6 \mid n \le j \le m\} \\ &= \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}. \end{aligned} \tag{2.8}$$

(III) Using the same way of the step **(II)**, we can obtain

$$\begin{aligned} D_3' &= \max\{p(Tx_j, x_i), p(Ty_j, x_i) \mid n < i \le m, n \le j \le m\} \\ &\le \max\{p(Tx_j, x_n), p(Ty_j, x_n), \Phi(\delta(A_{n,m})), D_6' \mid n \le j \le m\} \\ &= \max\{D_1', \Phi(\delta(A_{n,m})), D_6'\}. \end{aligned} \tag{2.9}$$

(IV) For $n \le i, j \le m$, by (2.7) and (2.8) we have

$$\begin{aligned} p(y_i, Tx_j) &= p(W_p(x_i, Tx_i, v_i; \xi_i, \eta_i, \delta_i), Tx_j) \\ &\le \xi_i p(x_i, Tx_j) + \eta_i p(Tx_i, Tx_j) + \delta_i p(v_i, Tx_j) \\ &\le \max\{p(x_i, Tx_j), \Phi(\delta(A_{n,m})), D_6\} \\ &\le \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}. \end{aligned}$$

Similarly, we have

$$p(y_i, Ty_j) \le \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}.$$

Hence we have

$$\begin{aligned} D_4 &= \max\{p(y_i, Tx_j), p(y_i, Ty_j) \mid n \le i, j \le m\} \\ &\le \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}. \end{aligned} \tag{2.10}$$

(V) Similarly, from (2.7) and (2.9) we can obtain

$$\begin{aligned} D_4' &= \max\{p(Tx_j, y_i), p(Ty_j, y_i) \mid n \le i, j \le m\} \\ &\le \max\{D_1', \Phi(\delta(A_{n,m})), D_6'\}. \end{aligned} \tag{2.11}$$

(VI) Since

$$D_5 = \max\{p(x_i, x_j), p(x_i, y_j), p(y_i, y_j), p(y_j, x_i) \mid n \le i, j \le m\},$$

we can consider the following:
(a) We first make an estimation for $\max\{p(x_i,x_j) \mid n \le i,j \le m\}$. Let

$$A_1 = \max\{p(x_i,x_j) \mid n \le i,j \le m\}.$$

Then there exists k,ℓ with $n \le k,\ell \le m$ such that $A_1 = p(x_k,x_\ell)$ and

$$\text{if } n \le k \le \ell \le m \quad \Longrightarrow \quad p(x_k,x_{\ell-1}) < p(x_k,x_\ell) = A_1, \tag{2.12}$$

$$\text{if } n \le \ell \le k \le m \quad \Longrightarrow \quad p(x_{k-1},x_\ell) < p(x_k,x_\ell) = A_1. \tag{2.13}$$

If $n \le k \le \ell \le m$, then

$$\begin{aligned} A_1 &= p(x_k, W_p(x_{\ell-1}, Ty_{\ell-1}, u_{\ell-1}; \alpha_{\ell-1}, \beta_{\ell-1}, \gamma_{\ell-1})) \\ &\le \alpha_{\ell-1}p(x_k,x_{\ell-1}) + \beta_{\ell-1}p(x_k,Ty_{\ell-1}) + \gamma_{\ell-1}p(x_k,u_{\ell-1}) \\ &\le \alpha_{\ell-1}p(x_k,x_{\ell-1}) + \beta_{\ell-1}D_1 + \gamma_{\ell-1}D_6'. \end{aligned} \tag{2.14}$$

If $\alpha_{\ell-1} = 0$, then from (2.14), $A_1 \le \max\{D_1, D_6'\}$.
If $\alpha_{\ell-1} \ne 0$, then from (2.12) and (2.14),

$$\begin{aligned} A_1 &< \alpha_{\ell-1}p(x_k,x_\ell) + \beta_{\ell-1}D_1 + \gamma_{\ell-1}D_6' \\ &\le \max\{A_1, D_1, D_6'\}. \end{aligned}$$

Therefore,

$$A_1 \le \max\{D_1, D_6'\} \qquad \text{for } n \le k \le \ell \le m. \tag{2.15}$$

If $n \le \ell \le k \le m$, then

$$\begin{aligned} A_1 &= p(W_p(x_{k-1}, Ty_{k-1}, u_{k-1}; \alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}), x_\ell) \\ &\le \alpha_{k-1}p(x_{k-1},x_\ell) + \beta_{k-1}p(Ty_{k-1},x_\ell) + \gamma_{k-1}p(u_{k-1},x_\ell) \\ &\le \alpha_{k-1}p(x_{k-1},x_\ell) + \beta_{k-1}D_1' + \gamma_{k-1}D_6. \end{aligned} \tag{2.16}$$

If $\alpha_{k-1} = 0$, then from (2.16), $A_1 \le \max\{D_1', D_6\}$.
If $\alpha_{k-1} \ne 0$, then from (2.13) and (2.16),

$$\begin{aligned} A_1 &< \alpha_{k-1}p(x_k,x_\ell) + \beta_{k-1}D_1' + \gamma_{k-1}D_6 \\ &\le \max\{A_1, D_1', D_6\}. \end{aligned}$$

Therefore,

$$A_1 \le \max\{D_1', D_6\}, \qquad \text{for } n \le \ell \le k \le m. \tag{2.17}$$

Hence, from (2.15) and (2.17),

$$A_1 \leq \max\{D_1, D_1', D_6, D_6'\}. \tag{2.18}$$

(b) Secondly, we make an estimation for $\max\{p(x_i, y_j) \mid n \leq i, j \leq m\}$. Let

$$A_2 = \max\{p(x_i, y_j) \mid n \leq i, j \leq m\}.$$

Since $y_j = W_p(x_j, Tx_j, v_j; \xi_j, \eta_j, \delta_j)$, using (2.18) and (2.8),

$$\begin{aligned} A_2 &= \max\{p(x_i, W_p(x_j, Tx_j, v_j; \xi_j, \eta_j, \delta_j)) \mid n \leq i, j \leq m\} \\ &\leq \{\xi_j p(x_i, x_j) + \eta_j p(x_i, Tx_j) + \delta_j p(x_i, v_j) \mid n \leq i, j \leq m\} \\ &\leq \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \end{aligned} \tag{2.19}$$

(c) And also, we can do it for $\max\{p(y_j, x_i) \mid n \leq i, j \leq m\}$ and then we obtain same estimation (2.19). Let

$$A_3 = \max\{p(y_j, x_i) \mid n \leq i, j \leq m\}.$$

Then from (2.18) and (2.9),

$$\begin{aligned} A_3 &= \max\{p(W_p(x_j, Tx_j, v_j; \xi_j, \eta_j, \delta_j), x_i) \mid n \leq i, j \leq m\} \\ &\leq \{\xi_j p(x_j, x_i) + \eta_j p(Tx_j, x_i) + \delta_j p(v_j, x_i) \mid n \leq i, j \leq m\} \\ &\leq \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \end{aligned} \tag{2.20}$$

(d) Finally, we make an estimation for $\max\{p(y_i, y_j) \mid n \leq i, j \leq m\}$. Let

$$A_4 = \max\{p(y_i, y_j) \mid n \leq i, j \leq m\}.$$

Then by using (2.20) and (2.10), we have

$$\begin{aligned} A_4 &= \max\{p(y_i, W_p(x_j, Tx_j, v_j; \xi_j, \eta_j, \delta_j)) \mid n \leq i, j \leq m\} \\ &\leq \max\{\xi_j p(y_i, x_j) + \eta_j p(y_i, Tx_j) + \delta_j p(y_i, v_j) \mid n \leq i, j \leq m\} \\ &\leq \max\{p(y_i, x_j), p(y_i, Tx_j), p(y_i, v_j) \mid n \leq i, j \leq m\} \\ &\leq \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \end{aligned} \tag{2.21}$$

Similarly, using (2.19) and (2.11) we obtain the same result (2.21)

$$\begin{aligned} A_4 &= \max\{p(W_p(x_i, Tx_i, v_i; \xi_i, \eta_i, \delta_i), y_j) \mid n \leq i, j \leq m\} \\ &\leq \max\{\xi_j p(x_i, y_j) + \eta_j p(Tx_i, y_j) + \delta_j p(v_i, y_j) \mid n \leq i, j \leq m\} \\ &\leq \max\{p(x_i, y_j), p(Tx_i, y_j), p(v_i, y_j) \mid n \leq i, j \leq m\} \\ &\leq \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \end{aligned}$$

Consequently, we have

$$A_4 \le \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \tag{2.22}$$

It follows from (2.18)∼(2.20) and (2.22),

$$\begin{aligned} D_5 &= \max\{p(x_i, x_j), p(x_i, y_j), p(y_j, x_i), p(y_i, y_j) \mid n \le i, j \le m\} \\ &\le \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \end{aligned} \tag{2.23}$$

Combining (2.7)∼(2.11) and (2.23), it follows from (2.5),

$$\begin{aligned} \delta(A_{n,m}) &= \max\{D_1, D_2, D_3, D_4, D_5, D_6, D_1', D_3', D_4', D_6'\} \\ &\le \max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}. \end{aligned} \tag{2.24}$$

If $\max\{D_1, D_1'\} < \Phi(\delta(A_{n,m}))$, $\max\{D_1, D_1'\} < D_6$ or $\max\{D_1, D_1'\} < D_6'$, then $\delta(A_{n,m}) > 0$. In fact, $\max\{D_1, D_1', D_6, D_6', \Phi(\delta(A_{n,m}))\}$ is $\Phi(\delta(A_{n,m}))$, D_6 or D_6'. Since Φ satisfying the condition (C_Φ), by using (2.4) we have

$$\max\{D_1, D_1'\} < \Phi(\delta(A_{n,m})) < \delta(A_{n,m}),$$

$$\max\{D_1, D_1'\} < D_6 < \delta(A_{n,m})$$

or

$$\max\{D_1, D_1'\} < D_6' < \delta(A_{n,m}).$$

Therefore, from (2.24) we have $\delta(A_{n,m}) < \delta(A_{n,m})$, which is a contradiction. So, we have $\delta(A_{n,m}) \le \max\{D_1, D_1'\}$. However, it is obvious that $\max\{D_1, D_1'\} \le \delta(A_{n,m})$. Therefore we have

$$\delta(A_{n,m}) = \max\{D_1, D_1'\}.$$

This completes the proof of conclusion (2.6).
Taking $n = 0$ in (2.6), we have

$$\begin{aligned} &\delta(A_{0,m}) \\ &= \max\{p(x_0, Tx_i), p(x_0, Ty_i), p(Tx_i, x_0), p(Ty_i, x_0) \mid 0 \le i \le m\} \\ &\le p(x_0, Tx_0) + p(Tx_0, x_0) \\ &\quad + \max\{p(Tx_0, Tx_i), p(Tx_0, Ty_i), p(Tx_i, Tx_0), p(Ty_i, Tx_0) \mid 0 \le i \le m\} \\ &\le p(x_0, Tx_0) + p(Tx_0, x_0) + \Phi(\delta(A_{0,m})). \end{aligned}$$

From the condition (C_Φ) (see Definition 1.3), there exists the $(I - \Phi)^{-1}$. Therefore, we have

$$\delta(A_{0,m}) \le (I - \Phi)^{-1} \left(p(x_0, Tx_0) + p(Tx_0, x_0)\right), \quad \forall m \ge 0. \tag{2.25}$$

This implies that the sequence $\{\delta(A_{0,m})\}$ is bounded.

On the other hand, for any positive integers n, m, $1 \le n < m$, it follows from (2.6)

$$\begin{aligned}
&\delta(A_{n,m})\\
&= \max\{D_1, D_1'\}\\
&= \max\{p(x_n, Tx_i), p(x_n, Ty_i), p(Tx_i, x_n), p(Ty_i, x_n) \mid n \le i \le m\}\\
&= \max_{n\le i\le m}\{p(W_p(x_{n-1}, Ty_{n-1}, u_{n-1}; \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}), Tx_i),\\
&\qquad\qquad p(W_p(x_{n-1}, Ty_{n-1}, u_{n-1}; \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}), Ty_i),\\
&\qquad\qquad p(Tx_i, W_p(x_{n-1}, Ty_{n-1}, u_{n-1}; \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1})),\\
&\qquad\qquad p(Ty_i, W_p(x_{n-1}, Ty_{n-1}, u_{n-1}; \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}))\}\\
&\le \max_{n\le i\le m}\{\alpha_{n-1}p(x_{n-1}, Tx_i) + \beta_{n-1}p(Ty_{n-1}, Tx_i) + \gamma_{n-1}p(u_{n-1}, Tx_i),\\
&\qquad\qquad \alpha_{n-1}p(x_{n-1}, Ty_i) + \beta_{n-1}p(Ty_{n-1}, Ty_i) + \gamma_{n-1}p(u_{n-1}, Ty_i),\\
&\qquad\qquad \alpha_{n-1}p(Tx_i, x_{n-1}) + \beta_{n-1}p(Tx_i, Ty_{n-1}) + \gamma_{n-1}p(Tx_i, u_{n-1}),\\
&\qquad\qquad \alpha_{n-1}p(Ty_i, x_{n-1}) + \beta_{n-1}p(Ty_i, Ty_{n-1}) + \gamma_{n-1}p(Ty_i, u_{n-1})\}\\
&\le \alpha_{n-1}\delta(A_{n-1,m}) + \beta_{n-1}\Phi(\delta(A_{n-1,m})) + \gamma_{n-1}\delta(A_{n-1,m})\\
&= (1-\beta_{n-1})\delta(A_{n-1,m}) + \beta_{n-1}\Phi(\delta(A_{n-1,m}))\\
&= (I - \beta_{n-1}(I-\Phi))\delta(A_{n-1,m}).
\end{aligned}$$

By induction and using (2.25),

$$\begin{aligned}
\delta(A_{n,m}) &\le (I - \beta_{n-1}(I-\Phi))\delta(A_{n-1,m})\\
&\le (I - \beta_{n-1}(I-\Phi))(I - \beta_{n-2}(I-\Phi))\delta(A_{n-2,m})\\
&\le \cdots\\
&\le \prod_{j=0}^{k-1}(I - \beta_j(I-\Phi))\delta(A_{j,m})\\
&\le \cdots\\
&\le \prod_{j=0}^{n-1}(I - \beta_j(I-\Phi))\delta(A_{0,m})\\
&\le \prod_{j=0}^{n-1}(I - \beta_j(I-\Phi))(t_0),
\end{aligned}$$

where $t_0 = (I-\Phi)^{-1}(p(x_0, Tx_0) + p(Tx_0, x_0))$. Let

$$a_n = \prod_{j=0}^{n-1}(I - \beta_j(I-\Phi))(t_0), \qquad \forall n \in \mathbb{N}.$$

Then for any $n, m \in \mathbb{N}$,

$$\delta(A_{n,m}) \leq a_n.$$

And, since $\Phi : [0, \infty) \to [0, \infty)$ satisfies the condition (C_Φ), for any $t > 0$, $\Phi(t) < t$, i.e., $(I - \Phi)(t) > 0$, for all $t > 0$, and $\sum_{j=0}^{\infty} \beta_j = \infty$, the sequence $\{a_n\}$ converges to zero as $n \to \infty$. Therefore, by Lemma 2.1 (ii), this implies that $\{x_n\}$ is a Cauchy sequence in K.

Let $\lim_{n\to\infty} x_n = z$ $(\in K)$. Then, since $p(x_n, \cdot)$ is lower semi-continuous,

$$\begin{aligned} p(x_n, z) &\leq \liminf_{m\to\infty} p(x_n, x_m) \\ &\leq \liminf_{m\to\infty} \delta(A_{n,m}) \\ &\leq a_n. \end{aligned} \tag{2.26}$$

Note that for any n,

$$p(x_n, Tx_n) \leq a_n. \tag{2.27}$$

Assume that $z \neq Tz$. Then by (2.2), (2.26) and (2.27), we have

$$\begin{aligned} 0 &< \inf\{p(x_n, z) + p(x_n, Tx_n) \mid n \in \mathbb{N}\} \\ &= 2 \inf_{n\in\mathbb{N}} a_n = 0. \end{aligned}$$

This is a contradiction. Therefore, we have $z = Tz$.

If $z, v \in F(T)$, then, since T is a generalized p-quasicontractive mapping,

$$\begin{aligned} p(z, z) &\leq \Phi(p(z, z)), \\ p(v, v) &\leq \Phi(p(v, v)), \\ p(z, v) &\leq \Phi(\max\{p(z, v), p(v, z), 0\}) \end{aligned} \tag{2.28}$$

and

$$p(v, z) \leq \Phi(\max\{p(z, v), p(v, z), 0\}). \tag{2.29}$$

Without loss of generality, if $0 < p(z, v) \leq p(v, z)$, then $p(z, v) \leq \Phi(p(v, z))$ and $p(v, z) \leq \Phi(p(v, z))$. This implies

$$p(z, z) = p(v, v) = p(z, v) = p(v, z) = 0.$$

Then by Lemma 2.1 (i), $z = v$. Therefore, $\{x_n\}$ converges to a unique fixed point z of T in K. This completes the proof. $\square$

We also, easily can get the following theorem for a p-quasicontractive mapping from Theorem 2.1.

Theorem 2.2. *Let E and K be the same as in Theorem 2.1, $T : K \to K$ be a p-quasicontractive mapping defined by*

$$\begin{aligned} &p(Tx, Ty) \\ &\leq k\Big(\max\{p(x,x),\ p(x,y),\ p(x,Tx),\ p(y,Ty),\ p(x,Ty),\ p(y,Tx), \\ &\qquad p(y,y),\ p(y,x),\ p(Tx,x),\ p(Ty,y),\ p(Ty,x),\ p(Tx,y)\}\Big), \end{aligned}$$

for all $x, y \in K$, $k \in [0,1)$, and (2.2) is satisfied. Let $\{x_n\}$ be the Ishikawa type iterative sequence with errors of T defined by (2.3) and (2.4) is satisfied. Then there is a unique fixed point z of T in K and the sequence $\{x_n\}$ converges to z.

For a Mann type iterative sequence with errors, we have the following result.

Theorem 2.3. *Let E and K be the same as in Theorem 2.1, $T : K \to K$ be a generalized p-quasicontractive mapping (2.1) satisfying condition (2.2). Let $\{x_n\}$ be a Mann type iterative sequence with errors of T defined by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = W_p(x_n, Tx_n, u_n; \alpha_n, \beta_n, \gamma_n), \qquad n \geq 0. \end{cases} \tag{2.30}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{u_n\}$ are satisfied the conditions of in Theorem 2.1 and (2.4) is satisfied. Then there is a unique fixed point z of T in K and the sequence $\{x_n\}$ converges to z.

Remark 2.1. *Since a convex structure W is a p-convex structure W_p with a w-distance $p = d$, the Ishikawa type iterative sequence with errors (2.3) becomes to (1.4) and especially, if $\eta_n = 0$ and $\delta_n = 0$ $\forall n \geq 0$, it follows from the definition of p-convex structure W_p that $y_n = x_n$ and hence (2.3) reduces to the Mann type iterative sequence with errors (2.30).*

Acknowledgements. This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund)(KRF-2006-311-C00201).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.2,163-172,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

On the analogue of Bernoulli polynomials

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Abstract In this paper we define the analogue of Bernoulli polynomials. We investigate some properties of the analogue of Bernoulli polynomials. Furthermore, some new relations, related to Bernoulli numbers and Euler numbers, are given. Finally, we consider the reflection symmetries of the analogue of Bernoulli polynomials.

2000 Mathematics Subject Classification - 11B68, 11S40

Key words- Bernoulli numbers, Bernoulli polynomials, Euler polynomials, Euler numbers

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we

*This work was supported by Hannam University Research Fund, 2007

normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the below notation:

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned}
&X = \varprojlim_{N} (\mathbb{Z}/dp^N\mathbb{Z}),\\
&X^* = \bigcup_{\substack{0<a<dp\\(a,p)=1}} a + dp\mathbb{Z}_p,\\
&a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \quad \text{cf. [1-4]}
\end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]}$$

and this is known to be a distribution on X due to Kim [4, 5].

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1 \text{ for some } N \geq 0\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$ (see [3, 6, 14]). Then ϕ_w has continuation to a continuous group homomorphism. For $f \in UD(\mathbb{Z}_p)$, the Kim's p-adic q-integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p]_q^N} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ see [2, 3, 4, 5] }.$$

Now we consider $I_1(f) = \lim_{q\to 1} I_q(f)$. From this, we can derive the below

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{ see [4, 6, 14] }.$$

From the above definition, we can also derive $I_1(f_1) = I_1(f) + f'(0)$, where $f_1(x) = f(x+1)$ (see [2, 4, 6, 14]. By using $I_1(f)$-integral, many authors

are studied the analogs of Bernoulli numbers and polynomials, cf.[1, 6, 7, 8, 10, 11, 14]. The remainder of the paper is organized as follows: In Section 2, we define the analogue of Bernoulli polynomials. We investigate some properties of the analogue of Bernoulli polynomials. In Section 3, we consider the reflection symmetries of the analogue of Bernoulli polynomials.

2. The analogue of Bernoulli numbers and polynomials

The purpose of this section is to introduce the analogue of Bernoulli numbers and polynomials. By using these numbers, we will give relations between Bernoulli numbers and Euler numbers. First, we start from the definition of the analogue of Bernoulli numbers as follows:

$$\frac{t}{we^t-1}=\sum_{n=0}^{\infty}B_n(w)\frac{t^n}{n!},\quad w\in T_p. \tag{1}$$

where $B_n(w)$ are called analogue of nth Bernoulli numbers. Since $I_1(f_1)=I_1(f)+f'(0)$, if we take $f(x)=e^{tx}w^x$, we easily see that

$$I_1(w^xe^{xt})=\frac{t}{we^t-1}.$$

Hence we have

$$\int_{\mathbb{Z}_p}w^xx^nd\mu_1(x)=B_n(w).$$

Now we define the analogue of Bernoulli polynomials $B_n(w,x)$ as

$$e^{xt}\frac{t}{we^t-1}=\sum_{n=0}^{\infty}B_n(w,x)\frac{t^n}{n!}. \tag{2}$$

By (1) and (2), it is not difficult to see that

$$B_n(w,x)=\sum_{l=0}^{n}\binom{n}{l}B_l(w)x^{n-l}.$$

By (2), we also have

$$\int_{\mathbb{Z}_p}w^t(x+t)^nd\mu_1(t)=B_n(w,x). \tag{3}$$

Let u be algebraic in complex number field. Then Frobenius-Euler numbers are defined by

$$e^{H(u)t} = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \tag{4}$$

This relation can be written as

$$H_0(u) = 1, \quad (H(u)+1)^k - uH_k(u) = 0 \quad (1 \le k).$$

Therefore we have

$$uH_k(u) = \sum_{i=0}^{k} \binom{k}{i} H_i(u), H_k(u) = \frac{1}{u-1}\sum_{i=0}^{k-1}\binom{k}{i} H_i(u), \text{ for } u \neq 1.$$

By (3) and (4), we give a interesting formula on relationship between the $B_n(w)$ and $H_n(w)$. Since

$$\frac{t}{we^t-1} = \frac{t}{w}\frac{1}{e^t - w^{-1}} = \frac{t}{w}\frac{1}{1-w^{-1}}\frac{1-w^{-1}}{e^t-w^{-1}} = \frac{t}{w-1}\frac{1-w^{-1}}{e^t-w^{-1}},$$

we have

$$\begin{aligned}\frac{1}{w-1}\sum_{n=0}^{\infty} H_n(w^{-1})\frac{t^n}{n!} &= \frac{1}{t}\sum_{n=0}^{\infty} B_n(w)\frac{t^n}{n!} = \frac{1}{t}\sum_{n=1}^{\infty} B_n(w)\frac{t^n}{n!} \\ &= \frac{1}{t}\sum_{n=0}^{\infty} B_{n+1}(w)\frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}\frac{B_{n+1}(w)}{n+1}\frac{t^n}{n!}.\end{aligned}$$

Hence we have the following theorem.

Theorem 1. For $n \ge 1$, we have

$$(1)\ B_n(w) = \frac{n}{w-1}H_{n-1}(w^{-1}), w \neq 1,$$

$$(2)\ B_n(w) = I_1(\phi_w(x)x^n),$$

$$(3)\ B_n(w) = \frac{1}{n+1}\lim_{N\to\infty}\frac{1}{Cp^N}\sum_{x=0}^{Cp^N-1} w^x x^{n+1}.$$

In [6, 14], the I_1-integral transform of f is the function $\widehat{f} : T_p \to \mathbb{C}_p$ defined by

$$\widehat{f}(w) = I_1(f\phi_w) \text{ for all } w \in T_p, f \in UD(\mathbb{Z}_p).$$

Now, we consider the I_q-integral transform by using p-adic q-integral on $\mathbb{Z}_p$ for a variable $q \in \mathbb{C}_p$ (see [7]) . For $f \in UD(\mathbb{Z}_p)$ the p-adic q-integral was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ cf. } [6] \ .$$

By simple calculation, we have

$$\begin{aligned} &\lim_{N\to\infty} \sum_{x=0}^{p^N-1} w^{-x} \int_{\mathbb{Z}_p} f(x) w^x q^x d\mu_1(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} w^{-x} \lim_{N\to\infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} f(y) w^y q^y \qquad (5) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} f(y) q^y \sum_{x=0}^{p^N-1} w^{y-x} = f(x) q^x = \phi_q(x) f(x), \text{ see } [8] \ . \end{aligned}$$

Since $I_1(f\phi_{wq}) = \int_{\mathbb{Z}_p} f(x) w^x q^x d\mu_1(x)$, we also have

$$\lim_{N\to\infty} \sum_{x=0}^{p^N-1} w^{-x} \int_{\mathbb{Z}_p} f(x) w^x q^x d\mu_1(x) = \lim_{N\to\infty} \sum_{w \in C_{p^N}} \phi_{w^{-1}} I_q(f\phi_{qw}) = \sum_{w\in T_p} \phi_{w^{-1}} I_1(f\phi_{qw}). \tag{6}$$

By (5), (6), we obtain

$$\frac{\log q}{q-1} \sum_{w\in T_p} \phi_{w^{-1}} \frac{q-1}{\log q} I_1(f\phi_{qw}) = \frac{\log q}{q-1} \sum_{w\in T_p} \phi_{w^{-1}} I_q(f\phi_w) = \phi_q(x) f(x).$$

Therefore, we obtain the following I_q-integral transform.

Theorem 2. For $f \in UD(\mathbb{Z}_p), w \in T_p$, we have [7]

$$\widehat{f}(qw) = \sum_{w\in T_p} I_q(f\phi_w)\phi_{w^{-1}} = \frac{q-1}{\log q} \phi_q(x) f(x).$$

Now we introduce the convolution for any $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ due to Woodcock as follows [14] :

$$(f \otimes g)(n) = \sum_{k=0}^{n} f(k)g(n-k), n \geq 0.$$

$$(f * g)(x) = \sum_{w} \widehat{f}_w \widehat{g}_w \phi_{w^{-1}}(x),$$

where the Fourier transform $\widehat{f}_w = I_1(f\phi_q)$. From Kim and Woodcock [4, 6, 7, 14], we have

$$\Delta^{n+1}(f \otimes g)(x) = (f \otimes \Delta^{n+1} g)(x) = \sum_{j=0}^{n} \Delta^j f(x+1)\Delta^{n-j} g(0).$$

If $g(0) = 0$, then we obtain

$$\Delta(f \otimes g)(x) = (f \otimes \Delta g)(x).$$

Since $I_1(f_1) = I_1(f) + f'(0)$, we have

$$I_1(\Delta f) = f'(0).$$

Hence we obtain

$$I_1(\Delta(f \otimes g))(x)) = I_1(f \otimes \Delta g)(x) = (f \otimes g)'(0).$$

On the other hand, Woodcock [8] introduced the following results.

$$(f \otimes g)' = (f \otimes g') + (f' \otimes g) + f * g,$$

$$(f * g)(z) = I_1(f(x)g(z-x)) - (f \otimes g').$$

By definition, we have $(f \otimes g)(0) = f(0)g(0)$. Hence

$$\begin{aligned}(f \otimes g)'(0) &= (f \otimes g')(0) + f'(0) \otimes g(0) + (f * g)(0) \\ &= f(0)g'(0) + f'(0) + (f * g)(0) \\ &= f(0)g'(0) + (f * g)(0).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} I_1(f\otimes\Delta g) &= (f\otimes g)'(0) \\ &= f(0)g'(0) + (f*g)(0) \\ &= f(0)g'(0) + I_1(fg_-) - (f*g')(0) \\ &= f(0)g'(0) - f(0)g'(0) + I_1(fg_-) \\ &= I_1(fg_-), \end{aligned}$$

where $g_-(x) = g(-x)$. For $w \in T_p$, let $f = z^m\phi_w(z), g = z^n$. Then we have

$$\begin{aligned} I_1(f\otimes\Delta g)(z) &= I_1(z^m\phi_w(z)(-z)^n) \\ &= (-1)^n I_1(z^{m+n}\phi_w(z)) \\ &= (-1)^n B_{n+m}(w). \end{aligned}$$

Since $I_1(\phi_w(x)) = 0$ and

$$e^{tx} = \lim_{N\to\infty}\sum_{w\in C_{p^N}}\frac{t\phi_w(x)}{we^t-1} = \sum_{n=0}^{\infty}\lim_{N\to\infty}\sum_{w\in C_{p^N}} I_1(x^n\phi_w(x))\phi_w(x),$$

we obtain

$$x^n = B_n(1) + \sum_{w\in T_p, w\neq 1}\frac{1}{w-1}H_{n-1}(w^{-1})\phi_w(x).$$

Therefore we have the following theorem.

Theorem 3. For $m, n \geq 1$, we have

$$(1)\ (-1)^n B_{m+n}(w) = \sum_{k=0}^{n-1}\binom{n}{k} I_1(z^m\phi_w(z)\otimes z^k),$$

$$\begin{aligned} (2)\ x^n &= B_n(1) + \sum_{w\in T_p, w\neq 1}\frac{1}{w-1}H_{n-1}(w^{-1})\phi_w(x) \\ &= B_n(1) + \sum_{w\in T_p, w\neq 1}\frac{B_n(w)}{n}\phi_w(x). \end{aligned}$$

3. The reflection symmetries of the analogue of Bernoulli polynomials

In this section we consider the reflection symmetries of the analogue of Bernoulli polynomials. Let $\mathbb{R}$ be the field of real numbers and let w be the p^N-th root of unity. For $x \in \mathbb{R}$, we consider the Bernoulli polynomials $B_n(x)$ as follows:

$$F(t,x) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \text{ see [9-13] .}$$

Since

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(1-x) \frac{(-t)^n}{n!} &= F(-t, 1-x) \\
&= \frac{-t}{e^{-t} - 1} e^{(1-x)(-t)} \\
&= \frac{t}{e^t - 1} e^{xt} \\
&= F(t,x) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},
\end{aligned}$$

we obtain that

$$B_n(1-x) = (-1)^n B_n(x). \tag{7}$$

Hence $B_n(x), x \in \mathbb{C}$, has $Re(x) = 1/2$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. What happens with the reflection symmetry (7), when one considers the analogue of Bernoulli polynomials ? We are going now to reflection at $1/2$ of x on the analogue of Bernoulli polynomials. Since

$$F_w(t,x) = \frac{t}{we^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(w,x) \frac{t^n}{n!},$$

by simple calculation, we have

$$\begin{aligned}
F_{w^{-1}}(-t, 1-x) &= \frac{-t}{w^{-1}e^{-t} - 1} e^{(1-x)(-t)} \\
&= \frac{-t}{w^{-1}e^{-t} - 1} e^{(-t)} e^{xt} \\
&= w \frac{t}{we^t - 1} e^{xt} \\
&= wF_w(t,x).
\end{aligned}$$

Hence we obtain the following theorem.

Theorem 4. For $n \geq 0$, we have

$$B_n(w,x) = (-1)^n w^{-1} B_n(w^{-1}, 1-x). \tag{8}$$

We have the following corollary.

Corollary 5. If $B_n(w,x) = 0$, then $B_n(w^{-1}, 1-x) = 0$.

Finally, we shall consider the more general problems. Prove or disprove: Since n is the degree of the polynomial $B_n(w,x)$, the number of real zeros $re_{B_n(w,x)}$ lying on the real plane $Im(x) = 0$ is then $re_{B_n(w,x)} = n - c_{B_n(w,x)}$, where $c_{B_n(w,x)}$ denotes complex zeros. In general, how many roots does $B_n(w,x)$ have ? Find the numbers of complex zeros $c_{B_n(w,x)}$ of the $B_n(w,x), Im(x) \neq 0$. Using numerical experiments, we hope to investigate the structure of the complex roots of the analogue of Bernoulli polynomials $B_n(w,x)$. For related topics the interested reader is referred to [9]. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the analogue of Bernoulli polynomials $B_n(w,x)$ to appear in mathematics and physics.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.2,173-178,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

TWO GENERAL FIXED POINT THEOREMS ON THREE COMPLETE METRIC SPACES

DURAN TURKOGLU

ABSTRACT. In this paper two general fixed point theorems on three complete metric spaces which generalize the results from [1] and [2] using functions are proved.

1. INTRODUCTION

The following fixed point theorems were proved by [1-2].

Theorem 1. [2]*Let* (X,d), $(Y,\rho),(Z,\sigma)$ *be complete metric spaces and suppose* T *is a mapping of* X *into* Y, S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X *satisfying the inequalities*

$$d(RSy,RSTx)\max\{d(x,RSy),d(x,RSTx)\}\leq c\sigma(Sy,STx)\max\{\sigma(Sy,STx),\\ d(x,RSTx)\}$$

$$\rho(TRz,TRSy)\max\{\rho(y,TRz),\rho(y,TRSy)\}\leq cd(Rz,RSy)\max\{d(Rz,RSy),\\ \rho(y,TRSy)\}$$

$$\sigma(STx,STRz)\max\{\sigma(z,STx),\sigma(z,STRz)\}\leq c\rho(Tx,TRz)\max\{\rho(Tx,TRz),\\ \sigma(z,STRz)\}$$

for all x *in* X, y *in* Y *and* z *in* Z, *where* $0\leq c<1$.
If one of the mappings R, S, T *is continuous, then* RST *has a unique fixed point* u *in* X, TRS *has a unique fixed point* v *in* Y *and* STR *has a unique fixed point* w *in* Z. *Further,*$Tu=v$, $Sv=w$, $Rw=u$.

Theorem 2. [1]*Let* (X,d), $(Y,\rho),(Z,\sigma)$ *be complete metric spaces and suppose* T *is a continious mapping of* X *into* Y, S *is a continuous mapping of* Y *into* Z *and* R *is a continuous mapping of* Z *into* X *satisfying the inequalities*

$$d(RSTx,RSy)\leq c\max\{\rho(y,Tx),d(x,RSTx),d(x,RSy),\sigma(Sy,STx)\}$$

$$\rho(TRSy,TRz)\leq c\max\{\sigma(z,Sy),\rho(y,TRSy),\rho(y,TRz),d(Rz,RSy)\}$$

$$\sigma(SRSz,STx)\leq c\max\{d(x,Rz),\sigma(z,STRz),\sigma(z,STx),\rho(Tx,TRz)\}$$

for all x *in* X, y *in* Y *and* z *in* Z, *where* $0\leq c<1$. *Then* RST *has a unique fixed point* u *in* X, TRS *has a unique fixed point* v *in* Y *and* STR *has a unique fixed point* w *in* Z. *Further,* $Tu=v$, $Sv=w$, $Rw=u$.

2000 *Mathematics Subject Classification.* Primary 54H25; Secondary, 47H10.
Key words and phrases. Fixed points, complete metric spaces.

Throughout this paper, $\Re_+$ stands for the non-negative reals. We will also denote by $\Im$ the set of all real functions $f : \Re_+^3 \to \Re_+$ such that:

(i) f is upper semi-continuous in each coordinate variable;

(ii) If either $u \le f(v, u, 0)$ or $u \le f(v, 0, u)$ for all $u, v \ge 0$, then there exists a real constant $0 \le c < 1$ such that $u \le cv$.

2. Main Results

We now generalize Theorem1 as follows:

Theorem 3. *Let (X, d), $(Y, \rho), (Z, \sigma)$ be complete metric spaces and suppose T is a mapping of X into Y, S is a mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities*

$$d(RSy, RSTx) \le f(\sigma(Sy, STx), d(x, RSTx), , d(x, RSy)) \tag{2.1}$$

$$\rho(TRz, TRSy) \le g(d(Rz, RSy), \rho(y, TRSy), \rho(y, TRz)) \tag{2.2}$$

$$\sigma(STx, STRz) \le h(\rho(Tx, TRz), \sigma(z, STRz), \sigma(z, STx)) \tag{2.3}$$

for all x in X, y in Y and z in Z, where $f, g, h \in \Im_3$. If one of the mappings R, S, T is continuous, then RST has a unique fixed point u in X, TRS has a unique fixed point v in Y and STR has a unique fixed point w in Z. Further, $Tu = v$, $Sv = w$, $Rw = u$.

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in X, Y and Z, respectively by $x_n = (RST)^n x_0, y_n = Tx_{n-1}, z_n = Sy_n$ for $n = 1, 2, ...$

Applying inequality (2.1) for $y = y_n$ and $x = x_n$ we have

$$d(x_n, x_{n+1}) \le f(\sigma(z_n, z_{n+1}), d(x_n, x_{n+1}), 0)$$

which implies by (ii) that

$$d(x_n, x_{n+1}) \le c_1 \sigma(z_n, z_{n+1}) \tag{2.4}$$

where $c_1 \in [0, 1)$. Applying inequality (2.3) for $x = x_{n-1}$ and $z = z_n$ we have

$$\sigma(z_n, z_{n+1}) \le h(\rho(y_n, y_{n+1}), \sigma(z_n, z_{n+1}), 0)$$

which implies by (ii) that

$$\sigma(z_n, z_{n+1}) \le c_3 \rho(y_n, y_{n+1}) \tag{2.5}$$

where $c_3 \in [0, 1)$. Applying inequality (2.2) for $z = z_{n-1}$ and $y = y_n$ we have

$$\rho(y_n, y_{n+1}) \le g(d(x_{n-1}, x_n), \rho(y_n, y_{n+1}), 0)$$

which implies by (ii) that

$$\rho(y_n, y_{n+1}) \le c_2 d(x_{n-1}, x_n) \tag{2.6}$$

where $c_2 \in [0, 1)$. It follows from inequalities (2.4), (2.5) and (2.6) that

$$d(x_{n+1}, x_n) \le c_1 \sigma(z_n, z_{n+1}) \le c_1 c_3 \rho(y_n, y_{n+1}) \le ... \le (c_1 c_2 c_3)^n d(x_0, x_1).$$

Since $0 \le c_1 c_2 c_3 < 1$, $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequence with the limits u, v and w in X, Y and Z respectively.

Now suppose that S is continuous. Then $\lim_{n\to\infty} Sy_n = \lim_{n\to\infty} z_n$ and so

$$Sv = w. \tag{2.7}$$

Applying inequality (2.1) we now have

$$d(RSv, x_{n-1}) \leq f(\sigma(Sv, z_n), d(x_{n-1}, x_n), d(x_{n-1}, RSv)).$$

Letting n tend to infinity and using (i), it follows

$$d(RSv, u) \leq f(\sigma(Sv, w), 0, d(u, RSv))$$

using equation (2.7) we have

$$d(RSv, u) \leq f(0, 0, d(u, RSv)).$$

By (ii) follows that $d(u, RSv) \leq c.0$ which implies by (2.7) that

$$u = RSv = Rw. \tag{2.8}$$

Applying inequality (2.2) we have

$$\rho(Tu, y_{n+1}) \leq g(d(u, x_n), \rho(y_n, y_{n+1}), \rho(y_n, TRw)).$$

Letting n tend to infinity and using (i), it follows that

$$\rho(Tu, v) \leq g(0, 0, \rho(v, TRw))$$

which implies by (2.8) that

$$\rho(Tu, v) \leq g(0, 0, \rho(v, Tu)).$$

By (ii) follows that

$$Tu = v. \tag{2.9}$$

It now follows from equations (2.7), (2.8) and (2.9)

$$\begin{aligned} TRSv &= TRw = Tu = v, \\ STRw &= STu = Sv = w, \\ RSTu &= RSv = Rw = u. \end{aligned}$$

The same results of course will hold if R or T is continuous instead of S.

We now prove the uniqueness of the fixed point u. Supoose that RST has a second fixed point u' . Then using inequality (2.1), we have

$$d(RSTu, RSTu') \leq f(\sigma(STu', STu), d(u, RSTu), d(u, RSTu'))$$
$$d(u, u) \leq f(\sigma(STu, STu'), 0, d(u, u')).$$

By (ii) we have

$$d(u, u') \leq c_1 \sigma(STu, STu'). \tag{2.10}$$

Further, using inequality (2.3), we have succesively

$$\begin{aligned} \sigma(STRSTu, STu') &\leq h(\rho(Tu', TRSTu), 0, \sigma(STu, STu')) \\ \sigma(STu, STu') &\leq h(\rho(Tu', Tu), 0, \sigma(STu, STu')). \end{aligned}$$

By (ii) we have

$$\sigma(STu, STu') \leq c_2 \rho(Tu, Tu'). \tag{2.11}$$

Finally, using inequality (2.2), we have

$$\rho(Tu, Tu') \leq c_2 d(u, u'). \tag{2.12}$$

By (2.10), (2.11) and (2.12) we have

$$d(u, u') \leq (c_1 c_2 c_3) d(u, u')$$

which implies $u = u'$. The fixed point u of RST is therefore unique. Similarly, it can be proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of the theorem. $\square$

Remark 1. *Letting*

$$f(t_1,t_2,t_3) = g(t_1,t_2,t_3) = h(t_1,t_2,t_3) = \frac{ct_1 \max\{t_1,t_2\}}{\max\{t_2,t_3\}}$$

with $(0 \le c < 1)$*, then we see that Theorem1 is a consequence of Theorem 3.*

We will also denote by $\Im$ the set of all real functions $f : \Re_+^4 \to \Re_+$ such that:
(iii) f is upper semi-continuous in each coordinate variable;
(iv) If either $u \le f(v,u,0,w)$ or $u \le f(v,0,u,w)$ for all $u,v \ge 0$, then there exists a real constant $0 \le c < 1$ such that $u \le c\max\{v,w\}$.
We now generalize Theorem 2 as follows:

Theorem 4. *Let* (X,d)*,* $(Y,\rho),(Z,\sigma)$ *be complete metric spaces and suppose* T *is a mapping of* X *into* Y*,* S *is a mapping of* Y *into* Z *and* R *is a mapping of* Z *into* X *satisfying the inequalities*

$$d(RSTx,RSy) \le f(\rho(y,Tx),d(x,RSTx),d(x,RSy),\sigma(Sy,STx)) \tag{2.13}$$

$$\rho(TRz,TRSy) \le g(\sigma(z,Sy),\rho(y,TRSy),\rho(y,TRz),d(Rz,RSy)) \tag{2.14}$$

$$\sigma(STRz,STx) \le h(d(x,Rz),\sigma(z,STRz),\sigma(z,STx),\rho(Tx,TRz)) \tag{2.15}$$

for all x *in* X*,* y *in* Y *and* z *in* Z*, where* $f,g,h \in \Im_4$*. If one of the mappings* R*,* S*,* T *is continuous, then* RST *has a unique fixed point* u *in* X*,* TRS *has a unique fixed point* v *in* Y *and* STR *has a unique fixed point* w *in* Z*. Further,* $Tu = v$*,* $Sv = w$*,* $Rw = u$*.*

Proof. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\},\{y_n\}$ and $\{z_n\}$ in X,Y and Z, respectively, by

$$x_n = (RST)^n x_0\ ,\ y_n = Tx_{n-1},\ z_n = Sy_n \text{ for } n = 1,2,3,....$$

Applying inequality (2.14) and using property (iv) for $z = z_{n-1}$ and $y = y_n$ we have

$$\rho(y_n,y_{n+1}) = \rho(TRz_{n-1},TRSy_n) \le g(\sigma(z_{n-1},z_n),\rho(y_n,y_{n+1}),0,d(x_{n-1},x_n))$$

and it follows that

$$\rho(y_n,y_{n+1}) \le c\max\{d(x_{n-1},x_n),\sigma(z_{n-1},z_n)\} \tag{2.16}$$

which implies by (iv) and inequality (2.16) that

$$\begin{aligned}\sigma(z_n,z_{n+1}) &\le c\max\{d(x_{n-1},x_n),\rho(y_n,y_{n+1})\} \\ &\le c\max\{d(x_{n-1},x_n),\sigma(z_{n-1},z_n)\}.\end{aligned} \tag{2.17}$$

Applying inequality (2.13) for $y = y_n$ and $x = x_n$ we have

$$d(x_n,x_{n+1}) = d(RSTx_n,RSy_n) \le f(\rho(y_n,y_{n+1}),d(x_n,x_{n+1}),0,\sigma(z_n,z_{n+1}))$$

which implies by (iv) and inequality (2.16) and (2.17) that

$$\begin{aligned}d(x_n,x_{n+1}) &\le c\max\{\rho(y_n,y_{n+1}),\sigma(z_n,z_{n+1})\} \\ &c\max\{d(x_{n-1},x_n),\sigma(z_{n-1},z_n)\}.\end{aligned} \tag{2.18}$$

It now follows easily by induction on using inequalities (2.16), (2.17) and (2.18) that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq c^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}, \\ \rho(y_n, y_{n+1}) &\leq c^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\} \\ \sigma(z_n, z_{n+1}) &\leq c^{n-1} \max\{d(x_1, x_2), \sigma(z_1, z_2)\}. \end{aligned}$$

Since $0 \leq c < 1$, it follows that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences with limits u, v and w in X, Y and Z respectively.
Now suppose that S is continuous. Then $\lim Sy_n = \lim z_n$ and so

$$Sv = w. \tag{2.19}$$

Applying inequality (2.13) for $y = v$ and $x = x_n$ we now have

$$d(RSv, x_{n+1}) \leq f(\rho(v, Tx_n), d(x_n, x_{n+1}), d(x_n, RSv), \sigma(Sv, STx_n)).$$

Letting n tend to infinity and using (iii) it follows

$$d(RSv, u) \leq f(0, 0, d(RSv, u), 0)$$

which implies by (iv) that $d(RSv, u) = 0$ and so

$$RSv = u. \tag{2.20}$$

Using equation (2.19) this gives us

$$Rw = u. \tag{2.21}$$

Using equation (2.20) and inequality (2.14) for $z = Sv$ and $y = y_n$, we have

$$\rho(Tu, y_{n+1}) \leq g(\sigma(Sv, Sy_n), \rho(y_n, TRSy_n), \rho(y_n, TRSv), d(RSv, RSy_n)).$$

Letting n tend to infinity and using (iii) it follows

$$\rho(Tu, v) \leq g(0, 0, \rho(v, Tu), 0)$$

which implies (ii) that $\rho(Tu, \ddot{u}v) = 0$ and so

$$Tu = v. \tag{2.22}$$

It follows from equations (2.19),(2.21) and (2.22) that

$$\begin{aligned} TRSv &= TRw = Tu = v, \\ STRw &= STu = Sv = w, \\ RSTu &= RSv = Rw = u. \end{aligned}$$

The same results of course hold if R or T is continuous instead of S.We now prove the uniqueness of the fixed point u. Supoose that RST has a second fixed point $u^{'}$. Then using inequality (2.13) for $y = Tu$ and $x = u^{'}$ we have

$$d(u, u^{'}) = d(RSTu, RSTu') \leq f(\rho(Tu, Tu'), 0, d(u, u^{'}), \sigma(STu, STu^{'}))$$

which implies by (ii) that

$$d(u, u^{'}) \leq c \max\{\rho(Tu, Tu'), \sigma(STu, STu^{'})\}. \tag{2.23}$$

Further, using inequality (2.14) for $z = STu$ and $y = Tu'$ we have

$$\rho(Tu, Tu') \leq g(\sigma(STu, STu^{'}), 0, \rho(Tu, Tu'), d(u, u'))$$

which implies by (ii) that

$$\rho(Tu, Tu') \leq c \max\{\sigma(STu, STu^{'}), d(u, u')\}. \tag{2.24}$$

inequalities (2.23) and (2.24) implies that

$$d(u,u') \leq c\sigma(STu, STu'). \tag{2.25}$$

Finally, using inequality (2.15) and property (ii), we have

$$\sigma(STu, STu') \leq h(d(u,u'), 0, \sigma(STu, STu'), \rho(Tu, Tu'))$$

which implies by (ii)

$$\sigma(STu, STu') \leq c \max\{d(u,u'), \rho(Tu, Tu')\}. \tag{2.26}$$

It now follows from inequalities (2.24),(2.25) and (2.26) that

$$d(u,u') \leq c\sigma(STu, STu') \leq c^2\sigma(STu, STu'),$$

and so $u = u'$, since $0 \leq c < 1$. The fixed point u of RST is therefore unique. Similarly, it can proved that v is the unique fixed point of TRS and w is the unique fixed point of STR. This completes the proof of theorem. □

Remark 2. *Letting*

$$f(t_1,t_2,t_3,t_4) = g(t_1,t_2,t_3,t_4) = h(t_1,t_2,t_3,t_4) = c \max\{t_1,t_2,t_3,t_4\}$$

with $(0 \leq c < 1)$*, then we see that Theorem 2 is a consequence of Theorem 4.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.2,179-186,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Explicit Quasiconformal Extensions of Planar Harmonic Mappings

Andrzej Ganczar

Abstract Harmonic mappings appear in connection with the theory of minimal surfaces – a nonparametric minimal surface expressed in terms of isothermal parameters and projected onto its base plane induces a harmonic mapping. Intensive studies of these mappings were initiated by J. Clunie and T. Sheil-Small (see [3] for details). They have been interested in harmonic mappings as generalizations of conformal mappings. The aim of this paper is to find explicite quasiconformal extensions, to the extended plane $\overline{\mathbb{C}}$, for some special harmonic mappings defined on the unit disk.

Keywords harmonic mapping · quasiconformal extension

Mathematics Subject Classification (2000) MSC 30C62 · 30C45

1 Introduction

A complex–valued function f defined on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ which is injective and satisfies the equation $f_{z\overline{z}} = 0$ will be called a harmonic map of $\mathbb{D}$. The statement that f is harmonic on $\mathbb{D}$ implies that f_z is analytic and $f_{\overline{z}}$ is anti–analytic. According to the above remark, we have

$$f = h + \overline{g} \tag{1}$$

where h and g are analytic on $\mathbb{D}$. In view of Lewy's theorem (see [7]), the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ doesn't vanish on $\mathbb{D}$ and we may assume that $J_f > 0$ (i.e. f to be sense–preserving).

The function $\omega_f = g'/h' = \overline{f_{\overline{z}}}/f_z$ has special significance. If f is a sense–preserving harmonic mapping, we see that ω_f is analytic and satisfies $|\omega_f(z)| < 1$ on $\mathbb{D}$. Moreover, ω_f agrees in modulus with the first complex dilatation $\mu_f = f_{\overline{z}}/f_z$, so it may be

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called the analytic (or second complex) dilatation of f. On each compact subset of the unit disk ω_f is bounded away form one, and so f is locally quasiconformal. It is clear that a sense–preserving harmonic mapping f is quasiconformal on $\mathbb{D}$ if and only if $\|\omega_f\|_\infty = \sup_{z\in\mathbb{D}} |\omega_f(z)| < 1$.

In this paper we are going to find conditions for a given harmonic mapping f of the disk $\mathbb{D}$ to have a quasiconformal extension to $\overline{\mathbb{C}}$, and to give an explicite construction of an extension. To this end we need the notions of quasicircle and quasiconformal reflection due to Ahlfors. Recall that, a Jordan curve Γ in $\overline{\mathbb{C}}$ is called a quasicircle if it is the image line of a circle under a quasiconformal automorphism of $\overline{\mathbb{C}}$. Let $\mathcal{G}$, $\mathcal{G}^*$ be complementary domains of a Jordan curve Γ in $\overline{\mathbb{C}}$. We say that ϕ is a quasiconformal reflection in Γ if ϕ is a sense-reversing quasiconformal mapping of $\mathcal{G}$ onto $\mathcal{G}^*$ whose homeomorphic extension to the closure $\overline{\mathcal{G}}$ keeps the points on Γ fixed. It is known (see e.g. [6]) that a Jordan curve Γ admits a quasiconformal reflection ϕ if and only if it is a quasicircle.

In the last fifteen years, many papers have been published on harmonic mappings satisfying certain coefficient conditions. For a fixed sequence $(\psi_n)_{n=2,3,\dots}$ of positive real numbers, we denote by $\mathcal{H}^0(\psi_n)$ the class of mappings f of the form (1) where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{2}$$

that satisfy the condition

$$\sum_{n=2}^{\infty} \psi_n(|a_n| + |b_n|) \le 1. \tag{3}$$

It has been shown (see [2]) that each function of the class $\mathcal{H}^0(n)$ is a sense–preserving and starlike harmonic mapping. The class $\mathcal{H}^0(n^2)$ was also considered by Avci and Złotkiewicz ([2]) and it is well-known that $f \in \mathcal{H}^0(n^2)$ is a convex harmonic mapping of $\mathbb{D}$.

A natural question arises in connection with this, i.e., to characterize harmonic mappings from $\mathcal{H}^0(\psi_n)$ that admit a quasiconformal extension to $\overline{\mathbb{C}}$ and find an explicit construction of such an extension.

2 Main results

Consider a sequence $(\psi_n)_{n=2,3,\dots}$ of positive real numbers satisfying the condition

$$\frac{\psi_n}{n} \ge \frac{\psi_2}{2}, \quad n = 3, 4, \dots\ . \tag{4}$$

Notice that if $\psi_2 < 2$, then $f \in \mathcal{H}^0(\psi_n)$ need not be univalent on $\mathbb{D}$ (see e.g. [5, Theorem 3]), so we will restrict our attention to the case $\psi_2 \ge 2$. Clearly, the last inequality implies that $\mathcal{H}^0(\psi_n) \subset \mathcal{H}^0(n)$, and we obtain

Lemma 1 *Assume that $f \in \mathcal{H}^0(\psi_n)$, where $(\psi_n)_{n=2,3,\dots}$ is a sequence of positive real numbers satisfying the condition (4) and such that $\psi_2 \ge 2$. Then f is a sense–preserving harmonic mapping of the disk $\mathbb{D}$ onto a starlike domain.*

The following technical lemma, provides some information on a sequence $(\psi_n)_{n=2,3,\dots}$ satisfying (4), which will be relevant later on.

Lemma 2 *Let $(\psi_n)_{n=2,3,\ldots}$ be a sequence of positive real numbers satisfying the condition (4).*

(i) If $\psi_2 \in [4, +\infty)$, then

$$\frac{\psi_n}{n} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right) > \frac{\psi_2}{2\log_2 \psi_2} \geq 1, \quad n = 2, 3, \ldots .$$

(ii) If $\psi_2 \in (2, 4)$, then

$$\frac{\psi_n}{n} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right) > 1 > \frac{\psi_2}{2\log_2 \psi_2}, \quad n = 2, 3, \ldots .$$

Proof If $\psi_2 \in [4, +\infty)$, obviously by Jordan's inequality

$$\frac{\psi_n}{n} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right) > \frac{\psi_n}{n \log_2 \psi_2}$$

and then (i) follows from the condition (4).
Now we prove (ii). According to the condition (4), we have

$$\frac{\psi_n}{n} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right) \geq \frac{\psi_2}{2} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right), \quad n = 2, 3, \ldots ,$$

so it suffices to prove

$$\frac{\psi_2}{2} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right) > 1$$

for $\psi_2 \in (2, 4)$. Consider the function

$$G(x) = x \sin\left(\frac{\pi}{2\log_2 x}\right) .$$

Clearly $G(2) = 2$, and it is enough to show that G is strictly increasing on the interval $(2, 4)$. But the inequality (see e.g. [8, p. 246])

$$\frac{4}{\pi} \frac{x}{\pi - 2x} < \tan x,$$

where $x \in (0, \frac{\pi}{2})$, implies

$$G'(x) > \frac{1}{2\pi \log_2 x} \cos\left(\frac{\pi}{2\log_2 x}\right) \frac{(4\ln 2)\log_2^2 x - \pi^2(\log_2 x - 1)}{(\ln 2)(\log_2 x - 1)\log_2 x} > 0$$

for any $x \in (2, 4)$, and the proof is complete.

□

The next step is an estimation of the modulus $|\omega_f(z)|$ for $f \in \mathcal{H}^0(\psi_n)$.

Lemma 3 *Suppose that $f = h + \overline{g} \in \mathcal{H}^0(\psi_n)$ and $(\psi_n)_{n=2,3,\ldots}$ is a sequence of positive real numbers satisfying the condition (4).*

(i) If $\psi_2 \in [4, +\infty)$, then

$$\|\omega_f\|_\infty = \sup_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| < \frac{2\log_2 \psi_2}{\psi_2} \sin\left(\frac{\pi}{2\log_2 \psi_2}\right) .$$

(ii) *If* $\psi_2 \in (2,4)$, *then*
$$\|\omega_f\|_\infty < \sin\left(\frac{\pi}{2\log_2\psi_2}\right).$$

Proof We shall justify the case (i) only. For the case (ii) the proof may be done similarly. Let us first observe that for $f = h + \overline{g} \in \mathcal{H}^0(\psi_n)$, where h and g are of the form (2), we have
$$|\omega_f(z)| = \left|\frac{g'(z)}{h'(z)}\right| \le \frac{\sum\limits_{n=2}^{\infty} n|b_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|}, \quad z \in \mathbb{D}.$$
By the above, it is enough to show that the condition (4) implies
$$\frac{\sum\limits_{n=2}^{\infty} n|b_n|}{1 - \sum\limits_{n=2}^{\infty} n|a_n|} < \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right).$$
Since $\psi_2 \in [4, +\infty)$, we conclude that
$$\sum_{n=2}^{\infty} n|b_n| + \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right)\sum_{n=2}^{\infty} n|a_n| \le \sum_{n=2}^{\infty} n(|a_n| + |b_n|) <$$
$$\frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right)\sum_{n=2}^{\infty}\psi_n(|a_n| + |b_n|),$$
the last inequality being a consequence of Lemma 2. Condition (3) now yields
$$\sum_{n=2}^{\infty} n|b_n| + \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right)\sum_{n=2}^{\infty} n|a_n| < \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right),$$
and (i) is proved.

□

Corollary 1 *Assume that* $(\psi_n)_{n=2,3,\ldots}$ *is a sequence of positive real numbers satisfying the condition (4) and* $\psi_2 > 2$, *then* $f \in \mathcal{H}^0(\psi_n)$ *is quasiconformal on* $\mathbb{D}$.

Observe that a necessary condition for a harmonic mapping $f \in \mathcal{H}^0(\psi_n)$ to have a quasiconformal extension to the whole plane is the following: the image curve $f(\mathbb{T})$ is a quasicircle, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. On account of the above remark, we now prove

Theorem 1 *Suppose that* $f \in \mathcal{H}^0(\psi_n)$, *where* $(\psi_n)_{n=2,3,\ldots}$ *is a sequence of positive real numbers satisfying the condition (4) and* $\psi_2 > 2$. *Then the curve* $f(\mathbb{T})$ *is a quasicircle.*

Proof Let $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n + \overline{\sum\limits_{n=2}^{\infty} b_n z^n}$, for all $z \in \mathbb{D}$. By Lemma 2 and the condition (3), we have
$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) < \sin\left(\frac{\pi}{2\log_2\psi_2}\right)\sum_{n=2}^{\infty}\psi_n(|a_n| + |b_n|) \le \sin\left(\frac{\pi}{2\log_2\psi_2}\right).$$

Thus, for $z_1, z_2 \in \mathbb{D}$ such that $z_1 \neq z_2$

$$|f(z_1) - f(z_2)| = \left| z_1 - z_2 + \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n) + \overline{\sum_{n=2}^{\infty} b_n (z_1^n - z_2^n)} \right| \leq \tag{5}$$

$$|z_1 - z_2| \left(1 + \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \right) < |z_1 - z_2| \left(1 + \sin\left(\frac{\pi}{2\log_2 \psi_2} \right) \right)$$

and

$$|f(z_1) - f(z_2)| \geq |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \right) > |z_1 - z_2| \left(1 - \sin\left(\frac{\pi}{2\log_2 \psi_2} \right) \right) > 0. \tag{6}$$

From (5) and (6) we deduce that f has homeomorphic extension on $\overline{\mathbb{D}}$, which also satisfies (5) and (6). Therefore the image line $\Gamma = f(\mathbb{T})$ is a Jordan curve.
According to Ahlfors (see [1]), a Jordan curve γ is a quasicircle if and only if

$$K(\gamma) = \sup \frac{|w_1 - w_2| \cdot |w_3 - w_4| + |w_1 - w_4| \cdot |w_2 - w_3|}{|w_1 - w_3| \cdot |w_2 - w_4|}$$

is finite, where supremum being taken over the set of all ordered quadruples $\{w_1, w_2, w_3, w_4\}$ of points on γ. We conclude from (5) and (6) that

$$K(\Gamma) \leq \left(\frac{1 + \sin\left(\frac{\pi}{2\log_2 \psi_2} \right)}{1 - \sin\left(\frac{\pi}{2\log_2 \psi_2} \right)} \right)^2 \cdot K(\mathbb{T}) < \infty,$$

and finally that Γ is a quasicircle.

□

Remark 1. *Possibility of homeomorphic extension on $\overline{\mathbb{D}}$ in the proof above follows immediately from the fact that f, according to Lemma 3, is quasiconformal on $\mathbb{D}$ (see e.g. [6, Theorem I.8.2]).*

W can now formulate our main theorem.

Theorem 2 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$ belong to $\mathcal{H}^0(\psi_n)$, where $(\psi_n)_{n=2,3,\ldots}$ is a sequence of positive real numbers satisfying the condition (4) and $\psi_2 > 2$. Then the mapping*

$$F(z) = \begin{cases} f(z), & \text{for } |z| \leq 1, \\ z + \sum_{n=2}^{\infty} a_n \overline{z^{-n}} + \sum_{n=2}^{\infty} \overline{b_n} z^{-n}, & \text{for } |z| \geq 1, \end{cases} \tag{7}$$

is a quasiconformal extension of f onto $\overline{\mathbb{C}}$. Moreover, its complex dilatation μ_F satisfies $|\mu_F(z)| < \sin\left(\frac{\pi}{2\log_2 \psi_2} \right)$ for any $\psi_2 \in (2, 4)$, and $|\mu_F(z)| < \frac{2\log_2 \psi_2}{\psi_2} \sin\left(\frac{\pi}{2\log_2 \psi_2} \right)$ in the case $\psi_2 \in [4, +\infty)$.

Proof We justify the case $\psi_2 \in [4,+\infty)$ only. On account of Lemma 3 f is quasiconformal on $\mathbb{D}$ and $|\mu_F(z)| < \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right)$ for all $z \in \mathbb{D}$. Moreover, this map has a homeomorphic extension on $\overline{\mathbb{D}}$.
Next, in the same way as in Theorem 1, for z_1, $z_2 \in \mathbb{C}\setminus\overline{\mathbb{D}}$ such that $z_1 \neq z_2$, we obtain

$$|f(z_1)-f(z_2)| \leq |z_1-z_2| + |z_1^{-1}-z_2^{-1}|\sum_{n=2}^{\infty} n(|a_n|+|b_n|) \leq$$

$$|z_1-z_2|\left(1+\sum_{n=2}^{\infty} n(|a_n|+|b_n|)\right) < |z_1-z_2|\left(1+\sin\left(\frac{\pi}{2\log_2\psi_2}\right)\right),$$

and

$$|f(z_1)-f(z_2)| \geq |z_1-z_2|\left(1-\frac{\sum\limits_{n=2}^{\infty} n(|a_n|+|b_n|)}{|z_1z_2|}\right) > |z_1-z_2|\left(1-\frac{\sin\left(\frac{\pi}{2\log_2\psi_2}\right)}{|z_1z_2|}\right) > 0.$$

In view of the part (i) of Lemma 2, for any $z \in \mathbb{C}\setminus\mathbb{D}$

$$|F_z| = |1-\sum_{n=2}^{\infty} n\overline{b_n}z^{-n-1}| \geq 1-\sum_{n=2}^{\infty} n|b_n| \geq 1-\sum_{n=2}^{\infty} n(|a_n|+|b_n|) > 1-\sin\left(\frac{\pi}{2\log_2\psi_2}\right) > 0,$$

and

$$|\mu_F(z)| = \left|\frac{F_{\overline{z}}}{F_z}\right| = \left|\frac{\sum\limits_{n=2}^{\infty} na_n\overline{z^{-n-1}}}{1-\sum\limits_{n=2}^{\infty} n\overline{b_n}z^{-n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n|a_n|}{1-\sum\limits_{n=2}^{\infty} n|b_n|} < \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right) < 1.$$

This leads to

$$J_F(z) = |F_z|^2 - |F_{\overline{z}}|^2 = |F_z|^2(1-|\mu_F(z)|^2) > 0$$

for $z \in \mathbb{C}\setminus\mathbb{D}$.
By the above considerations, F given by (7) is continuous and locally univalent on $\mathbb{C}$, moreover $\lim\limits_{z\to\infty} f(z) = \infty$. Therefore (see e.g. [9, Theorem 2.7.2]), the mapping F is a sense-preserving homeomorphism of $\overline{\mathbb{C}}$ onto itself. Its complex dilatation μ_F satisfies

$$|\mu_F(z)| < \frac{2\log_2\psi_2}{\psi_2}\sin\left(\frac{\pi}{2\log_2\psi_2}\right)$$

for any $z \in \mathbb{C}\setminus\mathbb{T}$. Since $\mathbb{T}$ is a removable set for F (see [6, p. 44]) it follows that F is a quasiconformal in the whole plane.

A trivial verification shows that $f(z) = z + \frac{1}{2}z^2$ belongs to $\mathcal{H}^0(n)$ but has no quasiconformal extension on $\overline{\mathbb{C}}$. For this reason, if $\psi_2 = 2$ then the condition (4) does not imply the possibility of quasiconformal extension of $f \in \mathcal{H}^0(\psi_n)$. Therefore, given $k \in (0,1)$ we consider the class $\mathcal{H}^0(\psi_n, k)$ of functions $f = h + \overline{g}$, where h and g are of the form (3), that satisfying the condition

$$\sum_{n=2}^{\infty} \psi_n(|a_n|+|b_n|) \leq k < 1.$$

We can now state the analogy of Lemma 3 and Theorem 1.

Theorem 3 *Suppose that $f = h + \overline{g} \in \mathcal{H}^0(\psi_n, k)$, where $(\psi_n)_{n=2,3...}$ is a sequence of positive real numbers satisfying the condition (4) and $\psi_2 = 2$. Then*

(i) $\|\omega_f\|_\infty \le k$,
(ii) $f(\mathbb{T})$ *is a quasicircle.*

Analysis similar to that in the proof of Theorem 2 shows that

Theorem 4 *Let $(\psi_n)_{n=2,3,...}$ be a sequence of positive real numbers such that the condition (4) holds and $\psi_2 = 2$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n} \in \mathcal{H}^0(\psi_n, k)$, then the function F given by (7) is a quasiconformal extension of f onto $\overline{\mathbb{C}}$ whose complex dilatation μ_F satisfies $|\mu_F(z)| \le k$.*

The following theorem gives a non–trivial way of producing quasiconformal extensions.

Theorem 5 *Let $f : \mathbb{D} \to \mathbb{C}$ be a sense–preserving harmonic mapping such that $\|\omega_f\|_\infty < 1$ and $\Gamma = f(\mathbb{T})$ is a quasicircle. If ϕ is a quasiconformal reflection in Γ, then*

$$G(z) = \begin{cases} f(z), & \text{for } |z| \le 1, \\ \phi(f(1/\overline{z})), & \text{for } |z| \ge 1, \end{cases} \tag{8}$$

is a quasiconformal extension of f onto $\overline{\mathbb{C}}$.

Proof The proof of Theorem follows immediately from the fact that a Jordan curve Γ admits a quasiconformal reflection ϕ if and only if it is a quasicircle. Moreover, G defined by (8) is a homeomorphism of $\overline{\mathbb{C}}$ onto itself which is quasiconformal in $\mathbb{D}$ and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We omit the details.

As a direct consequence of Theorem 5, we obtain

Corollary 2 *If f is a sense–preserving harmonic automorphism of the disk $\mathbb{D}$ such that $\|\omega_f\|_\infty < 1$, then the mapping*

$$G(z) = \begin{cases} f(z), & \text{for } |z| \le 1, \\ 1/\overline{f(1/\overline{z})}, & \text{for } |z| \ge 1, \end{cases}$$

defines a quasiconformal extension of f onto $\overline{\mathbb{C}}$.

It is known (see [4, Corollary 2]) that a convex Jordan curve contained in the annulus $\{w : r \le |w| \le R\}$ is a quasicircle. According to the above remark, we have

Lemma 4 *Suppose that $f : \mathbb{D} \to \mathbb{C}$ is a sense-preserving harmonic mapping and $f(\mathbb{D})$ is a bounded convex region. If $\|\omega_f\|_\infty < 1$, then the mapping G given by (8), where ϕ denotes a quasiconformal reflection in $f(\mathbb{T})$, is a quasiconformal extension of f onto $\overline{\mathbb{C}}$.*

Notice that, if $(\psi_n)_{n=2,3,...}$ is a sequence of positive real numbers satisfying the condition $\psi_n \ge n^2$, $n = 2, 3, \dots$, then $f \in \mathcal{H}^0(\psi_n)$ is a convex harmonic mapping.

Remark 2. *If the co-analytic part of $f = h + \overline{g} \in \mathcal{H}^0(\psi_n)$ is zero, i.e. the function g is identically zero, then our results describe analogous properties of analytic univalent functions (see e.g. [4]).*

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Numerical solution of integral equations by using local polynomial regression

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Abstract

In this paper, we find numerical solution of

$$x(t) + \lambda \int_a^b k(t,s)x(s)ds = y(t), \qquad a \le t \le b$$

or

$$x(t) + \lambda \int_a^t k(t,s)x(s)ds = y(t), \qquad a \le t \le b, a \le s \le b$$

by Local Polynomial Regression(LPR). We shown that, present new method is powerful in solving both Fredholm and Volterra integral equations. The method is tested on some model problems to demonstrate its usefulness. The convergence of the method is discusses.

Keywords: Integral equations,Local polynomial regression, Kernel functions.

1. Introduction

Consider the linear Fredholm integral equation ,

$$x(t) + \lambda \int_a^b k(t,s)x(s)ds = y(t), \qquad a \le t \le b \tag{1}$$

and the linear Volterra integral equation

$$x(t) + \lambda \int_a^t k(t,s)x(s)ds = y(t), \qquad a \le t \le b, a \le s \le b \tag{2}$$

Where the parameter λ , the kernel k and the function y are given and x(t) is the unknown function to be determined. Many different techniques have been presented so far for solving the above mentioned integral equation[4,5,6]. The theory of integral equations can be found in [1,2]. The subject of this paper is to present new method by use of local polynomial regression for the numerical solution of Fredholm and Volterra integral equations, and finally, we shown that the method to achive the desired accuracy. Details of the structure of the present method is explained in section 2. We apply method for linear Fredholm and Volterra integral equations.In Section 3 for showing the efficiency of numerical method. Finally, in Section 4 contains conclussion and direction for future research.

2. Numerical method

In this section, we describe our method to find the approximating solution of integral Eqs.(1) and Eqs.(2). In our study We have opted for using local polynomial regression. The fundamential idea of this methodology appears in the [3]. The following is the mathematical formulation of the local polynomial regression.

2.1. Local polynomial regression

Suppose that the $(p+1)$th derivative of $x(t)$ at point t_0 exists. We approximate the unknown regression function $x(t)$ locally at t_0 by a polynomial of order p. The theoretical justification is that we can approximate, in a neighborhood of t_0 , x(t) using a Taylor expansion

$$x(t) \approx \textstyle\sum_{k=0}^{p} \beta_k (t-t_0)^k \tag{3}$$

where

$$\beta_k = \frac{x^{(k)}(t_0)}{k!} \tag{4}$$

This polynomial, used to approximate the unknown function locally at t_0, is obtained by solving a locally weighted least squares regression problem, i.e. by minimizing

$$\sum_{i=1}^{n} \left\{ Y_i - \sum_{k=0}^{p} \beta_k (t_i - t_0)^k \right\}^2 K(\tfrac{t_i - t_0}{h}) \tag{5}$$

where h is a parameter called bandwidth (also called a smoothing parameter), K is a weighting function called the kernel function. Let $\beta_k, k = 0, 1, ...p$ be the solution of the minimizing problem. From Eqs.(4), it is clear that j!β_j is an estimator for the derivatives $x^{(j)}(t_0), j = 0, 1, ...p$. Thus, the estimation obtained, of both the regression function and its derivatives, is local, and therefore, the process must be repeated at all points where an estimation is of interest. Let us see the analytical expression of the solution $\beta_k, k = 0, 1, ...p$ of the locally weighted least squares regression problem. Let X be the n x $(p + 1)$ matrix

$$\text{X}=\begin{pmatrix} 1 & (t_1 - t_0) & ... & (t_1 - t_0)^p \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & (t_n - t_0) & ... & (t_n - t_0)^p \end{pmatrix} \tag{6}$$

and the vectors $y = (Y_1, Y_2, ..., Y_n)'$ and $\beta = (\beta_0, \beta_1, ...\beta_p)'$. Finally, denote by W the nxn diagonal matrix of weights $W = diag\{K_h(t_i - t_0)\}$. Then, the solution is

$$\beta = (X^T W X)^{-1} X^T W y. \tag{7}$$

The selection of K does not influence the results much. We selected the quartic kernel as follows

$$K(u) = \left\{ \begin{array}{cc} \frac{15}{16}(1 - u^2)^2 & if|u| \leq 1 \\ 0 & otherwise \end{array} \right\} \tag{8}$$

2.2. Illustration of the method

In this section the LPR method for solving Eqs(1) and Eqs(2) is outlined. Let Eqs(3) be an approximate solution of Volterra integral equation(2)

$$x(t) = \sum_{j=0}^{p} \beta_j (t - x_0)^j \tag{9}$$

where

$t_1 = a, t_2, ...t_n = b$

and It is required that the approximate solution(9) satisfies the integral equation at the points t=t_i. Putting (9) in (2), it is follows that

$$\sum_{j=0}^{p} \beta_j (t-t_0)^j + \lambda \sum_{j=0}^{p} \beta_j \int_a^t k(t,s)(s-t_0)^j ds = y(t),$$

$$a \leq t \leq b, a \leq s \leq b \tag{10}$$

Then, matrix form(6) can be written as

$$\text{X}=\begin{bmatrix} 1 & (t_1-t_0)+\lambda\int_a^t k(t,s)(s-x_0)ds & ... & (t_1-t_0)^p+\lambda\int_a^t k(t,s)(s-x_0)^p ds \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & (t_n-t_0)+\lambda\int_a^t k(t,s)(s-x_0)ds & ... & (t_n-t_0)^p+\lambda\int_a^t k(t,s)(s-x_0)^p ds \end{bmatrix}$$

$$\text{Y}=\begin{bmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{bmatrix} \tag{11}$$

Putting (11) in (7), then estimated set of coefficients β_i are obtained by solving matrix system solution. Therefore, approximate solution (9) is obtained. Same procedure can be used for Eqs(1).

3. Numerical examples

In this section we consider some examples of Fredholm and Volterra type that demonstrate the accurancy and effectiveness of the present method. All computations were carried out using MATLAB 6.5.

Example 1.

We first consider the following integral equation

$x(s) = sin(s) - s + \int_0^{\frac{\pi}{2}} stx(t)dt \ , \ 0 \leq t \leq \frac{\pi}{2}$

with the exact solution: $x(s) = sin(s)$

The numerical results are shown in Table 1 and illustruate Fig.1. The maximum absolute error is given in Table 2.

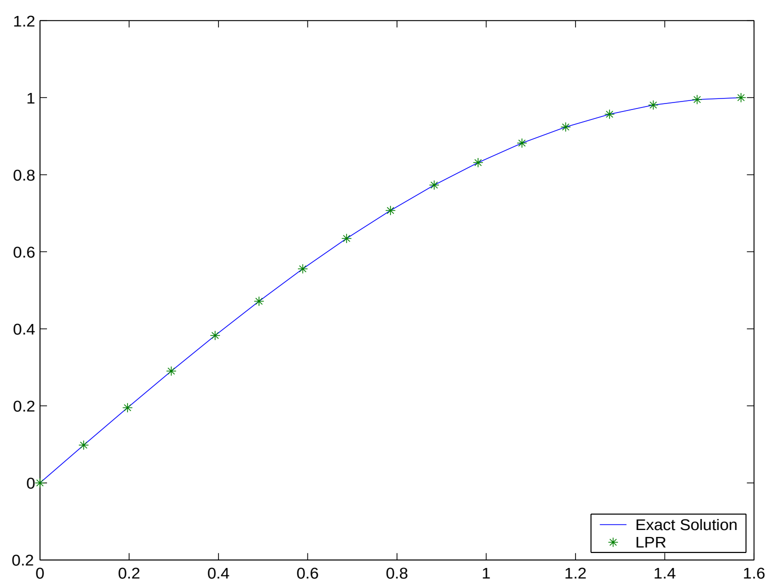

Figure 1: Results for Example 1 with x(s)=sin(s)

Table 1: $y_i : ExactSolution; Y_i : LPRSolution$

$\mathbf{y}_i$	$\mathbf{Y}_i$
0	-0.00000000120664
0.09801714032956	0.09801716335287
0.19509032201613	0.19509030895861
0.29028467725446	0.29028466194663
0.38268343236509	0.38268343573862
0.47139673682600	0.47139675044662
0.55557023301960	0.55557023796561
0.63439328416365	0.63439327155392
0.70710678118655	0.70710675890148
0.77301045336274	0.77301043768579
0.83146961230255	0.83146961261548
0.88192126434835	0.88192127296150
0.92387953251129	0.92387952957580
0.95694033573221	0.95694031039742
0.98078528040323	0.98078525344602
0.99518472667220	0.99518473630261
1.00000000000000	0.99999998107762

Example 2.

$$x(s) = e^s - \frac{e^{s+1}-1}{s+1} + \int_0^1 e^{st}x(t)dt, 0 \leq t \leq 1$$

Exact solution: $x(s) = e^s$
The maximum absolute error is given in Table.2. Also numerical results are shown Fig.2.

Example 3.

$$x(s) = s + \int_0^1 K(s,t)x(t)dt, 0 \leq t \leq 1$$

$$K(s,t) = \begin{cases} s, & s \leq t \\ t, & s \geq t \end{cases}$$

Exact solution: $x(s) = sec(1)sin(s)$

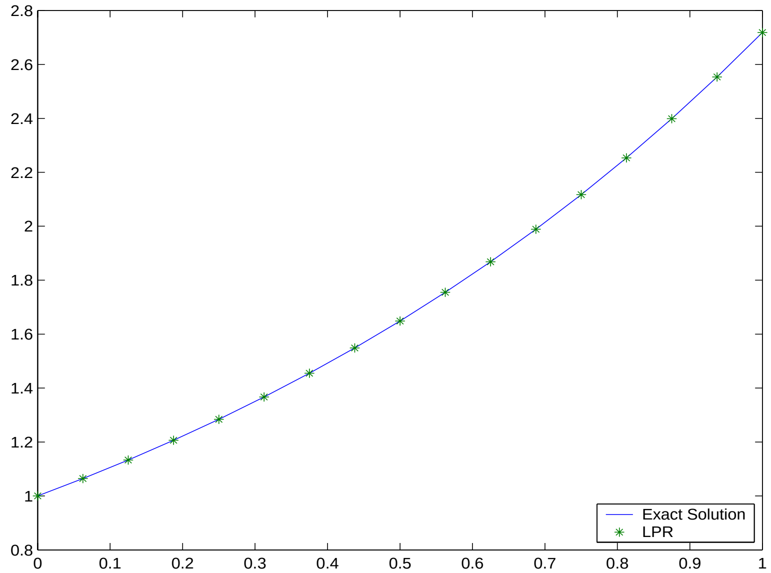

Figure 2: Results for Example 2 with x(s)=exp(s)

The maximum absolute error is given in Table 2. See Fig.3.

Example 4.

$x(s) = -\int_0^s stx(t)dt + \frac{(2-s)e^{-s^2}+x}{2}, 0 \leq t \leq \ 1$

Exact solution: $x(s) = e^{-s^2}$
The maximum absolute error is given in Table 2. See Fig.4.

4. Conclusions

In this study, we introduced new method for solve the integral equations. We shown that the method is a very fast convergent for solving integral equations.Numerical results showed that the present method approximents the exact solution very well. The Method can also be extended to the different bandwidth and kernel functions.

Table 2: The Maximum Errors

Example	**n,p**	**Absolute Error**
1	11,9	1.686115×10^{-7}
1	21,9	3.004420×10^{-9}
2	11,8	1.716089×10^{-7}
2	21,8	1.938185×10^{-8}
3	11,8	1.365100×10^{-7}
3	21,8	1.007513×10^{-9}
4	21,9	2.188129×10^{-6}
4	41,9	1.489571×10^{-7}

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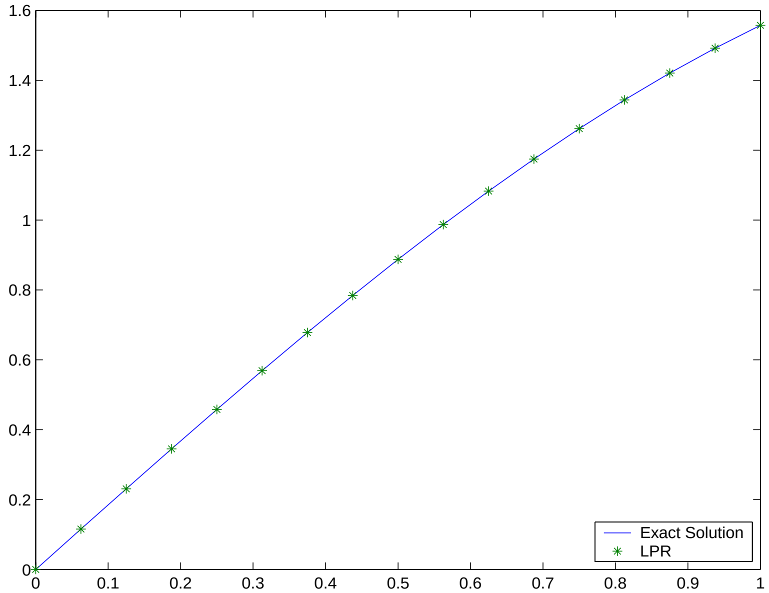

Figure 3: Results for Example 3 with with x(s)=sec(1)sin(s)

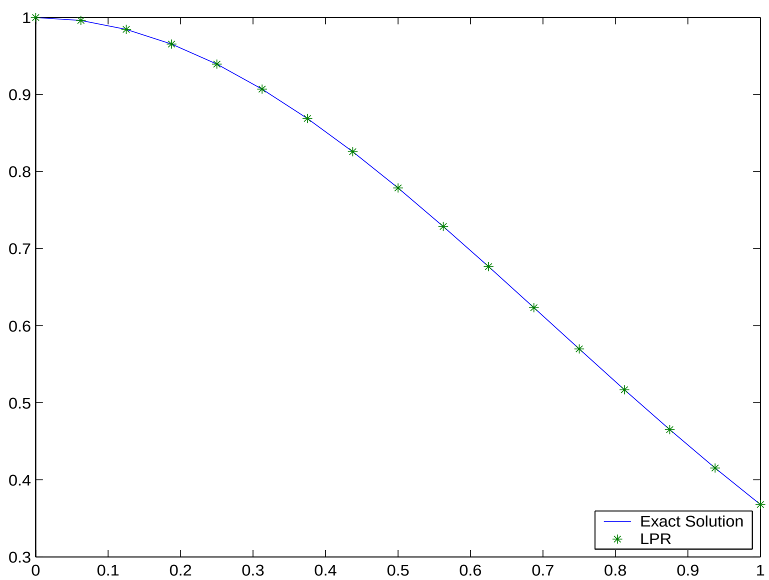

Figure 4: Results for Example 4 with with $x(s) = e^{-s^2}$

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Bayesian Classifier based on the Multivariate Normal Distribution

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Abstract

The aim of this paper is the Bayesian estimation techniques to obtain the form of the a posteriori density $p(\mu|D)$ and the desired probability density $p(X|D)$ in the case when $p(X|\mu) \sim N(\mu, \Sigma)$. The treatment of the multivariate case in which Σ is known but μ is not represents the generalization of the univariate case (see [5]).

This approach is then used in Bayesian classification.

AMS Subject Classification: 62C10, 62E17

Keywords: Bayesian estimation, a posteriori probability, a priori probability, Bayes theorem, conditional distribution, posterior distribution

1 The aim of the paper

Bayesian decision theory is a fundamental statistical approach to the problem of pattern classification.

We consider M classes $\omega_1, \ldots, \omega_M$ with *a priori* probabilities (the probabilities of each class occurring) $P(\omega_1), \ldots, P(\omega_M)$ assumed known.

Let be X a d- dimensional vector, called the *feature vector* which is normally distributed. The general multivariate normal density is given by a d-dimensional mean vector and a d-by-d covariance matrix:

$$p(X) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(X-\mu)^t \Sigma^{-1}(X-\mu)\right],$$

where the covariance matrix Σ is assumed to be known and the mean vector μ is unknown.

We wish to assign X to one of the M classes. We know that the Bayes rule is to assign X to class ω_j if

$$P(\omega_j|X) > P(\omega_k|X),\ k = \overline{1, M};\ k \neq j, \tag{1}$$

namely if the probability of class ω_j given the feature vector X, $P(\omega_j|X)$ is greatest over all classes $\omega_1, \ldots, \omega_M$.

The *a posteriori* probability $P(\omega_j|X)$ represents the probability so that X belongs to class ω_j, after X was classified.

The a posteriori probabilities $P(\omega_j|X)$ may be expressed in terms of the a priori probabilities $P(\omega_j)$ and the class-conditional density functions $p(X|\omega_j)$ using Bayes theorem as

$$P(\omega_j|X) = \frac{p(X|\omega_j)P(\omega_j)}{p(X)} \tag{2}$$

so the relation (1) may be written

$$p(X|\omega_j)P(\omega_j) > p(X|\omega_k)P(\omega_k),\ k = \overline{1, M};\ k \neq j. \tag{3}$$

Let us consider the sample set $D = \{X_1, X_2, \ldots, X_n\}$ on a random vector X.

We consider that the a priori multivariate density for the mean μ, $p(\mu)$ is also normal

$$p(\mu) \sim N(\mu_0, \Sigma_0),$$

namely

$$p(\mu) = \frac{1}{(2\pi)^{d/2}|\Sigma_0|^{1/2}} \exp\left[-\frac{1}{2}(\mu - \mu_0)^t {\Sigma_0}^{-1}(\mu - \mu_0)\right],$$

where both the mean vector μ_0 and the covariance matrix Σ_0 are known.

Our goal is to estimate the probability density $p(X|D)$ because it represents the desired class conditional density $p(X|\omega_j)$ which multiplied by the a prior probability $P(\omega_j)$ it gives the probabilistic information needed to design the classifier.

From [2] we know that

$$p(X|D) = \int_D p(X|\mu)p(\mu|D)d\mu, \tag{4}$$

where:

- the function $p(X|\mu)$ represents the parametric form of the probability density p(X)
- $p(\mu|D)$ is a posteriori density of the random vector μ,
- $p(X|D)$ is the desired class-conditional density.

Therefore, for realizing our goal are necessary two stages:
Stage I. We estimate $p(\mu|D)$,
Stage II. We estimate $p(X|D)$ using the relation (4).

2 Bayesian estimation approach in the gaussian case, used in pattern recognition: the multivariate hypothesis

Theorem 1 *[2] The desired class-conditional density is*

$$p(X|D) \sim N(\mu_n, \Sigma_n + \Sigma),$$

where:

$$\begin{array}{l} {\Sigma_n}^{-1} = {\Sigma_0}^{-1} + n\Sigma^{-1} \\ \mu_n = \Sigma_n \left({\Sigma_0}^{-1}\mu_0 + \Sigma^{-1} \sum_{k=1}^{n} X_k\right). \end{array}$$

Proof:

Stage I. In order to estimate $p(\mu|D)$ we shall use the Bayes theorem [2], i.e

$$p(\mu|D) = \frac{p(D|\mu)p(\mu)}{\int_D p(D|\mu)p(\mu)d\mu}. \tag{5}$$

As $X_1, X_2, \ldots, X_n$ are stochastically independent, we have:

$$T = p(X_1, X_2, \ldots, X_n|\mu)p(\mu) = p(\mu) \prod_{k=1}^{n} p(X_k|\mu). \tag{6}$$

Thus,

$$T = \frac{1}{(2\pi)^{d/2}|\Sigma_0|^{1/2}} \exp\left[-\frac{1}{2}(\mu - \mu_0)^t {\Sigma_0}^{-1}(\mu - \mu_0)\right] \cdot$$

$$\cdot \prod_{k=1}^{n} \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left[-\frac{1}{2}(X_k - \mu)^t \Sigma^{-1}(X_k - \mu)\right].$$

The last formula can be written as

$$T = \frac{1}{(2\pi)^{\frac{d+nd}{2}}|\Sigma_0|^{1/2}|\Sigma|^{n/2}} \exp\left\{-\frac{1}{2}[(\mu-\mu_0)^t{\Sigma_0}^{-1}(\mu-\mu_0)+\sum_{k=1}^{n}(X_k-\mu)^t\Sigma^{-1}(X_k-\mu)]\right\}.$$

Denoting

$$A = \frac{1}{(2\pi)^{\frac{d+nd}{2}}|\Sigma_0|^{1/2}|\Sigma|^{n/2}}$$

we shall have

$$T = A\cdot\exp\Big\{-\frac{1}{2}\Big[\mu^t{\Sigma_0}^{-1}\mu-\mu^t{\Sigma_0}^{-1}\mu_0-\mu_0^t{\Sigma_0}^{-1}\mu+\mu_0^t{\Sigma_0}^{-1}\mu_0+$$

$$+\sum_{k=1}^{n}(X_k^t\Sigma^{-1}X_k-X_k^t\Sigma^{-1}\mu-\mu^t\Sigma^{-1}X_k+\mu^t\Sigma^{-1}\mu)\Big]\Big\}=$$

$$=A\exp\Big\{-\frac{1}{2}\Big[\mu^t{\Sigma_0}^{-1}\mu-2\mu^t{\Sigma_0}^{-1}\mu_0+\mu_0^t{\Sigma_0}^{-1}\mu_0+\sum_{k=1}^{n}X_k^t\Sigma^{-1}X_k-2\mu^t\Sigma^{-1}\sum_{k=1}^{n}X_k+n\mu^t\Sigma^{-1}\mu\Big]\Big\}$$

and, finally:

$$T=A\cdot\exp\Big\{-\frac{1}{2}\Big[\mu^t(n\Sigma^{-1}+{\Sigma_0}^{-1})\mu-2\mu^t({\Sigma_0}^{-1}\mu_0+\Sigma^{-1}\sum_{k=1}^{n}X_k)+\mu_0^t{\Sigma_0}^{-1}\mu_0+\sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k\Big]\Big\}$$

Using the notation

$$\hat{\mu_n}=\frac{1}{n}\sum_{k=1}^{n}X_k \tag{7}$$

we shall obtain

$$T=A\cdot\exp\Big\{-\frac{1}{2}\Big[\mu^t(n\Sigma^{-1}+{\Sigma_0}^{-1})\mu-2\mu^t({\Sigma_0}^{-1}\mu_0+n\Sigma^{-1}\hat{\mu_n})+\mu_0^t{\Sigma_0}^{-1}\mu_0+\sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k\Big]\Big\}$$

Using the notation

$${\Sigma_n}^{-1}={\Sigma_0}^{-1}+n\Sigma^{-1} \tag{8}$$

we shall have

$$T=A\cdot\exp\Big\{-\frac{1}{2}\Big[\mu^t{\Sigma_n}^{-1}\mu-2\mu^t({\Sigma_0}^{-1}\mu_0+n\Sigma^{-1}\hat{\mu_n})+\mu_0^t{\Sigma_0}^{-1}\mu_0+\sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k\Big]\Big\};$$

therefore

$$T = A\cdot\exp\left\{-\frac{1}{2}\left[\mu^t{\Sigma_n}^{-1}\left(\mu - 2(\Sigma_n{\Sigma_0}^{-1}\mu_0 + n\Sigma_n\Sigma^{-1}\hat{\mu_n})\right) + \mu_0^t{\Sigma_0}^{-1}\mu_0 + \sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k\right]\right\}.$$

Introducing the new notation

$$\mu_n = \Sigma_n\left({\Sigma_0}^{-1}\mu_0 + n\Sigma^{-1}\hat{\mu_n}\right) \tag{9}$$

it results

$$T = A\cdot\exp\left\{-\frac{1}{2}\left[\mu^t{\Sigma_n}^{-1}(\mu - 2\mu_n) + \mu_0^t{\Sigma_0}^{-1}\mu_0 + \sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k\right]\right\},$$

namely

$$T = A\cdot\exp\left\{-\frac{1}{2}\left[\mu_0^t{\Sigma_0}^{-1}\mu_0 + \sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k + (\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n) - \mu_n^t{\Sigma_n}^{-1}\mu_n\right]\right\}.$$

Taking into account the notation

$$k_n = \mu_0^t{\Sigma_0}^{-1}\mu_0 + \sum_{k=1}^{n}{X_k}^t\Sigma^{-1}X_k - \mu_n^t{\Sigma_n}^{-1}\mu_n \tag{10}$$

we obtain the shorter form

$$T = A\exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]\exp\left(-\frac{k_n}{2}\right) \tag{11}$$

Using equation (5) as in terms of the normal case notations

$$p(\mu|D) = \frac{A\exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]\exp\left(-\frac{k_n}{2}\right)}{A\exp\left(-\frac{k_n}{2}\right)\int_D\exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]d\mu} \tag{12}$$

we obtain the posterior distribution as

$$p(\mu|D) = \frac{1}{(2\pi)^{d/2}|\Sigma_n|^{1/2}}\exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]. \tag{13}$$

We used the fact that

$$\int\frac{1}{(2\pi)^{d/2}|\Sigma_n|^{1/2}}\exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]d\mu = 1$$

implies

$$\int \exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]d\mu = (2\pi)^{d/2}|\Sigma_n|^{1/2}$$

From (13) it results that $p(\mu|D)$ is a normal density with mean vector μ_n and covariance matrix Σ_n, i.e.

$$p(\mu|X_1, X_2, \ldots, X_n) \sim N(\mu_n, \Sigma_n).$$

Stage II. Let's determine now, the conditional distribution $p(X|X_1, X_2, \ldots, X_n)$. Using (4) we obtain

$$B = p(X|X_1, X_2, \ldots, X_n) = \int_D p(X|\mu)p(\mu|X_1, X_2, \ldots, X_n)d\mu.$$

Using the formula (13), the previous relation becomes

$$B = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}(2\pi)^{d/2}|\Sigma_n|^{1/2}}\cdot$$

$$\cdot\int_D \exp\left[-\frac{1}{2}(X-\mu)^t\Sigma^{-1}(X-\mu)\right]\exp\left[-\frac{1}{2}(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]d\mu$$

or

$$B = \frac{1}{(2\pi)^{d}|\Sigma|^{1/2}|\Sigma_n|^{1/2}}\cdot\int_D \exp\left\{-\frac{1}{2}\left[(X-\mu)^t\Sigma^{-1}(X-\mu)+(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n)\right]\right\}d\mu \tag{14}$$

Let us consider the expression

$$C = (X-\mu)^t\Sigma^{-1}(X-\mu)+(\mu-\mu_n)^t{\Sigma_n}^{-1}(\mu-\mu_n),$$

which can be developed as follows

$$\begin{aligned}C &= X^t\Sigma^{-1}X - 2\mu^t\Sigma^{-1}X + \mu^t\Sigma^{-1}\mu + \mu^t{\Sigma_n}^{-1}\mu - 2\mu^t{\Sigma_n}^{-1}\mu_n + \mu_n^t{\Sigma_n}^{-1}\mu_n = \\ &= X^t\Sigma^{-1}(\Sigma+\Sigma_n)(\Sigma+\Sigma_n)^{-1}X - 2\mu^t\Sigma^{-1}(\Sigma+\Sigma_n)(\Sigma+\Sigma_n)^{-1}X + \\ &\quad + \mu^t\Sigma^{-1}(\Sigma+\Sigma_n)(\Sigma+\Sigma_n)^{-1}\mu + \mu^t\Sigma_n^{-1}(\Sigma+\Sigma_n)(\Sigma+\Sigma_n)^{-1}\mu - \\ &\quad - 2\mu^t\Sigma_n^{-1}(\Sigma+\Sigma_n)(\Sigma+\Sigma_n)^{-1}\mu_n + \mu_n^t\Sigma_n^{-1}(\Sigma+\Sigma_n)(\Sigma+\Sigma_n)^{-1}\mu_n = \\ &= X^t(\Sigma+\Sigma_n)^{-1}X + X^t\Sigma^{-1}\Sigma_n(\Sigma+\Sigma_n)^{-1}X - 2\mu^t(\Sigma+\Sigma_n)^{-1}X -\end{aligned}$$

$$-2\mu^t\Sigma^{-1}\Sigma_n(\Sigma+\Sigma_n)^{-1}X+\mu^t(\Sigma+\Sigma_n)^{-1}\mu+\mu^t\Sigma^{-1}\Sigma_n(\Sigma+\Sigma_n)^{-1}\mu+$$

$$+\mu^t\Sigma_n^{-1}\Sigma(\Sigma+\Sigma_n)^{-1}\mu+\mu^t(\Sigma+\Sigma_n)^{-1}\mu-2\mu^t\Sigma_n^{-1}\Sigma(\Sigma+\Sigma_n)^{-1}\mu_n-$$

$$-2\mu^t(\Sigma+\Sigma_n)^{-1}\mu_n+\mu_n^t\Sigma_n^{-1}\Sigma(\Sigma+\Sigma_n)^{-1}\mu_n+\mu_n^t(\Sigma+\Sigma_n)^{-1}\mu_n-$$

$$-2\mu_n^t(\Sigma+\Sigma_n)^{-1}X+2\mu_n^t(\Sigma+\Sigma_n)^{-1}X.$$

Thus, finally we have

$$C=(X-\mu_n)^t(\Sigma+\Sigma_n)^{-1}(X-\mu_n)-2\mu^t(\Sigma^{-1}X+\Sigma_n^{-1}\mu_n)+\mu^t(\Sigma^{-1}+\Sigma_n^{-1})\mu+$$

$$+(X^t\Sigma^{-1}+\mu_n^t\Sigma_n^{-1})\Sigma_n(\Sigma+\Sigma_n)^{-1}\Sigma(\Sigma^{-1}X+\Sigma_n^{-1}\mu_n)$$

One obtains

$$p(X|X_1,X_2,\ldots,X_n)=\frac{1}{(2\pi)^{d/2}|\Sigma+\Sigma_n|^{1/2}}\exp\left\{-\frac{1}{2}\left[(X-\mu_n)^t(\Sigma+\Sigma_n)^{-1}(X-\mu_n)\right]\right\}. \tag{15}$$

and hence $p(X|D)$ is normally distributed with mean vector μ_n and covariance matrix $\Sigma+\Sigma_n$, i.e.

$$p(X|D)\sim N(\mu_n,\Sigma+\Sigma_n).$$

3 Conclusions

The Bayesian approach described above involves two stages. The first stage is concerned on the calculation of the posterior density $p(\mu|X_1,X_2,\ldots,X_n)$ for a specified prior with (13). The second stage is the integration over μ to obtain the conditional density $p(X|X_1,X_2,\ldots,X_n)$ which may be viewed as making allowance for the variability in the estimate due to sampling.

In conclusion, as in the univariate case (see [5]), to obtain the class conditional density $p(X|D)$, whose parametric form is known to be $p(X|\mu)\sim N(\mu,\Sigma)$, we have to replace μ by μ_n and Σ by $\Sigma+\Sigma_n$. In effect, the mean vector μ_n is treated as if it were the true mean vector and the known covariance matrix is increased to Σ_n .

The density $p(X|D)$ is the desired class conditional density $p(X|\omega_j)$ which multiplied by the a prior probability $P(\omega_j)$ it gives the probabilistic information needed to design the classifier.

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On Harmonic Function Spaces II

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Abstract. In this paper we investigate several spaces of harmonic functions, such as: a-Bloch, Hardy, the weighted Bergman, Besov and Dirichlet on the open unit ball in $\mathbb{R}^n$.

MSC 2000: 31B05.

Keywords: Harmonic function, Hardy space, Bergman space, Besov space, Hadamard gaps.

1 Introduction and preliminaries

Throughout this paper G is a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 1$, $B(a,r) = \{x \in \mathbb{R}^n \,|\, |x-a| < r\}$ denotes the open ball centered at $a \in \mathbb{R}^n$ of radius $r > 0$, where $|x|$ denotes the norm of $x \in \mathbb{R}^n$ and B is the open unit ball in $\mathbb{R}^n$. Also, $S = \partial B = \{\, x \in \mathbb{R}^n \,|\, |x| = 1\}$ is the boundary of B.

Let dV denote the Lebesgue measure on $\mathbb{R}^n$, v_n the volume of B, $d\sigma$ the surface measure on S, σ_n the surface area of S, dV_N the normalized Lebesgue measure on B, $d\sigma_N$ the normalized surface measure on S, and $V_{a,r} = V(B(a,r))$. Let $\mathcal{H}(B)$ denote the set of complex valued harmonic functions on B.

Hardy type spaces. For $p > 0$, let $\mathcal{H}^p(B)$ denote the set of harmonic functions u on B such that

$$\|u\|_{\mathcal{H}^p(B)} = \sup_{0<r<1} M_p(u,r) = \sup_{0<r<1} \left(\int_S |u(r\zeta)|^p d\sigma_N(\zeta) \right)^{1/p} < +\infty.$$

For $p \geq 1$, $\mathcal{H}^p(B)$ is a Banach space and for $p \in (0,1)$ it is a complete metric space with the following translation invariant metric

$$d(u,v) = \sup_{0<r<1} \int_S |u(r\zeta) - v(\zeta)|^p d\sigma_N(\zeta).$$

Elements of $\mathcal{H}^p(B)$ theory can be found in [3, Chapter VI].

The Hardy space $\mathcal{H}^p(B)$ is contained in the spaces $\mathcal{H}^p_\beta(B)$, consisting of all harmonic functions u on B such that

$$\|u\|_{\mathcal{H}^p_\beta(B)} = \sup_{0<r<1} (1-r)^\beta \left(\int_S |u(r\zeta)|^p d\sigma_N(\zeta) \right)^{1/p} < +\infty,$$

where $\beta \in [0,\infty)$. These spaces we call Hardy type spaces or weighted Hardy spaces.

a-Bloch space. Let $a > 0$. A function $f \in C^1(B)$ is said to be an a-Bloch function if

$$\|f\|_{\mathcal{B}^a} = \sup_{x\in B}(1-|x|)^a|\nabla f(x)| < +\infty.$$

The space of a-Bloch functions is denoted by $\mathcal{B}^a(B) = \mathcal{B}^a$. If $a = 1$, $\mathcal{B}^a$ just becomes the Bloch space $\mathcal{B}$. Note that $\|\cdot\|_{\mathcal{B}^a}$ is a seminorm on $\mathcal{B}^a(B)$, that is, $\mathcal{B}^a(B)$ is a normed space, modulo constant functions. The norm is usually defined by the quantity $|f(0)|+\|f\|_{\mathcal{B}^a}$. With this norm $\mathcal{B}^a(B)$ is a Banach space. Let $\mathcal{H}_{\mathcal{B}^a}(B)$ denote the space which consists of all harmonic a-Bloch functions on the unit ball, i.e., $\mathcal{H}(B)\cap\mathcal{B}^a(B)$. Basic results on Bloch functions can be found, for example, in [2, 14, 18, 20, 32, 42, 44, 46]. In [18] it was proved that for $p \geq 1$, there is a positive constant $c(p,n)$, depending on p and n, such that for every $u \in \mathcal{H}(B)$

$$\frac{1}{c(p,n)}\|u\|_{BMO_p} \leq \|u\|_{\mathcal{H},n} := \sup_{x\in B}\frac{1}{2}(1-|x|^2)|\nabla u(x)| \leq c(p,n)\|u\|_{BMO_p}.$$

In the case $n = 2$, this result was essentially obtained in [8]. For some information on BMO space, see, for example [4]. In [29, Theorems 2 and 3] we proved that Muramoto's result is true also for $p \in (0,1)$. This Muramoto paper inspired us to calculate exactly the BMO_p norm for harmonic functions (see [29]) where we essentially proved a generalization of the Hardy-Stein identity (see, for example, [10, p.42]). Further applications of this identity can be found in [31] and [37], where among others some results for analytic functions on the unit disk in [45] and [47] are extended.

Square area space. In the main result of [37] a space of functions on B, consisting of all $u \in \mathcal{H}(B)$ such that

$$\int_0^1(1-r)^\gamma\mathcal{A}^{p/2}(r,u)dr < \infty,$$

where $\gamma \in (-1,\infty)$, $p > 0$ and

$$\mathcal{A}(r,u) = \int_{rB}|\nabla u(x)|^2 dV(x)$$

is introduced. We will denote this space by $\mathcal{S}_{p,\gamma}(B)$. It can be considered as a natural generalization of a space of analytic functions on the unit disk which appears in [13]. When $p \geq 1$ it is easy to see that

$$\|u\|_{\mathcal{S}_{p,\gamma}} = |u(0)| + \left(\int_0^1(1-r)^\gamma\mathcal{A}^{p/2}(r,u)dr\right)^{1/p},$$

defines a norm on $\mathcal{S}_{p,\gamma}(B)$ and that with this norm it is a Banach space. For $p \in (0,1)$, $\mathcal{S}_{p,\gamma}(B)$ becomes a complete metric space with the following metric

$$d_{\mathcal{S}_{p,\gamma}}(u,v) = |u(0)-v(0)|^p + \int_0^1(1-r)^\gamma\mathcal{A}^{p/2}(r,u-v)dr.$$

Weighted Bergman space. Let $\omega(r)$, $0<r<1$, be a positive weight function which is integrable on $(0,1)$. We extend ω on B by setting $\omega(x)=\omega(|x|)$.

For $0<p<\infty$ the weighted Bergman space $b^p_\omega(B)$ is the space of all harmonic functions u on B such that

$$\|u\|_{b^p_\omega}=\left(\int_B |u(x)|^p\omega(x)dV(x)\right)^{1/p}<+\infty.$$

If $\omega(r)=(1-r)^\alpha, \alpha>-1$, we denote the norm by $\|u\|_{b^p_\alpha}$ and the corresponding space by $b^p_\alpha(B)$. When $p\geq 1$, $b^p_\omega(B)$ is a Banach space with the norm $\|\cdot\|_{b^p_\omega}$ and when $p\in(0,1)$ a complete metric space with the metric defined by

$$d_b(u,v)=\int_B |(u-v)(x)|^p\omega(x)dV(x).$$

Recently there has been a great interest in studying the weighted Bergman type spaces of analytic or harmonic functions, see, for example, [1], [7], [16], [21], [22], [26], [27], [30], [33], [34], [38], [39] and the references therein.

Dirichlet type space. For $\alpha\in(-1,\infty)$ let $\mathcal{D}^p_\alpha(B)=\mathcal{D}^p_\alpha$ be the class of all harmonic functions u on the unit ball obeying

$$\|u\|^p_{\mathcal{D}^p_\alpha}=|u(0)|^p+\int_B |\nabla u(x)|^p(1-|x|)^\alpha dV(x)<\infty.$$

For $p=2$ and $\alpha=0$ we obtain the classical Dirichlet space.

Harmonic Besov space. The Harmonic Besov space $\mathcal{B}_p(B)=\mathcal{B}_p$ is the space of all $u\in\mathcal{H}(B)$ such that

$$\int_B (1-|x|^2)^p|\nabla u(x)|^p d\tau(x)<\infty,$$

where

$$d\tau(x)=\frac{dV(x)}{(1-|x|^2)^n}.$$

For $p>1$, the Besov space with the following norm

$$\|u\|_{\mathcal{B}_p}=|u(0)|+\left(\int_B (1-|x|^2)^p|\nabla u(x)|^p d\tau(x)\right)^{1/p},$$

becomes a Banach space.

We say that a locally integrable function f on B possesses the $HL-$property, with a constant $c>0$ if

$$f(a)\leq\frac{c}{r^n}\int_{B(a,r)} f(x)dV(x)\quad\text{whenever } \overline{B}(a,r)\subset B.$$

Every subharmonic function ([11]) possesses the $HL-$property when $c=1/v_n$. In [9] Hardy and Littlewood proved that $|u|^p$, $p>0$, $n=2$, also possesses the $HL-$property whenever u is a harmonic function in B. In the case $n\geq 3$ a

generalization was made by Fefferman and Stein [5]. In this paper we need the following generalization of the Fefferman and Stein result, see [23].

Lemma A. *Let f be a non-negative subharmonic function on a proper open domain $G \subset \mathbb{R}^n$ and $p > 0$. Then there is a constant C depending only on n and p, such that*

$$f^p(x) \le \frac{C}{r^n} \int_{B(x,r)} f^p(y) dV(y),$$

where $0 < r < d(x, \partial G)$ (the distance from the point x to the boundary of G).

The following lemma could be folklore. We omit its proof.

Lemma B. *The following statements are true.*

(a) If $u \in \mathcal{H}_{\mathcal{B}^a}(B)$, then there is a positive constant C independent of u and x, such that

$$|u(x)| \le \begin{cases} C\|u\|_{\mathcal{B}^a} & , \quad 0 \le a < 1, \\ C\|u\|_{\mathcal{B}^a} \ln \frac{2}{1-|x|} & , \quad a = 1, \\ C(1-|x|)^{1-\alpha}\|u\|_{\mathcal{B}^a} & , \quad a > 1. \end{cases}$$

(b) If $u \in \mathcal{H}^p_\beta(B)$, $p, \beta \in (0, \infty)$, then there is a positive constant C independent of u and x, such that

$$|u(x)| \le C \frac{\|u\|_{\mathcal{H}^p_\beta}}{(1-|x|)^{\frac{n+p\beta-1}{p}}}.$$

This paper can be considered as a continuation of our investigations devoted to harmonic functions on the unit ball, see [1, 29, 31, 33, 35, 37, 40, 41, 42].

The paper is organized as follows. Section 2 is devoted to the study of some relationships among the functions which belong to the above mentioned spaces. Some new equivalent conditions for harmonic Bergman functions are presented in Section 3. Equivalent conditions for harmonic Besov spaces are presented in Section 4. In Section 5 we initiate the study of harmonic functions with Hadamard gaps.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2 Some relationships among spaces of harmonic functions

In this section we investigate some relationships among the spaces of harmonic functions mentioned in Section 1.

2.1 Fejér-Riesz type inequality

In this subsection we will extend our Theorem 2 in [37], which is an inequality of Riesz-Fejér type, see [6, 11, 19].

Theorem 1. *Suppose that* $u \in \mathcal{H}(B)$ *and* $\varepsilon > 0$. *Then*

$$\int_0^r (r-\rho)^{n-2+\varepsilon} M(\rho)^p d\rho \le c_{p,n,\varepsilon}\, r^{n-1+\varepsilon} \sup_{0<\rho<r} M_p^p(u,\rho), \quad p>0, \tag{1}$$

for some $c_{p,n,\varepsilon} > 0$, *which depends only on* p, n *and* ε, *and all* $r \in (0,1)$, *where*

$$M(r) = M(r,u) = \sup\{\, |u(x)| \mid |x| = r \,\}.$$

Proof. Without loss of generality we may assume that $r = 1$. Applying Lemma B *(b)* with $\beta = 0$, we have that

$$(1-|x|)^{n-1} |u(x)|^p \le C \|u\|^p_{\mathcal{H}^p},$$

which implies that

$$(1-\rho)^{n-1} M^p(\rho) \le C \|u\|^p_{\mathcal{H}^p}.$$

Multiplying the last formula by $(1-\rho)^{-1+\varepsilon}$ and then integrating from 0 to 1, we obtain inequality (1), as desired.

Remark 1. Note that Theorem 1 extends our result from [37] for the case $p \in (0,1)$, and that in the proof of Theorem 2 in [37] we have used the Poisson integral formula

$$u(x) = \int_{\partial B} \frac{1-|x|^2}{|x-\zeta|^n} u(\zeta) d\sigma_N(\zeta), \quad x \in B$$

and Jensen's and Harnack's inequalities, to obtain

$$|u(x)|^p \le \frac{2\|u\|^p_{\mathcal{H}^p(B)}}{(1-|x|)^{n-1}},$$

however Jensen's inequality holds only for $p \ge 1$.

Remark 2. We do not know, at the moment, whether or not Theorem 1 holds for $\varepsilon = 0$. However, it is interesting that for the case of analytic functions on the unit disk the result also holds for this case.

2.2 Inclusion theorems

In this section we prove several inclusion results regarding spaces of harmonic functions. Some of them could be known but we cannot find any specific references in the literature. Before formulating the main result of this section we quote the following known result, see, for example, [15].

Lemma C. *Suppose* $0 \le p < \infty$ *and* $\alpha > -1$. *Then there is a constant* $C = C(p, \omega)$ *such that*

$$|u(0)|^p + \int_B |\nabla u(x)|^p (1-|x|)^{p+\alpha} dV(x) \asymp \int_B |u(x)|^p (1-|x|)^\alpha dV(x),$$

for all $u \in \mathcal{H}(B)$.

Now we formulate and prove the main result of this section.

Theorem 2. *The following statements hold.*

(a) *If* $a \neq 1$ *and* $\alpha > -1$, *then* $\mathcal{B}^a \subset b^p_\alpha$ *if* $a < 1 + \frac{1+\alpha}{p}$.

(b) *If* $a = 1$, *then* $\mathcal{B}^a \subset b^p_\alpha$ *for every* $p \in (0, \infty)$ *and* $\alpha > -1$.

(c) $b^p_\alpha \subset \mathcal{B}^{\frac{n+\alpha}{p}+1}$.

(d) *If* $a \in (0,1)$, *then* $\mathcal{B}^a \subset \mathcal{H}^p_\beta$, *for every* $p > 0$ *and* $\beta \in [0, \infty)$.

(e) $\mathcal{H}^p_\beta \subset \mathcal{B}^{\frac{n-1}{p}+\beta+1}$, $p \in (0, \infty)$.

(f) *If* $p > 2$, $\beta \ge 0$, *and* $\beta_1 = 2p\beta/(2-p)$, *then* $\mathcal{S}_{p,(n-2+\beta_1+\varepsilon)(2-p)/2} \subset \mathcal{H}^p_\beta$, *for every* $\varepsilon > 0$.

(g) *If* $a > p + 1$ *then* $\mathcal{D}^p_\alpha = b^p_{\alpha - p}$.

Proof. *(a)* By the proof of Lemma B *(a)*, we have that

$$|f(x)| \le |f(0)| + \frac{\|f\|_{\mathcal{B}^a}}{a-1} \frac{1}{(1-|x|)^{a-1}}.$$

From this and a well-known inequality, we obtain

$$|f(x)|^p \le c_p \left(|f(0)|^p + \frac{\|f\|^p_{\mathcal{B}^a}}{(a-1)^p} \frac{1}{(1-|x|)^{p(a-1)}} \right),$$

and consequently

$$\int_B |f(x)|^p (1-|x|)^\alpha dV(x) \le C \left(|f(0)|^p + \left(\frac{\|f\|_{\mathcal{B}^a}}{a-1} \right)^p \int_B (1-|x|)^{\alpha - p(a-1)} dV(x) \right).$$

The last integral converges if and only if $\alpha + 1 > p(a-1)$, from which the result follows.

(b) By Lemma B *(a)* case $a = 1$ and a known inequality, it follows that

$$\int_B |f(x)|^p (1-|x|)^\alpha dV(x) \le c_p \left(|f(0)|^p + \|f\|^p_{\mathcal{B}^1} \int_B (1-|x|)^\alpha \ln^p \frac{1}{1-|x|} dV(x) \right).$$

By polar coordinates, and the change of variables $r = 1 - \rho$, it is easy to see that the last integral is equiconvergent to the following integral

$$\int_0^{1/2} t^\alpha \ln^p \frac{1}{t} dt.$$

Since this integral converges for every $p \in (0, \infty)$, and $\alpha > -1$, it follows that $\mathcal{B}^1(B) \subset b^p_\alpha(B)$, for every $p \in (0, \infty)$, and $\alpha > -1$.

(c) By HL-property of the function $|\nabla u|^p$, $p > 0$, we have that

$$|\nabla u(z)|^p \leq \frac{C}{(1-|x|)^n} \int_{B(x,(1-|x|)/2)} |\nabla u(y)|^p dV(y).$$

For $y \in B(x, (1 - |x|)/2)$, we have $\frac{1}{2}(1 - |x|) < (1 - |y|) < \frac{3}{2}(1 - |x|)$. Thus

$$|\nabla u(x)|^p \leq \frac{C}{(1-|x|)^{n+\alpha+p}} \int_{B(x,(1-|x|)/2)} |\nabla u(y)|^p (1-|y|)^{\alpha+p} dV(y).$$

From this and by Lemma C, it follows that

$$\begin{aligned}
(1-|x|)^{\frac{n+\alpha}{p}+1} |\nabla u(x)| &\leq C \left(\int_B |\nabla u(y)|^p (1-|y|)^{\alpha+p} dV(y) \right)^{1/p}, \\
&\leq C_1 \left(\int_B |u(y)|^p (1-|y|)^{\alpha} dV(y) \right)^{1/p}.
\end{aligned}$$

Hence $b^p_\alpha \subset \mathcal{B}^{\frac{n+\alpha}{p}+1}$, from which the result follows.

(d) Assume that $u \in \mathcal{B}^a$ and $a \in (0, 1)$. Then by Lemma 6.4.8 in [24], $u \in Lip_{1-a}(B)$, which implies that $u \in Lip_{1-a}(\overline{B})$.

We have

$$\begin{aligned}
|u(r_1\zeta) - u(r_2\zeta)| &= \left| \int_{r_1}^{r_2} \langle \nabla u(t\zeta), \zeta \rangle dt \right| \leq \int_{r_1}^{r_2} |\nabla u(t\zeta)| dt \\
&\leq \|u\|_{\mathcal{B}^a} \int_{r_1}^{r_2} \frac{1}{(1-|t\zeta|)^a} dt \\
&= \|u\|_{\mathcal{B}^a} \frac{1}{1-a} \left((1-r_1)^{1-a} - (1-r_2)^{1-a} \right) \to 0,
\end{aligned}$$

uniformly on $\zeta \in S$. Since the functions $u_r(\zeta) = u(r\zeta)$ are continuous on S it follows that the limit function $u(\zeta)$ is also continuous on S.

Of course, every continuous function on $\overline{B}$, belongs to $\mathcal{H}^p_\beta(B)$ for every $p \in (0, \infty)$, which is what we need to prove.

(e) By Lemma B *(b)* it follows that there is a positive constant C such that

$$|u(x)| \leq \frac{C}{(1-|x|)^{\frac{n-1}{p}}} \|u\|_{\mathcal{H}^p} \tag{2}$$

for every $u \in \mathcal{H}^p(B)$.

On the other hand, by the Cauchy's estimate we have that

$$|\nabla u(x)| \le \frac{C}{1-|x|} \sup_{y\in B(x,(1-|x|)/2)} |u(y)|. \tag{3}$$

Applying (2) to the harmonic function $u\big(\frac{(1+|x|)}{2}y\big)$, (3) and the fact that

$$1-|x| \asymp 1-|y| \quad \text{if} \quad y \in B(x,(1-|x|)/2),$$

it follows that there is a positive constant C independent of the function u such that

$$(1-|x|)^{\frac{n-1}{p}+\beta+1}|\nabla u(x)| \le C \sup_{y\in B}\left(1-\frac{1+|y|}{2}\right)^{\beta} M_p\left(u,\frac{1+|y|}{2}\right),$$

from which the result follows.

(f) This has been proved in [37, Theorem 2], case $p > 2$.

(g) This is a direct consequence of Lemma C.

From Theorem 2 we obtain the following corollary.

Corollary 1. *If $a \in (0,1)$. Then, $\mathcal{B}^a \subset b^p_\alpha$.*

Proof. Note that when $a \in (0,1)$, then $a < 1 + \frac{1+\alpha}{p}$.

Remark 3. Note that statements *(a)* and *(b)* in Theorem 2 hold for every $f \in C^{(1)}(B)$, that is, we have not used the harmonicity of function f in the proofs of these statements.

2.3 A little on the $\mathcal{H}^p_0(B)$ space

By $\mathcal{H}^p_0(B)$, we denote the subset of the Hardy space $\mathcal{H}^p(B)$ consisting of all harmonic Hardy functions u which satisfy the following condition

$$\lim_{r\to 1}\int_S |u(r\zeta)|^p d\sigma_N(\zeta) = 0.$$

If $p \ge 1$, then in view of the monotonicity of the integral means $M_p(u,r)$, it follows that $\mathcal{H}^p_0(B)$ is trivial, that is $\mathcal{H}^p_0(B) = \{0\}$. In view of this fact, a somewhat surprising fact is that $\mathcal{H}^p_0(B)$ is nontrivial when $p \in (0,1)$ (for the case $n = 1$, see, for example, [25]). We now quote some functions which belong to the space. Before that we cite a well known fundamental integral estimate on the unit ball B.

Lemma D. *Let $p > \frac{n-1}{n}$, $\zeta \in S$ and $I_p = \int_S \frac{d\sigma(\zeta)}{|x-\zeta|^{np}}$, then*

$$I_p < c_{p,n}(1-r)^{n-1-np}, \quad 0 \le r < 1 \tag{4}$$

for some $c_{p,n} > 0$, *where* $x = r\eta$, $\eta \in S$.

It is interesting that Lemma D can be proved by a somewhat forgotten Zonal region method (see [12]). Let $\gamma = \angle(\eta, \zeta)$, then by a simple calculation we have

$$I_p = \int_S \frac{d\sigma(\zeta)}{(1 - 2\langle x, \zeta\rangle + |x|^2)^{np/2}} = \int_S \frac{d\sigma(\zeta)}{(1 - 2r\ \cos\gamma + r^2)^{np/2}}.$$

Let $F(\gamma) = \int_{c(x,\gamma)} d\sigma(\zeta)$, where $c(x,\gamma)$ is the polar cap centered at x and with half angle γ (see, [12]). Then I_p can be written, in the following form

$$I_p = \int_0^\pi \frac{dF(\gamma)}{(1 - 2r\ \cos\gamma + r^2)^{np/2}}.$$

Employing integration by parts we obtain

$$I_p = \frac{F(\pi)}{(1+r)^{np}} + npr \int_0^\pi \frac{\sin\gamma\, F(\gamma) d\gamma}{(1 - 2r\ \cos\gamma + r^2)^{np/2+1}},$$

since $F(0) = \sigma(c(x,0)) = \sigma(\{x\}) = 0$. Note that $F(\pi) = \int_{c(x,\pi)} d\sigma(\zeta) = \int_S d\sigma(\zeta) = \sigma_n$. It is enough only to estimate the following integral

$$\int_0^\delta \frac{\sin\gamma\, F(\gamma) d\gamma}{(1 - 2r\ \cos\gamma + r^2)^{np/2+1}}.$$

Since

$$F(\gamma) = \sigma(c(x,\gamma)) = \sigma_{n-1} \int_0^\gamma \sin^{n-2}\theta\, d\theta,$$

we have $F(\gamma) \le \sigma_{n-1}\gamma^{n-1}/(n-1)$, by the well-known inequality $\sin\theta \le \theta$, $\theta \ge 0$. Thus, we have that

$$\int_0^\delta \frac{\sin\gamma\, F(\gamma) d\gamma}{(1 - 2r\ \cos\gamma + r^2)^{np/2+1}} \le \int_0^\delta \frac{\sigma_{n-1}}{n-1} \frac{\gamma^n\ d\gamma}{\left((1-r)^2 + \frac{4r}{\pi^2}\gamma^2\right)^{np/2+1}}. \tag{5}$$

Using the substitution $\frac{2}{\pi}\sqrt{r}\gamma = (1-r)\varphi$ we obtain that the last integral is less than

$$\frac{c_n}{r^{\frac{n+1}{2}}} \int_0^{\frac{2\sqrt{r}\delta}{\pi(1-r)}} \frac{(1-r)^{n+1}\varphi^n d\varphi}{(1-r)^{np+2}(1+\varphi^2)^{np/2+1}} \le \frac{C}{r^{\frac{n+1}{2}}}(1-r)^{n-1-np},$$

since for $p > (n-1)/n$ the last integral converges. Thus for $r \in [1/4, 1)$ we have the desired estimates. From this and since the integrand is bounded away from zero for $r \in [0, 1/4]$, the estimate follows.

Note that by using the same method, the following (local) estimate can be proved.

Lemma D a). *Let* $0 < p < \frac{n-1}{n}$ *and* $I_p(\delta) = \int_{c(\zeta,\delta)} \frac{d\sigma(\zeta)}{|x-\zeta|^{np}}$, $\delta \in (0, \pi]$, *then*

$$I_p(\delta) < c_{p,n}\delta^{n-1-np}, \quad 0 \le r < 1 \tag{6}$$

for some $c_{p,n} > 0$.

It may be noted that

$$I_p(\delta) = \int_0^\delta \frac{dF(\gamma)}{(1-2r\ \cos\gamma + r^2)^{np/2}},$$

where $\gamma = \angle(\eta, \zeta)$, and that by integration by parts it follows that

$$I_p(\delta) = \frac{F(\delta)}{(1+r)^{np}} + npr\int_0^\delta \frac{\sin\gamma\, F(\gamma)d\gamma}{(1-2r\ \cos\gamma + r^2)^{np/2+1}}.$$

Remark 4. By Lemma D can be proved that for $p \in (0,1)$ there exists a nonconstant function $u \in \mathcal{H}_0^p(B)$. Indeed, if $u(x) = P(x,\zeta)$ is a Poisson kernel, then by Lemma D we have that $\lim_{r\to 1-0}\int_S P(r\eta,\zeta)^p d\sigma(\eta) = 0$ for $p \in (\,(n-1)/n, 1)$. If $p \in (0,(n-1)/n]$ the result easily follows by Jensen's inequality. Lemma D also can be used in proving the following known result ([3, p. 167]), that a Poisson kernel $P(\cdot\,,\zeta)$ belongs to $b^p(B)$ for $p < n/(n-1)$, as well as that for $p < (n+\alpha)/(n-1)$, $\alpha > -1$, there exists a positive harmonic function u on B belonging to $b^p_\alpha(B)$.

The next result shows that $\mathcal{H}_0^p(B)$, $p \in (0,1)$ contains a large class of functions (this is a natural generalization of Theorem 1 in [25]).

Theorem 3. *Let* μ *be a complex Borel measure on* S, *singular with respect to Lebesgue measure, then* $P[\mu](x) \in \mathcal{H}_0^p(B)$, *for* $0 < p < 1$.

Proof. Let $u_r(\zeta) = u(r\zeta) = P[\mu](r\zeta)$, $r \in (0,1)$. This family of functions belong to $\mathcal{L}(S)$ and by a well-known theorem (see [12]) $\lim\limits_{r\to 1-0} u(r\zeta) = 0$ a.e. $\zeta \in S$. By Jensen's inequality we obtain

$$\left(\frac{1}{\sigma(E)}\int_E |u_r(\eta)|^p d\sigma(\eta)\right)^{1/p} \le \frac{1}{\sigma(E)}\int_E |u_r(\eta)| d\sigma(\eta)$$

for every $E \subset S$ and $p \in (0,1)$.

From this we have

$$\int_E |u_r(\eta)|^p d\sigma(\eta) \le \sigma(E)^{1-p}\left(\int_S |u_r(\eta)| d\sigma(\eta)\right)^p \le \sigma(E)^{1-p}\|u\|^p_{\mathcal{H}^1}.$$

Since $u = P[\mu]$ we have $u \in \mathcal{H}^1(B)$ ([3]) i.e. $\|u\|_{\mathcal{H}^1} < +\infty$. By this estimate we obtain

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall E \subset S)\left(\sigma(E) < \delta \Rightarrow \int_E |u_r(\eta)|^p d\sigma(\eta) < \varepsilon\right)$$

i.e. the family of functions $\{|u_r|^p,\ r \in (0,1)\}$ is uniformly integrable. By Vitali's theorem, we have that

$$\lim_{r\to 1-0}\int_S |u(r\eta)|^p d\sigma(\eta) = \int_S \lim_{r\to 1-0} |u(r\eta)|^p d\sigma(\eta) = 0,$$

finishing the proof of the theorem.

3 Equivalent conditions for the Bergman harmonic space

In [30], [33], [34], [38] and [39], we have investigated relationships among various type of integrals on the Bergman space on the unit disk, unit ball and unit polydisc. In talk [36] we posed several open problems and conjectures concerning this topic. Among others we posed the following conjecture.

Conjecture 1. *Let $p > 1$, $q \in [0,p]$, $\alpha \in (-1,\infty)$, and $u \in \mathcal{H}(B)$. Show that*

$$\int_B |u(x)|^p (1-|x|)^\alpha dV(x) \asymp |u(0)|^p + \int_B |u(x)|^{p-q} |\nabla u(x)|^q (1-|x|)^{\alpha+q} dV(x). \quad (7)$$

The above means that there are finite positive constants C and C' independent of u such that the left and right hand sides $L(u)$ and $R(u)$ satisfy

$$CR(u) \le L(u) \le C'R(u)$$

for every harmonic function u.

Remark 5. Note that for $q = 0$ the relationship (7) is obvious, and if $p = q$ then the result is contained in Lemma C.

In [41] we partially confirmed Conjecture 1, by proving the following result:

Theorem B. *Suppose that $u \in \mathcal{H}(B)$, $u(0) = 0$, $p \ge 2$ and $\alpha > 1$. Then the following quantities*

$$\int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1-|x|)^\alpha dV_N(x),$$

and

$$\int_B |u(x)|^{p-1} |\nabla u(x)| (1-|x|)^{\alpha-1} dV_N(x)$$

are equivalent.

If $\alpha > 2$, then these quantities are equiconvergent with

$$\int_B |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x).$$

Corollary 2. *Let $1 < p < \infty, \alpha > -1$ and $u \in \mathcal{H}(B)$, then the following relationship hold*

$$\begin{aligned} \|u\|_{p,\alpha}^p &\asymp |u(0)|^p + \int_0^1 M_p^{p-1}(u,r)M_p(\nabla u,r)(1-r)^{\alpha+1}dr \\ &\asymp |u(0)|^p + \int_0^1 M_p^{p-2}(u,r)M_p^2(\nabla u,r)(1-r)^{\alpha+2}dr. \end{aligned}$$

Proof. Assume that $q \in \{1, 2\}$. From the proof of Theorem B, we obtain that

$$\begin{aligned} \|u\|_{p,\alpha}^p &\le C\left(|u(0)|^p + \int_B |u(x)|^{p-q}|\nabla u(x)|^q(1-|x|)^{\alpha+1}dV(x)\right) \\ &\le C\left(|u(0)|^p + \int_0^1 M_p^{p-q}(u,r)M_p^q(\nabla u,r)(1-r)^{\alpha+1}dr\right). \end{aligned}$$

The reverse inequality, follows by applying Hölder's inequality with exponents $p/(p-q)$ and p/q to the integral

$$\int_0^1 M_p^{p-q}(u,r)M_p^q(\nabla u,r)(1-r)^{\alpha+q}dr,$$

and by Lemma C for $p = q$.

In this section we will extend Theorem B. Namely, we prove the following result.

Theorem 4. *Let G be a domain in $\mathbb{R}^n$, $u \in \mathcal{H}(G)$, $p \ge 2$, and $d(x) = dist(x, \partial G)$ be the distance from x to the boundary of G. Then*

$$\begin{aligned} \int_G |u(x)|^p d(x)^\alpha dV(x) &\asymp \int_G |\nabla u(x)|^p d(x)^{\alpha+p}dV(x) \\ &\asymp \int_G |\nabla u(x)|^2|u(x)|^{p-2}d(x)^{\alpha+2}dV(x) \\ &\asymp \int_G |\nabla u(x)||u(x)|^{p-1}d(x)^{\alpha+1}dV(x), \end{aligned}$$

if all of these integrals are convergent.

In order to prove Theorem 4 we need three auxiliary results, which are incorporated in lemmas which follows. The first one is a simple consequence of Fubini's theorem.

Lemma 1. *Let $f \in \mathcal{L}^1(G)$ and $\alpha \in \mathbb{R}$. Then*

$$\int_G d(x)^\alpha |f(x)| dV(x) \asymp \int_G d(w)^{\alpha-n} \int_{B(x,d(x)/2)} |f(x)| dV(x) dV(w).$$

The following lemma was proved in [41, Theorems 1 and 2].

Lemma 2. *Let $u \in \mathcal{H}(B)$. Then the following statements are true:*

(a) If $p > 1$ and $\alpha > 0$, then

$$\begin{aligned} &\int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1-|x|)^\alpha dV_N(x) \\ \leq\ & C \int_B |u(x)|^{p-1} |\nabla u(x)| (1-|x|)^{\alpha-1} dV_N(x), \end{aligned} \tag{8}$$

for some positive constant independent of u.

(b) If $2 \leq p < \infty$ and $\alpha > 1$, then

$$\int_B |u(x)|^{p-1} |\nabla u(x)| (1-|x|)^{\alpha-1} dV_N(x) \leq C \int_B |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x), \tag{9}$$

for some positive constant independent of u.

Lemma 3. *Let $u \in \mathcal{H}(\overline{B(0,r)})$, $r > 0$ and $p \geq 2$. Then*

$$\begin{aligned} r^p \int_{B_{r/8}} |\nabla u(x)|^p dV(x) &\leq Cr^2 \int_{B_{r/4}} |u(x)|^{p-2} |\nabla u(x)|^2 dV(x) \\ &\leq Cr \int_{B_{r/2}} |u(x)|^{p-1} |\nabla u(x)| dV(x) \\ &\leq C \int_{B_r} |u(x)|^p dV(x). \end{aligned}$$

Proof. Applying Lemma 2 *(a)*, with $\alpha = 1$, to the harmonic function $u_{r/2}(x) = u\big(\frac{r}{2}x\big)$ on the unit ball, and then using the change of variable $rx/2 \to x$, we obtain that

$$\begin{aligned} C \int_{B_{r/2}} |u(x)|^{p-1} |\nabla u(x)| dV_N(x) &\geq \frac{r}{2} \int_{B_{r/2}} |u(x)|^{p-2} |\nabla u(x)|^2 \Big(1 - \frac{2|x|}{r}\Big) dV_N(x) \\ &\geq \frac{r}{4} \int_{B_{r/4}} |u(x)|^{p-2} |\nabla u(x)|^2 dV_N(x), \end{aligned}$$

which implies the second inequality.

By using Lemma 2 *(b)*, with $\alpha = 2$, to the harmonic function $u_r(x) = u(rx)$ on the unit ball, and then using the change of variable $rx \to x$, we obtain that

$$\begin{aligned} C\int_{B_r} |u(x)|^p dV_N(x) &\geq r\int_{B_{r/2}} |u(x)|^{p-1}|\nabla u(x)|\Big(1-\frac{|x|}{r}\Big)dV_N(x) \\ &\geq \frac{r}{2}\int_{B_{r/2}} |u(x)|^{p-1}|\nabla u(x)|dV_N(x), \end{aligned}$$

which implies the third inequality.

The first inequality was proved in [43, Lemma 3].

3.1 Proof of Theorem 4

We are now in a position to prove Theorem 4.

Proof of Theorem 4. By Lemma 1 we have that

$$\int_G |\nabla u(x)|^p d(x)^{\alpha+p} dV(x) \asymp \int_G d(x)^{\alpha+p-n} \int_{B(w,d(w)/8)} |\nabla u(x)|^p dV(x) dV(w). \quad (10)$$

On the other hand, applying Lemma 3 to the harmonic function $u(y) = u(w+x)$, we have that

$$\begin{aligned} \int_{B(w,d(w)/8)} |\nabla u(y)|^p dV(y) &= \int_{B(0,d(w)/8)} |\nabla u(x)|^p dV(x) \\ &\leq Cd(w)^{2-p}\int_{B(0,d(w)/4)} |u(x)|^{p-2}|\nabla u(x)|^2 dV(x) \\ &= Cd(w)^{2-p}\int_{B(w,d(w)/4)} |u(y)|^{p-2}|\nabla u(y)|^2 dV(y). \end{aligned}$$

Using this inequality in (10), and then Fubini's theorem, it follows that

$$\begin{aligned} &\int_G |\nabla u(x)|^p d(x)^{\alpha+p} dV(x) \\ \leq\ & C\int_G d(x)^{\alpha+2-n}\int_{B(w,d(w)/4)} |u(y)|^{p-2}|\nabla u(y)|^2 dV(y)dV(w) \\ \leq\ & C\int_G d(x)^{\alpha+2}|u(x)|^{p-2}|\nabla u(x)|^2 dV(x). \end{aligned} \quad (11)$$

By Lemmas 1 and 3, similar to the just proven inequality, it can be obtained that the following inequalities hold:

$$\int_G |u(x)|^{p-2}|\nabla u(x)|^2 d(x)^{\alpha+2} dV(x) \leq C\int_G |u(x)|^{p-1}|\nabla u(x)| d(x)^{\alpha+1} dV(x) \quad (12)$$

and

$$\int_G |u(x)|^{p-1}|\nabla u(x)|d(x)^{\alpha+1}dV(x) \leq C\int_G |u(x)|^p dV(x). \tag{13}$$

The inequality

$$\int_G |u(x)|^p dV(x) \leq C\int_G |\nabla u(x)|^p d(x)^{\alpha+p}dV(x), \tag{14}$$

follows from the main result in [15]. From inequalities (11)-(14), the result follows.

3.2 A result concerning Conjecture 1

In this subsection we partially confirm Conjecture 1, for all positive values p, q such that $p \geq q$. We prove the following result.

Theorem 5. *Let $p > 1$, $q \in [0,p]$, $\alpha \in (-1,\infty)$, and $u \in \mathcal{H}(B)$ such that the functions $|u(y)|^{p-q}|\frac{\partial u}{\partial x_k}(y)|^q$, $k \in \{1,\dots,n\}$ are subharmonic. Then*

$$\int_B |u(x)|^p(1-|x|)^\alpha dV(x) \asymp |u(0)|^p + \int_B |u(x)|^{p-q}|\nabla u(x)|^q(1-|x|)^{\alpha+q}dV(x). \tag{15}$$

The proof of Theorem 5 follows the lines of the proof of Theorem 1 in [1], hence we omit some details.

Before proving Theorem 5 we need some auxiliary results. The following lemma was proved in [33].

Lemma E. *Suppose $0 < p < \infty$ and $u \in \mathcal{H}(B)$. Then*

$$\left|\frac{d}{dr}(|u(x)|^p)\right| \leq p|u(x)|^{p-1}|\nabla u(x)|, \tag{16}$$

for almost every $x = r\zeta \in B$.

The following lemma can be proved by Lemma E similar to Lemma 2 in [1].

Lemma 4. *Suppose $1 < q \leq p < \infty$ and $\alpha > -1$. Then, there is a constant $C = C(p,q,\alpha,n)$ such that*

$$M_\infty^p(u,1/2) \leq C\left(|u(0)|^p + \int_B |u(x)|^{p-q}|\nabla u(x)|^q(1-|x|)^{q+\alpha}dV(x)\right),$$

for all $u \in \mathcal{H}(B)$.

Lemma 5. *Suppose $0 < p < \infty$, $q \in [0,p]$, $0 \leq r < 1$ and a function u satisfies the conditions in Theorem 5. Then there is a constant C independent of u and r such that*

$$\int_S \sup_{0\le\tau<1} |u(\tau r\zeta)|^{p-q}|\nabla u(\tau r\zeta)|^q d\sigma(\zeta) \le C\int_S |u(r\zeta)|^{p-q}|\nabla u(r\zeta)|^q d\sigma(\zeta)$$

for all $u \in \mathcal{H}(B)$.

Proof. By [28, p.165] there is a positive constant C independent of the nonnegative subharmonic function u on the unit ball $B \subset \mathbb{R}^m$ such that

$$\int_S \sup_{0\le\tau<1} u(\tau r\zeta)\, d\sigma(\zeta) \le C\int_S u(r\zeta)\, d\sigma(\zeta)$$

for every $r \in (0,1)$. From this and employing the fact that the functions $|u(y)|^{p-q}|\frac{\partial u}{\partial x_k}(y)|^q$, $k \in \{1,\dots,n\}$ are subharmonic, we can easily obtain the result. $\Box$

By using Lemmas 4 and 5, some calculations and the fact that there is a positive constant C such that

$$\int_B |u(x)|^{p-q}|\nabla u(x)|^q(1-|x|)^{\alpha+q}dV(x) \le C\int_B |u(x)|^p(1-|x|)^\alpha dV(x)$$

(see [33], with $\omega(x) = (1-|x|)^\alpha$), Theorem 5 can be proved similar to Theorem 1 in [1].

4 Characterization of the Besov space

In this section, we give some characterizations of the harmonic Besov space. The following result is our main result in this section.

Theorem 6. *Assume that* $u \in \mathcal{H}(B)$, $r > 0$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$ *and* $2(n-1) < p < \infty$. *Then the following statements are equivalent:*

(a) $u \in B_p$;

(b)

$$\int_B \Big(\sup_{y\in B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|}{|x-y|}(1-|x|^2)^\alpha(1-|y|^2)^\beta\Big)^p d\tau(x) < \infty;$$

(c)

$$\int_B \Big(\frac{1}{V_{x,(1-|x|)/2}}\int_{B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|^p}{|x-y|^p}(1-|x|^2)^{p\alpha}(1-|y|^2)^{p\beta}dv(y)\Big)d\tau(x) < \infty;$$

(d)

$$\int_B \Big(\frac{1}{V_{x,(1-|x|)/2}}\int_{B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|}{|x-y|}(1-|x|^2)^\alpha(1-|y|^2)^\beta dv(y)\Big)^p d\tau(x) < \infty;$$

(e)

$$\int_B \int_{B\left(x, \frac{1-|x|}{2}\right)} (1-|x|^2)^{p\alpha}(1-|y|^2)^{p\beta} \frac{|u(x)-u(y)|^p}{|x-y|^p} d\tau(y) d\tau(x) < \infty.$$

Proof. $(a) \Leftrightarrow (e)$ was proved in [42].

$(d) \Rightarrow (a)$. By the Cauchy's inequality and subharmonicity we have that for every $\varepsilon \in (0, 1/2)$

$$\begin{aligned} & (1-|x|)|\nabla u(x)| \\ \leq\ & C \sup_{y \in B(x,(1-|x|)/4)} |u(y)-u(x)| \\ \leq\ & \frac{C}{(1-|x|)^n} \int_{B(x,(1-|x|)/2)} |u(y)-u(x)| dv(y) \\ \leq\ & C \int_{B(x,(1-|x|)/2)} |u(y)-u(x)| d\tau(y) \\ \leq\ & C \int_{B(x,(1-|x|)/2)} |u(y)-u(x)| \frac{(1-|x|^2)^\alpha (1-|y|^2)^\beta}{|x-y|} d\tau(y), \end{aligned} \tag{17}$$

where we have used the fact that

$$\frac{1-|y|}{|x-y|} > \frac{1-|x|}{2|x-y|} > 1, \quad \text{when} \quad y \in B(x,(1-|x|)/2),$$

as well as the condition $\alpha + \beta = 1$.

By (17) and the fact that

$$1-|y| \asymp 1-|x| \quad \text{if} \quad y \in B(x,(1-|x|)/2), \tag{18}$$

we have that

$$\begin{aligned} & (1-|x|)|\nabla u(y)| \\ \leq\ & C \int_{B(x,(1-|x|)/2)} |u(y)-u(x)| \frac{(1-|x|^2)^\alpha (1-|y|^2)^\beta}{|x-y|} d\tau(y) \\ \leq\ & \frac{C}{V_{x,(1-|x|)/2}} \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|}{|y-x|} (1-|x|^2)^\alpha (1-|y|^2)^\beta dv(y). \end{aligned}$$

Therefore

$$\begin{aligned} & \int_B (1-|x|)^p |\nabla u(x)|^p d\tau(x) \\ \leq\ & C \int_B \Big(\frac{1}{V_{x,(1-|x|)/2}} \int_{B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|}{|y-x|} (1-|x|^2)^\alpha (1-|y|^2)^\beta dv(y) \Big)^p d\tau(x) \\ <\ & \infty. \end{aligned}$$

$(a) \Rightarrow (b)$. By (18), the mean value theorem and the subharmonicity of $|\nabla u|^p$, $p > 1$, we have that

$$\begin{aligned}
&\Big(\sup_{y\in B(x,(1-|x|)/2)} \frac{|u(x)-u(y)|}{|x-y|}(1-|x|^2)^\alpha(1-|y|^2)^\beta\Big)^p\\
\leq\;& C\Big((1-|x|)\sup_{\zeta\in B(x,(1-|x|)/2)}|\nabla u(\zeta)|\Big)^p\\
\leq\;& C\Big(\sup_{\zeta\in B(x,(1-|x|)/2)}(1-|\zeta|)|\nabla u(\zeta)|\Big)^p\\
\leq\;& C\int_{\zeta\in B(x,1-|x|)}\Big((1-|\zeta|)|\nabla u(\zeta)|\Big)^p d\tau(\zeta). \qquad (19)
\end{aligned}$$

Multiplying (19) by $d\tau(x)$, then integrating over B and applying Fubini's theorem, we obtain

$$\begin{aligned}
&\int_B\Big(\sup_{y\in B(x,(1-|x|)/2)} \frac{|u(y)-u(x)|}{|y-x|}(1-|x|^2)^\alpha(1-|y|^2)^\beta\Big)^p d\tau(x)\\
\leq\;& C\int_B\int_{y\in B(x,1-|x|)}\Big((1-|\zeta|)|\nabla u(\zeta)|\Big)^p d\tau(\zeta)d\tau(x)\\
\leq\;& C\int_B\Big((1-|\zeta|)|\nabla u(\zeta)|\Big)^p\int_{x\in A(\zeta)} d\tau(x)d\tau(\zeta)\\
\leq\;& C\int_B\Big((1-|\zeta|)|\nabla u(\zeta)|\Big)^p d\tau(\zeta), \qquad (20)
\end{aligned}$$

where we have used the fact that the quantity $\int_{x\in A(\zeta)} d\tau(x)$ is bounded, since the set $A(\zeta)$ is contained in the ball $B(y, 1-|y|)$. Note that (b) implies (c) is trivial, and (c) implies (d) follows from Hölder inequality. The proof is completed.

5 Harmonic functions with Hadamard gaps

For $u \in \mathcal{H}(B)$, the radial derivative $\mathcal{R}u$ of u is $\mathcal{R}u(x) = \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}(x) = \sum_{k=0}^\infty kp_k(x)$, where $\sum_{k=0}^{+\infty} p_k(x)$ is the homogeneous polynomial expansion of u (see, [3]). For $p, \alpha \in (0,\infty)$, the Dirichlet type space B^p_α is defined to consist of those $u \in \mathcal{H}(B)$ for which

$$\|u\|_{B^p_\alpha} = \Big(\int_B |\mathcal{R}u(x)|^p(1-|x|)^{\alpha-1}dV(x)\Big)^{1/p} < \infty.$$

We say that $u \in \mathcal{H}(B)$ with the homogeneous polynomial expansion $u(x) = \sum_{k=0}^{+\infty} p_{m_k}(x)$, where $p_{m_k}(x)$ is a harmonic homogeneous polynomial of degree m_k on B, has Hadamard gaps if $m_{k+1}/m_k \geq \tau > 1$ for all $k \in \mathbb{N}$.

In order to prove the main result in this section we need the following lemma from [17].

Lemma F. *Suppose* $\alpha > 0$, $p > 0$, $n \geq 0$, $a_n \geq 0$, $I_n = \{k \mid 2^n \leq k < 2^{n+1}, k \in \mathbb{N}\}$, $t_n = \sum_{k\in I_n} a_k$ *and* $f(x) = \sum_{n=1}^{\infty} a_n x^n$. *Then there is a positive constant* K *depending only on* p *and* α *such that*

$$\frac{1}{K}\sum_{n=0}^{\infty}\frac{t_n^p}{2^{n\alpha}} \leq \int_0^1 (1-x)^{\alpha-1} f^p(x)dx \leq K\sum_{n=0}^{\infty}\frac{t_n^p}{2^{n\alpha}}.$$

Theorem 7. *Let* $p, \alpha \in (0,\infty)$, $u \in \mathcal{H}(B)$ *and* $u(x) = \sum_{k=0}^{+\infty} p_{m_k}(x)$, *have Hadamard gaps. Then* $u \in B^p_\alpha$ *if* $\sum_{k=0}^{+\infty} \|p_{m_k}\|^p_\infty m_k^{p-\alpha} < \infty$.

Proof. By polar coordinates and some calculations we obtain

$$\begin{aligned}
\|u\|^p_{B^p_\alpha} &= \int_B |\mathcal{R}u(x)|^p (1-|x|)^{\alpha-1} dV(x)\\
&= \int_0^1 \int_S \left|\sum_{k=0}^{+\infty} m_k p_{m_k}(r\zeta)\right|^p (1-r)^{\alpha-1} d\sigma(\zeta) r^{n-1} dr\\
&\leq \sigma_n \int_0^1 \left(\sum_{k=0}^{+\infty} m_k \|p_{m_k}\|_\infty\, r^{m_k}\right)^p (1-r)^{\alpha-1} dr\\
&\leq \sigma_n K \sum_{n=0}^{+\infty} 2^{-n\alpha}\left(\sum_{m_k\in I_n} m_k\|p_{m_k}\|_\infty\right)^p \qquad \text{(Lemma F)}\\
&\leq \sigma_n C 2^p([\log_\lambda 2]+1)^p \sum_{k=0}^{+\infty}\frac{\|p_{m_k}\|^p_\infty}{m_k^{\alpha-p}},
\end{aligned}$$

from which the result follows.

It is known that the norm in the a-Bloch spaces of harmonic functions is equivalent with the following norm

$$b_a(u) = |u(0)| + \sup_{x\in B}(1-|x|^2)^a|\mathcal{R}u(x)| = |u(0)| + \sup_{0<r<1}(1-r^2)^a M_\infty(\mathcal{R}u, r).$$

On the other hand, the quantity b_a, can be considered as border case for the following quantities $b_a^p(f) = \sup_{0<r<1}(1-r^2)^a\|\mathcal{R}u_r\|_p$, where $p \in (0,\infty)$, since clearly for every $u \in \mathcal{H}^\infty(B)$ and $p \in (0,\infty)$

$$\sup_{0<r<1}(1-r^2)^a\|\mathcal{R}u_r\|_p \leq \sup_{0<r<1}(1-r^2)^a\|\mathcal{R}u_r\|_\infty. \tag{21}$$

Now we consider harmonic functions with Hadamard gaps on the following spaces

$$\mathcal{B}_2^a = \left\{u \mid \sup_{0<r<1}(1-r^2)^a\|\mathcal{R}u_r\|_2 < \infty, u \in \mathcal{H}(B)\right\}$$

and

$$\mathcal{B}_{2,0}^a = \left\{u \mid \lim_{r\to 1}(1-r^2)^a\|\mathcal{R}u_r\|_2 = 0, u \in \mathcal{H}(B)\right\}.$$

Some characterization for the classes $\mathcal{B}_2^a$ and $\mathcal{B}_{2,0}^a$ are given in function of the sequence $(\|p_{m_k}\|_2)_{k\in\mathbb{N}}$.

Theorem 8. *Assume that $a>0$ and $u(x)=\sum_{k=1}^\infty p_{m_k}(x)$ is a harmonic function on B with Hadamard gaps. Then the following statements are equivalent:*

(a) $f\in\mathcal{B}_2^a$

(b) $\limsup_{k\to\infty}\|p_{m_k}\|_2 m_k^{1-a}<\infty.$

Proof. $(a)\Rightarrow(b)$ Let $u\in\mathcal{B}_2^a$. We have that

$$\begin{aligned}
\|u\|^2_{\mathcal{B}_2^a} &= \sup_{0<r<1}(1-r^2)^a\Big(\int_S|\mathcal{R}u(r\zeta)|^2d\sigma(\zeta)\Big)^{1/2}\\
&= \sup_{0<r<1}(1-r^2)^a\Big(\int_S\Big|\sum_{k=1}^\infty m_kp_{m_k}(\zeta)r^{m_k}\Big|^2d\sigma(\zeta)\Big)^{1/2}\\
&= \sup_{0<r<1}(1-r^2)^a\Big(\sum_{k=1}^\infty m_k^2\|p_{m_k}\|_2^2r^{2m_k}\Big)^{1/2}\\
&\geq \sup_{0<r<1}(1-r^2)^a m_k\|p_{m_k}\|_2r^{m_k}
\end{aligned}$$

for every $k\in\mathbb{N}$ and $r\in(0,1)$. Choosing $r=1-\frac{1}{m_k}$ we obtain

$$m_k^{1-a}\|p_{m_k}\|_2\leq C_1,$$

as desired.

$(b)\Rightarrow(a)$ Assume that $\limsup_{k\to\infty}\|p_{m_k}\|_2m_k^{1-a}<\infty$. We have that

$$\begin{aligned}
\|u\|^2_{\mathcal{B}_2^a} &= \sup_{0<r<1}(1-r^2)^a\Big(\sum_{k=1}^\infty m_k^2\|p_{m_k}\|_2^2r^{2m_k}\Big)^{1/2}\\
&\leq \sup_{0<r<1}(1-r^2)^a\sum_{k=1}^\infty m_k\|p_{m_k}\|_2r^{m_k}\\
&\leq \sup_{0<r<1}(1-r^2)^{a+1}\sum_{m=1}^\infty\Big(\sum_{m_k\leq m}m_k\|p_{m_k}\|_2\Big)r^m\\
&\leq C\sup_{0<r<1}(1-r^2)^{a+1}\sum_{m=1}^\infty\Big(\sum_{m_k\leq m}m_k^a\Big)r^m\\
&\leq C\sup_{0<r<1}(1-r^2)^{a+1}\sum_{m=1}^\infty m^ar^m\leq C,
\end{aligned}$$

where we have used the fact that there is a positive constant C independent of m such that

$$\sum_{m_k \le m} m_k^a \le C m^a \tag{22}$$

(here is used the assumption that $m_{k+1}/m_k \ge \lambda > 1$) and the following well known estimate

$$\sum_{m=1}^{\infty} m^a r^m \le C(1-r)^{-(a+1)}, \tag{23}$$

$a > 0, r \in [0,1)$, see, for example, [48].

Theorem 9. *Assume that $a > 0$ and $u(x) = \sum_{k=1}^{\infty} p_{m_k}(x)$ is a harmonic function on B with Hadamard gaps. Then the following statements are equivalent:*

(a) $f \in \mathcal{B}^a_{2,0}$

(b) $\lim_{k\to\infty} \|p_{m_k}\|_2 m_k^{1-a} = 0.$

Proof. $(a) \Rightarrow (b)$ Let $u \in \mathcal{B}^a_{2,0}$, then for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(1-r^2)^a \left(\int_S |\mathcal{R}u(r\zeta)| d\sigma(\zeta) \right)^{1/2} < \varepsilon, \tag{24}$$

whenever $\delta < r < 1$. From (24) we have that

$$\begin{aligned}
\varepsilon &> \sup_{\delta<r<1} (1-r^2)^a \left(\int_S |\mathcal{R}u(r\zeta)|^2 d\sigma(\zeta) \right)^{1/2} \\
&= \sup_{\delta<r<1} (1-r^2)^a \left(\int_S \Big| \sum_{k=1}^{\infty} m_k p_{m_k}(\zeta) r^{m_k} \Big|^2 d\sigma(\zeta) \right)^{1/2} \\
&= \sup_{\delta<r<1} (1-r^2)^a \left(\sum_{k=1}^{\infty} m_k^2 \|p_{m_k}\|_2^2 r^{2m_k} \right)^{1/2} \\
&\ge \sup_{\delta<r<1} (1-r)^a m_k \|p_{m_k}\|_2 r^{m_k}
\end{aligned}$$

for every $k \in \mathbb{N}$ and $r \in (\delta, 1)$. Choosing $r = 1 - \frac{1}{m_k}$ we obtain

$$m_k^{1-a} \|p_{m_k}\|_2 \le \varepsilon C_1,$$

from which the implication follows.

$(b) \Rightarrow (a)$ Assume that $\lim_{k\to\infty} \|p_{m_k}\|_2 m_k^{1-a} = 0$. Then for every $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that for every $k > k_0$

$$\|p_{m_k}\|_2 m_k^{1-a} < \varepsilon. \tag{25}$$

From (25) and some simple estimates we have that

$$\begin{aligned}
& (1-r^2)^a \Big(\int_S |\mathcal{R}u(r\zeta)|^2 d\sigma(\zeta) \Big)^{1/2} = (1-r^2)^a \Big(\sum_{k=1}^{\infty} m_k^2 \|p_{m_k}\|_2^2 r^{2m_k} \Big)^{1/2} \\
\leq \ & (1-r^2)^a \sum_{k=1}^{\infty} m_k \|p_{m_k}\|_2 r^{m_k} \leq (1-r^2)^{a+1} \sum_{m=1}^{\infty} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m \\
\leq \ & (1-r^2)^{a+1} \sum_{m=1}^{k_0} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m \\
& + (1-r^2)^{a+1} \sum_{m=k_0+1}^{\infty} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m \\
\leq \ & (1-r^2)^{a+1} \sum_{m=1}^{k_0} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m \\
& + \varepsilon C (1-r^2)^{a+1} \sum_{m=1}^{\infty} \Big(\sum_{m_k \leq m} m_k^a \Big) r^m \\
\leq \ & (1-r^2)^{a+1} \sum_{m=1}^{k_0} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m + \varepsilon C (1-r^2)^{a+1} \sum_{m=1}^{\infty} m^a r^m \\
\leq \ & (1-r^2)^{a+1} \sum_{m=1}^{k_0} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m + \varepsilon C, \qquad (26)
\end{aligned}$$

where we have used (22) and (23).

Now note that for every $\varepsilon > 0$ there is a $\delta \in (0,1)$ such that for every $r \in (\delta, 1)$

$$(1-r^2)^{a+1} \sum_{m=1}^{k_0} \Big(\sum_{m_k \leq m} m_k \|p_{m_k}\|_2 \Big) r^m < \varepsilon. \tag{27}$$

From (26) and (27) the result follows.

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CHARACTERIZATIONS OF LAGUERRE-HAHN AFFINE ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Abstract. In this work we characterize a monic polynomial sequence, orthogonal with respect to a hermitian linear functional u that satisfies a functional equation $\mathrm{D}(Au) = Bu + zH\mathcal{L}$, where A, B and H are polynomials and $\mathcal{L}$ is the Lebesgue functional, in terms of a first order linear differential equation for the Carathéodory function associated with u and in terms of a first order structure relation for the orthogonal polynomials.

keywords. Orthogonal polynomials on the unit circle, hermitian functionals, measures on the unit circle, semi-classical functionals, Carathéodory function.
AMS Subject Classifications (2000).Primary 33C47,42C05.

1. Introduction

Let u be a linear functional, defined in the linear space of polynomials with real coefficients. A linear functional u is Laguerre-Hahn affine if the corresponding formal Stieltjes function satisfies a first order linear differential equation,

$$\phi(x)S'(x) = B(x)S(x) + C(x) \tag{1}$$

with ϕ, B, C polynomials (*cf.* [11]). It is known that the Laguerre-Hahn affine class and the semi-classical class coincide. This result follows from [10, 12], where it is established that u is Laguerre-Hahn affine if, and only if, u satisfies a functional Pearson equation, $\mathrm{D}(\phi u) = \psi u$, where ϕ and ψ are polynomials (ϕ is the same polynomial as in (1)).

Laguerre-Hahn affine functionals on the real line are also characterized in terms of first order structure relations for the corresponding sequence of orthogonal polynomials on the real line, $\{P_n\}$,

$$\phi(x)P'_{n+1}(x) = C_n(x)P_{n+1}(x) + D_n(x)P_n(x),\ n = 0, 1, \ldots$$

where C_n, D_n are polynomials of bounded degree ([10, 11, 12]).

In [8, 18], an analogue theory for hermitian linear functionals defined in the linear space of Laurent polynomials with complex coefficients was outlined. The concept of semi-classical functional was extended to this set of functionals; a hermitian linear functional u is said to be semi-classical if it satisfies a Pearson equation $\mathrm{D}(Au) = Bu$, where A, B are polynomials (see, in section 3, the definition of the derivation operator D), and the corresponding sequences of orthogonal polynomials, semi-classical orthogonal polynomials on the unit circle, were defined; the Laguerre-Hahn

affine class on the unit circle was defined in terms of a first order linear differential equation with polynomial coefficients for the formal series $G(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$, where c_n is the n-th moment of the hermitian functional u,

$$A(z)G'(z) = B_1(z)G(z) + H(z). \tag{2}$$

Since then, the comparison between both theories, namely the characterization of the functionals in terms of differential properties for the corresponding sequences of orthogonal polynomials, as well the generating functions for the moments, has been the central theme in some works (see [1, 2, 9, 13, 16, 18]).

In [2, 13, 18], it is established that u is Laguerre-Hahn affine and the corresponding G satisfies (2) if, and only if, the corresponding u satisfies the generalized Pearson equation $\mathrm{D}\,(Au) = Bu + zH\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue operator and B is a polynomial depending on A, B_1. Moreover, in [2, 13], the authors obtain conditions on the coefficients of equation (2) and, also, on the polynomial coefficients of a differential equation for the formal Carathéodory function F,

$$A(z)F'(z) = B_1(z)F(z) + C(z) \tag{3}$$

in order to establish the semi-classical character of the corresponding functional. Some examples of functionals and the corresponding sequences of orthogonal polynomials that are not semi-classical are given, thus showing that, in the complex case, the Laguerre-Hahn affine class and the semi-classical class do not coincide.

In this work, following a different approach from the referred works, we study the relation between a first order differential equation for the Carathéodory function, (3), and the distributional equation for the corresponding u (we remind that, in many ways, the Carathéodory function is the analogue of the Stieltjes function (see [14])). We prove that if F satisfies a first order differential equation (3) in $|z| < 1$, then the corresponding linear functional u satisfies a generalized Pearson equation $\mathrm{D}\,(Au) = Bu + H\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue operator and B, H are polynomials given explicitly in terms of A, B_1, C. Then, we deduce first order structure relations for the corresponding sequences of orthogonal polynomials on the unit circle (analogue of the structure relations for orthogonal polynomials on the real line, studied in [10, 11, 12]). Finally, using these structure relations, we obtain a differential system for semi-classical orthogonal polynomials on the unit circle (the analogue of the result established for semi-classical orthogonal polynomials on the real line in [7], by Magnus).

This paper is organized as follows: in section 2 we give the definitions and state the main results which will be used in the next sections. In section 3 we study the relation between the first order linear differential equation for F, and the generalized Pearson equation for u (see theorem 3). In section 4, we establish the equivalence between a first order differential equation for F and a system of differential relations for the sequence of orthogonal polynomials, for the sequence of associated polynomials of the second kind and for the sequence of functions of the second kind. We deduce a differential system for sequences of semi-classical orthogonal polynomials on the unit circle.

2. Preliminary results

Let $\Lambda = \text{span}\,\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with complex coefficients, Λ' its algebraic dual space, $\mathbb{P} = \text{span}\,\{z^k : k \in \mathbb{N}\}$ the space of complex polynomials and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (or, using the parametrization $z = e^{i\theta}$, $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$) the unit circle.

Given a sequence of moments (c_n) of a linear functional $u : \Lambda \to \mathbb{C}$, $c_n = \langle u, \xi^{-n}\rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, the minors of the Toeplitz matrix are defined by

$$\Delta_k = \begin{vmatrix} c_0 & c_1 & \cdots & c_k \\ c_{-1} & c_0 & \cdots & c_{k-1} \\ \vdots & \vdots & & \vdots \\ c_{-k} & c_{-k+1} & \cdots & c_0 \end{vmatrix},\ \Delta_0 = c_0,\ \Delta_{-1} = 1,\ k \in \mathbb{N}$$

Definition 1. (*cf.* [17]). The linear functional u is:
a) *hermitian* if $c_{-n} = \overline{c_n}, \forall n \geq 0$,
b) *regular* or *quasi-definite* if $\Delta_n \neq 0, \forall n \geq 0$,
c) *positive definite* if $\Delta_n > 0, \forall n \geq 0$.

If u is a positive definite hermitian functional there exists a non-trivial probability measure μ supported on the unit circle such that

$$\langle u, \xi^{-n}\rangle = \frac{1}{2\pi}\int_0^{2\pi} \xi^{-n} d\mu(\theta),\ n \in \mathbb{Z},\ \xi = e^{i\theta}.$$

Hereafter we will use the notation $\langle u_\theta,\ .\ \rangle$ to denote the action of the linear functional u over the variable θ, $\theta \in [0, 2\pi[$.

Definition 2. Let $\{\phi_n\}$ be a sequence of complex polynomials with $\deg(\phi_n) = n$ and u a hermitian linear functional. We say that $\{\phi_n\}$ is a *sequence of orthogonal polynomials* with respect to u (or $\{\phi_n\}$ is a sequence of orthogonal polynomials on the unit circle) if

$$\langle u, \phi_n(\xi)\overline{\phi_m}(1/\xi)\rangle = K_n\delta_{n,m},\ K_n \neq 0,\ n, m \in \mathbb{N},\ \xi = e^{i\theta}.$$

If the leading coefficient of each ϕ_n is 1, then $\{\phi_n\}$ is said to be a *sequence of monic orthogonal polynomials*.

It is well known (see [3, 4, 5]) that a given hermitian linear functional u is regular if, and only if, there exists a sequence $\{\phi_n\}$ of orthogonal polynomials with respect to u. Sequences of monic orthogonal polynomials $\{\phi_n\}$ satisfy each of the following recurrence relations, for $n \geq 1$,

$$(R_1)\ \ \phi_n(z) = z\phi_{n-1}(z) + a_n\phi^*_{n-1}(z)$$
$$(R_2)\ \ \phi^*_n(z) = \phi^*_{n-1}(z) + \overline{a}_n z\phi_{n-1}(z)$$

with $a_n = \phi_n(0)$, and initial conditions $\phi_0(z) = 1,\ \phi_{-1}(z) = 0$, and the polynomials $\{\phi^*_n\}$ are defined by $\phi^*_n(z) = z^n\overline{\phi_n}(1/z),\ n = 0, 1, \ldots$, where $n = \deg(\phi_n)$. Also, $|a_n| \neq 1$, $\forall n \in \mathbb{N}$, in the regular case and $|a_n| < 1$, $\forall n \in \mathbb{N}$, in the positive-definite case.

We consider the formal series associated with the hermitian linear functional u (whose sequence of moments is (c_n) and $c_0 = 1$), and denote it by F,

$$F(z) = 1 + 2\sum_{k=1}^{+\infty} c_k z^k,\ |z| < 1\,, \quad F(z) = -1 - 2\sum_{k=1}^{+\infty} \overline{c}_k z^{-k}\,, \quad |z| > 1 \tag{4}$$

Since, for each $\theta \in [0, 2\pi[$, the following expansions take place,

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = 1 + 2\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k,\ |z|<1\ ,\quad \frac{e^{i\theta}+z}{e^{i\theta}-z} = -1 - 2\sum_{k=1}^{+\infty}(e^{-i\theta})^{k}z^{-k},\ |z|>1$$

then, formally,

$$\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = F(z) \tag{5}$$

Thus, we will also say that the series in (4) correspond (formally) to the function F defined by (5). In the positive definite case, F is the Carathéodory function corresponding to u, and is represented by

$$F(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta),\ z \in \mathbb{C}\setminus\mathbb{T}$$

where μ is the probability measure associated with u.

Given a sequence of monic orthogonal polynomials $\{\phi_n\}$ with respect to u, the *sequence of associated polynomials of the second kind* $\{\Omega_n\}$ are defined by

$$\begin{aligned}&\Omega_n(z) = \langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\rangle,\ n=1,2,\ldots\\ &\Omega_0(z) = 1.\end{aligned}$$

The associated polynomials $\{\Omega_n\}$ also satisfy recurrence relations,

$$\Omega_n(z) = z\Omega_{n-1}(z) - a_n\Omega^*_{n-1}(z),\ n=1,2,\ldots$$

with initial conditions $\Omega_0(z)=1,\ \Omega_{-1}(z)=0$.

The *functions of the second kind* associated with $\{\phi_n\}$ are defined by

$$\begin{aligned}&Q_n(z) = \langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\rangle,\ n=1,2,\ldots\\ &Q_0(z) = F(z)\end{aligned}$$

and $\{Q_n\}$ satisfy the following recurrence relations (*cf.* [15]),

$$Q_n(z) = zQ_{n-1} - a_nQ^*_{n-1}(z),\ n=1,2,\ldots$$

with $Q_0(z) = F(z)$ and $Q_0^*(z) = -F(z)$.

Theorem 1 (*cf.* [3, 4, 5]). *Let $\{\phi_n\}$ be a sequence of monic orthogonal polynomials on the unit circle and $\{\Omega_n\}$, $\{Q_n\}$ the sequence of the associated polynomials and the functions of the second kind, respectively. Then the following equations hold, $\forall n \geq 1$,*

$$Q_n(z) = \Omega_n(z) + F(z)\phi_n(z), \tag{6}$$
$$Q^*_n(z) = \Omega^*_n(z) - F(z)\phi^*_n(z) \tag{7}$$
$$\phi^*_n(z)\Omega_n(z) + \phi_n(z)\Omega^*_n(z) = 2h_nz^n \tag{8}$$
$$\phi^*_n(z)Q_n(z) + \phi_n(z)Q^*_n(z) = 2h_nz^n \tag{9}$$

with $h_n = \prod_{k=1}^n(1-|a_k|^2)$ and $Q^*_n = z^n\overline{Q_n}(1/z)$.

As a consequence we get the following results (see [15]).

Corollary 1. *Let $\{Q_n\}$ be the sequence of functions of the second kind associated with $\{\phi_n\}$. Then, the following holds, $\forall n \geq 1$,*

$$Q_n(z) = 2h_n z^n + \mathcal{O}(z^{n+1}),\ |z| < 1 \tag{10}$$

$$Q_n(z) = 2a_{n+1}h_n z^{-1} + \mathcal{O}(z^{-2}),\ |z| > 1\,. \tag{11}$$

Corollary 2. *Let $\{\phi_n\}$ be a sequence of monic orthogonal polynomials on the unit circle and $\{\Omega_n\}$ the sequence of associated polynomials of the second kind. Then, the following holds:*

a) If there exists $k \in \mathbb{N}$ such that $\phi_k(\alpha) = \Omega_k(\alpha) = 0$, then $\alpha = 0$;

b) If there exists $k \in \mathbb{N}$ such that $\phi_k(\alpha) = Q_k(\alpha) = 0$, then $\alpha = 0$.

3. The first order differential equation for the Carathéodory function

Let $u \in \Lambda'$ be a regular hermitian functional and $f \in \Lambda$. We define the linear functional $fu \in \Lambda'$ as

$$\langle fu, g(\xi)\rangle = \langle u, f(\xi)g(\xi)\rangle,\ g \in \Lambda,$$

and the derivative $\mathrm{D}u \in \Lambda'$ as

$$\langle \mathrm{D}\,u, f(\xi)\rangle = -i\langle \xi u, f'(\xi)\rangle = -i\langle u, \xi f'(\xi)\rangle.$$

In [2, 13, 18] it is established the equivalence between the Laguerre-Hahn affine character of a hermitian linear functional u, and the distributional equation

$$\mathrm{D}(Au) = Bu + zH\mathcal{L} \tag{12}$$

where $\mathcal{L}$ is the Lebesgue operator and A, B, H are polynomials. We remark that when $H = 0$ and $A \not\equiv 0$ in (12), u is said to be semi-classical.

In this section we study the relation between regular hermitian functionals u that satisfy (12) and a first order differential equation for the corresponding F.

We begin by establishing some properties for the function F. Throughout this section we will use the representation (5) for F.

Lemma 1. *If A and B are polynomials and u is a hermitian linear functional, the following relations hold, for $|z| \neq 1$:*

$$\langle B(e^{i\theta})u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = P(z) + B(z)F(z)\,, \tag{13}$$

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{zi}\langle D(Au), \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle \tag{14}$$

where P and Q are the polynomials defined by

$$P(z) = \langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(B(e^{i\theta}) - B(z)\right)\rangle \tag{15}$$

$$Q(z) = -A'(z) - \langle u_\theta, 2e^{i\theta}\sum_{k=2}^{\deg(A)} \frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}\rangle \tag{16}$$

Proof. Since

$$\langle B(e^{i\theta})u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = \langle u, \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(B(e^{i\theta}) - B(z)\right)\rangle + B(z)\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle$$

and $\langle u_\theta, \dfrac{e^{i\theta}+z}{e^{i\theta}-z}\big(B(e^{i\theta})-B(z)\big)\rangle$ is a polynomial, we get (13) with P defined as referred.

To obtain (14) we proceed as follows:

$$\begin{aligned} A(z)F'(z) &= \langle u_\theta, \frac{2e^{i\theta}}{(e^{i\theta}-z)^2}A(z)\rangle \\ &= -\langle u_\theta, \frac{2e^{i\theta}}{(e^{i\theta}-z)^2}(A(e^{i\theta})-A(z))\rangle + \langle u_\theta, \frac{2e^{i\theta}A(e^{i\theta})}{(e^{i\theta}-z)^2}\rangle \\ &= -\langle u_\theta, 2e^{i\theta}\sum_{k=1}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}\rangle + \langle u_\theta, \frac{2e^{i\theta}A(e^{i\theta})}{(e^{i\theta}-z)^2}\rangle \end{aligned}$$

But

$$\sum_{k=1}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2} = \frac{A'(z)}{e^{i\theta}-z} + \sum_{k=2}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}$$

and, multiplying and dividing by z in the second term and taking into account that

$$\langle u_\theta, \frac{2ze^{i\theta}A(e^{i\theta})}{(e^{i\theta}-z)^2}\rangle = -\langle A(e^{i\theta})u_\theta, e^{i\theta}\frac{\partial}{\partial\theta}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\rangle,$$

we get

$$\begin{aligned} A(z)F'(z) = -A'(z)\left(F(z)+c_0\right) - \langle u_\theta, 2e^{i\theta}\sum_{k=2}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}\rangle \\ - \frac{1}{z}\langle A(e^{i\theta})u, e^{i\theta}\frac{\partial}{\partial\theta}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\rangle \end{aligned}$$

Now we use the definition of D and assume that $c_0 = 1$, to get

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{zi}\langle \mathrm{D}\,(Au), \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle,$$

with Q given by (16). □

REMARK 1. Throughout this section we will use the notation

$$P_{A,B}(z) = zQ(z) - iP(z)$$

where the polynomials P and Q are defined in terms of A and B by (15) and (16), respectively.

Next, we recover a result established in [2, 13], but here a different approach is used.

THEOREM 2. *Let u be a regular hermitian functional. If u satisfies $D(Au) = Bu + \xi H(\xi)\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, and A, B, H are polynomials, then F satisfies the first order differential equations*

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + P_{A,B}(z) - 2izH(z), |z|<1 \quad (17)$$

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + P_{A,B}(z), \quad |z|>1 \quad (18)$$

Conversely, if F satisfies (17) and (18), then u satisfies the functional equation

$$D(Au) = Bu + \xi H(\xi)\mathcal{L}.$$

Proof. Let u satisfy $\mathrm{D}\,(Au) = Bu+\xi H(\xi)\mathcal{L}$. By substituting $\mathrm{D}\,(Au) = Bu+\xi H(\xi)\mathcal{L}$ in (14) we obtain

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{iz}\langle Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle + \frac{1}{iz}\langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle$$

From (13) follows

$$\begin{aligned} &A(z)F'(z) \\ &\quad = -A'(z)F(z) + Q(z) + \frac{P(z)+B(z)F(z)}{iz} + \frac{1}{iz}\langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle \end{aligned} \tag{19}$$

Let $e^{i\theta}H(e^{i\theta}) = h_1 e^{i\theta} + \cdots + h_l (e^{i\theta})^l$, for some $l \in \mathbb{N}$.

Since, for $|z|<1$,

$$\begin{aligned} \langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle &= \langle (h_1 e^{i\theta} + \cdots + h_l (e^{i\theta})^l)\mathcal{L}, 1 + 2\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k\rangle \\ &= 2\left(h_1 z + h_2 z^2 + \cdots + h_l z^l\right), \\ &= 2zH(z) \end{aligned}$$

then, for $|z|<1$, (19) is equivalent to

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{zi}\left(P(z)+B(z)F(z)\right) + \frac{1}{iz}2zH(z)$$

and we obtain the equation

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + C_1(z), \quad |z|<1,$$

with $C_1(z) = zQ(z) - iP(z) - 2izH(z) = P_{A,B}(z) - 2izH(z)$.

On the other hand, for $|z|>1$,

$$\langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = \langle (h_1 e^{i\theta} + \cdots + h_l (e^{i\theta})^l)\mathcal{L}, -1 - 2\sum_{k=1}^{+\infty}(e^{i\theta})^{k}z^{-k}\rangle = 0.$$

Therefore, for $|z|>1$, (19) is equivalent to

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + C_2(z),$$

with $C_2(z) = zQ(z) - iP(z) = P_{A,B}(z)$.

Conversely, let F satisfy equations (17) and (18). We observe that, if F satisfies a differential equation with polynomial coefficients

$$zA(z)F'(z) = (-iB(z) - zA'(z))F(z) + C(z),$$

then, from (14) and (13), we obtain

$$\langle Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle - P(z) + iC(z) = izQ(z) + \langle \mathrm{D}\,(Au), \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle$$

and the following equation follows

$$\langle \mathrm{D}\,(Au) - Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = R(z) \tag{20}$$

where $R(z) = iC(z) - P(z) - izQ(z) = iC(z) - iP_{A,B}(z)$.

We now study equation (20) in both domains, $|z|<1$ and $|z|>1$:
a) since, if $|z|<1$, equation (17) holds, then $C(z)=P_{A,B}(z)-2izH(z)$, and (20) becomes

$$\langle \mathrm{D}\,(Au)-Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = 2zH(z)$$

Since $\dfrac{e^{i\theta}+z}{e^{i\theta}-z} = 1+2\displaystyle\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k$, $|z|<1$, last equation is equivalent to

$$\langle \mathrm{D}\,(Au)-Bu,1\rangle + 2\sum_{k=1}^{+\infty}\langle \mathrm{D}\,(Au)-Bu,(e^{i\theta})^{-k}\rangle z^k = 2zH(z) \tag{21}$$

b) since, if $|z|>1$, equation (18) holds, then $C(z)=P_{A,B}(z)$ and (20) becomes

$$\langle \mathrm{D}\,(Au)-Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = 0$$

Since $\dfrac{e^{i\theta}+z}{e^{i\theta}-z} = -1-2\displaystyle\sum_{k=1}^{+\infty}(e^{i\theta})^{k}z^{-k}$, last equation is equivalent to

$$-\langle \mathrm{D}\,(Au)-Bu,1\rangle - 2\sum_{k=1}^{+\infty}\langle \mathrm{D}\,(Au)-Bu,(e^{i\theta})^{k}\rangle z^{-k} = 0 \tag{22}$$

Finally, from (21) and (22), we have

$$\begin{aligned}
&\langle \mathrm{D}\,(Au)-Bu,(e^{i\theta})^{k}\rangle = 0,\ \forall k\geq 0\\
&\langle \mathrm{D}\,(Au)-Bu,(e^{i\theta})^{-k}\rangle = 0,\ \forall k> \deg(H)+1\\
&\langle \mathrm{D}\,(Au)-Bu,(e^{i\theta})^{-k}\rangle = h_k,\ k=1,\ldots,\deg(H)+1
\end{aligned}$$

with $zH(z)=h_1z+h_2z^2+\cdots+h_lz^l$.
Therefore, we obtain the functional equation $\mathrm{D}\,(Au)-Bu=\xi H(\xi)\mathcal{L}$. □

Note that if u is a semi-classical functional such that $\mathrm{D}\,(Au)=Bu$, then the function F associated with u satisfies a first order linear differential equation with polynomial coefficients,

$$zA(z)F'(z) = (-iB(z)-zA'(z)))F(z)+C(z),\ |z|\neq 1,$$

as is stated in [18].

Corollary 3. *Let F satisfy*

$$zA(z)F'(z) = (-iB(z)-zA'(z))F(z)+C(z),\ |z|\neq 1\,.$$

Then, the corresponding linear functional u is semi-classical and satisfies $D\,(Au)=Bu$ if and only if $C(z)=P_{A,B}(z)$.

Finally, we study the case of one differential equation for F. We will need the following lemma, which is a generalization of a result from [1].

Lemma 2. *Let u be a regular hermitian functional. If there exist polynomials A,B,H such that $D\,(Au)=Bu+H\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, then the following equation holds,*

$$D\,(A+\overline{A})u = (B+\overline{B})u + (H+\overline{H})\mathcal{L} \tag{23}$$

Conversely, if (23) holds, then there exist polynomials $\tilde{A}, \tilde{B}, \tilde{H}$ such that $D(\tilde{A}u) = \tilde{B}u + \tilde{H}\mathcal{L}$.

Proof. If $\mathrm{D}\,(Au) = Bu + H\mathcal{L}$, then

$$\langle \mathrm{D}\,(Au), \xi^k \rangle = \langle Bu + H\mathcal{L}, \xi^k \rangle, \ \forall k \in \mathbb{Z}.$$

Applying conjugates, follows

$$\langle \mathrm{D}\,(\overline{A}u), \xi^{-k} \rangle = \langle \overline{B}u + \overline{H}\mathcal{L}, \xi^{-k} \rangle, \ \forall k \in \mathbb{Z}.$$

Therefore, we get

$$\langle \mathrm{D}\,((A+\overline{A})u), \xi^n \rangle = \langle (B+\overline{B})u + (H+\overline{H})\mathcal{L}, \xi^n \rangle, \ \forall n \in \mathbb{Z}$$

and (23) follows.

Conversely, if u satisfies (23), then

$$\langle \mathrm{D}\,((A+\overline{A})u), \xi^k \rangle = \langle (B+\overline{B})u, \xi^k \rangle + \langle (H+\overline{H})\mathcal{L}, \xi^k \rangle, \ \forall k \in \mathbb{Z}.$$

From the definition of D, we obtain for all $k \in \mathbb{Z}$,

$$-ik\langle u, (A(\xi) + \overline{A}(1/\xi))\xi^k \rangle = \langle u, (B(\xi) + \overline{B}(1/\xi))\xi^k \rangle + \langle \mathcal{L}, (H(\xi) + \overline{H}(1/\xi))\xi^k \rangle.$$

Let $s = \max\{\deg(A), \deg(B), \deg(H)\}$. Last equation can be written as

$$\begin{aligned} &-ik\langle u, \xi^s(A(\xi) + \overline{A}(1/\xi))\xi^{k-s} \rangle \\ &\qquad = \langle u, \xi^s(B(\xi) + \overline{B}(1/\xi))\xi^{k-s} \rangle + \langle \mathcal{L}, \xi^s(H(\xi) + \overline{H}(1/\xi))\xi^{k-s} \rangle, \ k \in \mathbb{Z}. \end{aligned}$$

If we write $k = s + m$ and

$$\begin{aligned} A_1(\xi) &= \xi^s(A(\xi) + \overline{A}(1/\xi)) \\ B_1(\xi) &= \xi^s(B(\xi) + \overline{B}(1/\xi)) \\ H_1(\xi) &= \xi^s(H(\xi) + \overline{H}(1/\xi)) \end{aligned}$$

then A_1, B_1, H_1 are polynomials, and last functional equation is

$$-i(s+m)\langle u, A_1(\xi)\xi^m \rangle = \langle u, B_1(\xi)\xi^m \rangle + \langle \mathcal{L}, H_1(\xi)\xi^m \rangle$$

which is equivalent to

$$-im\langle u, A_1(\xi)\xi^m \rangle = \langle u, (B_1(\xi) + isA_1(\xi))\xi^m \rangle + \langle \mathcal{L}, H_1(\xi)\xi^m \rangle.$$

From the definition of D, follows

$$\langle \mathrm{D}\,(A_1 u), \xi^m \rangle = \langle (B_1 + isA_1)u, \xi^m \rangle + \langle H_1\mathcal{L}, \xi^m \rangle, \ \forall m \in \mathbb{Z},$$

and we obtain the required result with $\tilde{A} = A_1, \ \tilde{B} = B_1 + isA_1, \ \tilde{H} = H_1$. □

THEOREM 3. *Let u be a regular hermitian functional. If F satisfies a first order differential equation with polynomial coefficients*

$$zA(z)F'(z) = (-iB(z) - zA'(z))\,F(z) + C(z), \ |z| < 1 \tag{24}$$

then there exist polynomials $\tilde{A}, \tilde{B}, \tilde{H}$ such that u satisfies $D(\tilde{A}u) = \tilde{B}u + \tilde{H}\mathcal{L}$, with $\mathcal{L}$ the Lebesgue functional.

Proof. If F satisfies (24), following the same steps as in the second part of the proof of theorem 2 (see (20)), we obtain the equation

$$\langle \mathrm{D}\,(Au) - Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = i(C(z) - P_{A,B}(z)),\ |z|<1 \tag{25}$$

Applying conjugates and the transformation $z \to 1/z$, follows

$$\langle \mathrm{D}\,(\overline{A}u) - \overline{B}u, \frac{e^{-i\theta}+1/z}{e^{-i\theta}-1/z}\rangle = -i(\overline{C}(1/z) - \overline{P}_{A,B}(1/z))\,.$$

Since $\dfrac{e^{-i\theta}+1/z}{e^{-i\theta}-1/z} = -\dfrac{e^{i\theta}+z}{e^{i\theta}-z}$, last equation is equivalent to

$$\langle \mathrm{D}\,(\overline{A}u) - \overline{B}u, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = i(\overline{C}(1/z) - \overline{P}_{A,B}(1/z)),\ |z|>1 \tag{26}$$

Now, since there exists an analytic continuation outside the unit disk, in (25), and inside the unit disk, in (26), we get, for $1-\epsilon_2 < |z| < 1+\epsilon_1,\ \epsilon_1,\epsilon_2>0$,

$$\langle \mathrm{D}\,((A+\overline{A})u) - (B+\overline{B})u, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = i(C(z) - P_{A,B}(z)) + i(\overline{C}(1/z) - \overline{P}_{A,B}(1/z)).$$

By computing the moments of the hermitian functional $\mathrm{D}\,((A+\overline{A})u) - (B+\overline{B})u$, from last equation follows $\mathrm{D}\,((A+\overline{A})u) - (B+\overline{B})u = (H+\overline{H})\mathcal{L}$, with $H(\xi) = i(C(\xi) - P_{A,B}(\xi))/2$. From previous lemma, we obtain the required result. $\square$

4. First order structure relations for orthogonal polynomials on the unit circle

In this section we establish the equivalence between the first order differential equation

$$zAF' + BF + C = 0,$$

for the Carathéodory function associated with a hermitian functional u, and first order structure relations for the corresponding orthogonal polynomials, the associated polynomials of the second kind and for the functions of the second kind. This will be done using the same ideas of [6].

Theorem 4. *Let u be a regular and hermitian functional, $\{\phi_n\}$ the sequence of monic orthogonal polynomials with respect to u, $\{\Omega_n\}$ the associated polynomials of the second kind and $\{Q_n\}$ the functions of the second kind. If there exist polynomials A, B, C such that F satisfies*

$$zA(z)F'(z) + B(z)F(z) + C(z) = 0,\ |z|<1$$

then there exist polynomials G_n and H_n with degrees not depending on n, such that the following relations holds, for all $n \in \mathbb{N}$,

$$zA(z)\phi_n'(z) = (G_n(z) + \frac{B}{2}(z))\phi_n(z) + H_n(z)\phi_n^*(z) \tag{27}$$

$$zA(z)\Omega_n'(z) = (G_n(z) - \frac{B}{2}(z))\Omega_n(z) - H_n(z)\Omega_n^*(z) + C(z)\phi_n(z) \tag{28}$$

$$zA(z)Q_n'(z) = (G_n(z) - \frac{B}{2}(z))Q_n(z) - H_n(z)Q_n^*(z) \tag{29}$$

Conversely, if equations (27), (28) and (29) hold, for all $n \in \mathbb{N}$, then F satisfies a first order linear differential equation with polynomial coefficients

$$zAF' + BF + C = 0.$$

Proof. Before going into the proof we remark that if $\phi_n(0) = 0,\ \forall n \in \mathbb{N},$ then $\phi_n(z) = \Omega_n(z) = z^n,\ Q_n(z) = 2z^n,\ \phi_n^*(z) = \Omega_n^*(z) = 1,\ Q_n^*(z) = 0,\ F(z) = 1$, and the result holds with $A = 1, B = 0, C = 0$, and the differential relations (27), (28) and (29) with $G_n = n, H_n = 0,\ \forall n \in \mathbb{N}$. So, in what follows we will assume that we are not in the case $\phi_n(0) = 0,\ \forall n \in \mathbb{N}$.

Using (6) in $zAF' + BF + C = 0$ we get

$$zA\left\{\frac{\Omega_n\phi_n' - \Omega_n'\phi_n}{\phi_n^2}\right\} + zA\left(\frac{Q_n}{\phi_n}\right)' - B\frac{\Omega_n}{\phi_n} + B\frac{Q_n}{\phi_n} + C = 0.$$

Therefore the following equation holds,

$$zA\left(\Omega_n\phi_n' - \Omega_n'\phi_n\right) - B\Omega_n\phi_n + C\phi_n^2 = \Theta_n(z) \tag{30}$$

with

$$\Theta_n(z) = \left\{-zA(z)\left(\frac{Q_n(z)}{\phi_n(z)}\right)' - B(z)\frac{Q_n(z)}{\phi_n(z)}\right\}\phi_n^2(z) \tag{31}$$

From the asymptotic expansion of Q_n in $|z| < 1$ (see (10)), and since the left side of (30) is a polynomial, we get $\Theta_n = z^n\Theta_{n,1}$, for some polynomial $\Theta_{n,1}$. Moreover, using the asymptotic expansion of Q_n in $|z| > 1$ (see (11)), we conclude that $\Theta_{n,1}$ has bounded degree,

$$\deg(\Theta_{n,1}) = \max\{\deg(A) - 1, \deg(B) - 1\},\ \forall n \in \mathbb{N}.$$

Thus, (30) becomes

$$-\phi_n\left\{zA\Omega_n' + B\Omega_n - C\phi_n\right\} + \Omega_n(zA\phi_n') = z^n\Theta_{n,1}(z).$$

Using (8) follows

$$-\phi_n\left\{zA\Omega_n' + B\Omega_n - C\phi_n\right\} + \Omega_n(zA\phi_n') = \Theta_{n,2}(z)\left(\phi_n^*\Omega_n + \phi_n\Omega_n^*\right),$$

with $\Theta_{n,2}(z) = \Theta_{n,1}(z)/(2h_n)$, and we obtain

$$\phi_n\left\{zA\Omega_n' + \frac{B}{2}\Omega_n - C\phi_n + \Theta_{n,2}\Omega_n^*\right\} = \Omega_n\left\{zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^*\right\} \tag{32}$$

We distinguish the following cases (see corollary 2):
a) ϕ_n and Ω_n have no common roots, $\forall n \in \mathbb{N}$, i.e., $\phi_n(0) \neq 0, \forall n \in \mathbb{N}$;
b) There exists a finite number of indexes $k \in \mathbb{N}$ such that ϕ_k and Ω_k have common roots, i.e., $\phi_k(0) = \Omega_k(0) = 0$ for a finite number of k's;
c) There exists $n_0 > 1$ such that $\phi_n(0) = 0,\ \forall n \geq n_0$.

Case a): If ϕ_n and Ω_n do not have common roots, then we conclude that there exists a polynomial l_n, $\forall n \in \mathbb{N}$, such that

$$\begin{cases} zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^* = l_n\phi_n \\ zA\Omega_n' + \frac{B}{2}\Omega_n - C\phi_n + \Theta_{n,2}\Omega_n^* = l_n\Omega_n \end{cases} \tag{33}$$

and we obtain (27) and (28) with $G_n = l_n$ and $H_n = \Theta_{n,2}$. Moreover, as $\deg(H_n)$ is bounded, then $\deg(G_n)$ is bounded,

$$\deg(G_n) = \max\{\deg(A), \deg(B)\},\ \forall n \in \mathbb{N}.$$

Case b): We first suppose $\phi_1(0) \neq 0, \dots, \phi_{k-1}(0) \neq 0$, and k is the first index such that $\phi_k(0) = 0$. Then, ϕ_n and Ω_n have no common roots, for $n = 1, \dots, k-1$. From case a), equations (33) hold for $n = 1, \dots, k-1$. Now we write equations (33) for $k-1$ and multiply by z, to obtain

$$\begin{cases} zAz\phi'_{k-1} - \frac{B}{2}z\phi_{k-1} - z\Theta_{k-1,2}\phi^*_{k-1} = l_{k-1}z\phi_{k-1} \\ zAz\Omega'_{k-1} + \frac{B}{2}z\Omega_{k-1} - Cz\phi_{k-1} + z\Theta_{k-1,2}\Omega^*_{k-1} = l_{k-1}z\Omega_{k-1} \end{cases}$$

By substituting

$$\phi_k(z) = z\phi_{k-1}(z),\ \phi^*_k(z) = \phi^*_{k-1}(z),\ z\phi'_{k-1}(z) = \phi'_k(z) - \phi_{k-1}(z)$$

and

$$\Omega_k(z) = z\Omega_{k-1}(z),\ \Omega^*_k(z) = \Omega^*_{k-1}(z),\ z\Omega'_{k-1}(z) = \Omega'_k(z) - \Omega_{k-1}(z)$$

in previous equations follows

$$\begin{cases} zA\phi'_k - \frac{B}{2}\phi_k - z\Theta_{k-1,2}\phi^*_k = (l_{k-1} + A)\phi_k \\ zA\Omega'_k + \frac{B}{2}\Omega_{k1} - C\phi_k + z\Theta_{k-1,2}\Omega^*_k = (l_{k-1} + A)\Omega_k \end{cases}$$

and we obtain (27) and (28) for $n = k$ with $G_k = l_{k-1} + A$ and $H_k = z\Theta_{k-1,2}$. Further, if $\phi_{k+1}(0) = \dots = \phi_{k+k_0}(0) = 0$, and $\phi_{k+k_0+1}(0) \neq 0$ for some $k_0 \in \mathbb{N}$, using the same procedure as before, the differential relations (27) and (28) are obtained for $n = k+1, \dots, k+k_0$, with

$$G_n = l_{k-1} + (n-k+1)A,\ H_n = z^{n-k+1}\Theta_{k-1,2}.$$

Case c): If $\phi_n(0) = 0,\ \forall n \geq n_0$, then ϕ_n and Ω_n are polynomials of the Bernstein-Szegő type,

$$\phi_n(z) = z^{n-n_0+1}\phi_{n_0-1}(z),\ \Omega_n(z) = z^{n-n_0+1}\Omega_{n_0-1}(z).$$

Applying the same procedure as before we conclude that equations (27) and (28) hold for $n \in \mathbb{N}$, and, $\forall n \geq n_0$, the polynomials G_n and H_n are given by

$$G_n = l_{n_0-1} + (n-n_0+1)A,\ H_n = z^{n-n_0+1}\Theta_{n_0-1,2}.$$

On the other hand, if we replace Θ_n by $2h_nz^n\Theta_{n,2}$ in (31) we get

$$\left\{-zA\left(\frac{Q_n}{\phi_n}\right)' - B\left(\frac{Q_n}{\phi_n}\right)\right\}\phi_n^2 = \Theta_{n,2}(z)2h_nz^n.$$

Using (9) we get

$$\left\{-zA\left(\frac{Q_n}{\phi_n}\right)' - B\left(\frac{Q_n}{\phi_n}\right)\right\}\phi_n^2 = \Theta_{n,2}(z)(\phi^*_nQ_n + \phi_nQ^*_n).$$

Therefore, $\forall n \in \mathbb{N}$,

$$\left\{zAQ'_n + \frac{B}{2}Q_n + \Theta_{n,2}Q^*_n\right\}\phi_n = \left\{zA\phi'_n - \frac{B}{2}\phi_n - \Theta_{n,2}\phi^*_n\right\}Q_n.$$

If we distinguish the two cases (see corollary 2):
a) ϕ_n and Q_n have no common roots, $\forall n \in \mathbb{N}$, i.e., $\phi_n(0) \neq 0, \forall n \in \mathbb{N}$;
b) ϕ_n and Q_n have the common root $z = 0$;

then, applying the same procedure as before, we conclude that, in both cases, there exists a polynomial L_n such that

$$\begin{cases} zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^* = L_n\phi_n \\ zAQ_n' + \frac{B}{2}Q_n + \Theta_{n,2}Q_n^* = L_nQ_n. \end{cases}$$

Since $L_n = l_n$, we obtain (29) with $G_n = l_n$ and $H_n = \Theta_{n,2}$.

To prove the converse result we use (6) and (7) in equation (28), thus obtaining

$$zA\left(Q_n' - \phi_n'F - \phi_nF'\right) = (G_n - \frac{B}{2})\left(Q_n - \phi_nF\right) - H_n\left(Q_n^* + \phi_n^*F\right) + C\phi_n,$$

i.e.,

$$\begin{aligned} zAQ_n' + (\frac{B}{2} - G_n)Q_n + H_nQ_n^* \\ = \left\{zA\phi_n' - G_n\phi_n + \frac{B}{2}\phi_n - H_n\phi_n^*\right\}F + \left\{zAF' + C\right\}\phi_n. \end{aligned}$$

From (27) and (29) we obtain $\{zAF' + BF + C\}\phi_n = 0$, and $zAF' + BF + C = 0$ follows. □

Remark 2. Moreover, from (7) and using the same reasoning as before, we deduce the equations for ϕ_n^* and Q_n^*, $\forall n \in \mathbb{N}$,

$$zA(\phi_n^*)' = (S_n + B/2)\phi_n^* - T_n\phi_n \tag{34}$$

$$zA(Q_n^*)' = T_nQ_n + (S_n - B/2)Q_n^* \tag{35}$$

where S_n, T_n are bounded degree polynomials.

From the differential equations (27), (29), (34) and (35), we obtain a differential system for semi-classical orthogonal polynomials on the unit circle (analogue to the equations deduced in [7], for the real case).

Theorem 5. *Let $\{\phi_n\}$ be a sequence of monic orthogonal polynomials with respect to a semi-classical functional u, such that $D(Au) = Bu$. If u is positive definite and w is the absolutely continuous part of the corresponding measure, then the following equations hold,*

$$zA\begin{bmatrix} \phi_n & Q_n/w \\ \phi_n^* & -Q_n^*/w \end{bmatrix}' = \begin{bmatrix} G_n - \tilde{B}/2 & H_n \\ -T_n & S_n - \tilde{B}/2 \end{bmatrix}\begin{bmatrix} \phi_n & Q_n/w \\ \phi_n^* & -Q_n^*/w \end{bmatrix}, \ \forall n \in \mathbb{N}$$

where $\tilde{B}(z) = -iB(z) - zA'(z)$, and G_n, H_n, S_n, T_n are bounded degree polynomials.

Proof. If u satisfies $\mathrm{D}(Au) = Bu$ then the corresponding F satisfies $zAF' = \tilde{B}F + C$, with $\tilde{B} = -iB - zA'$, and C a polynomial (see corollary 3 of theorem 2).

From theorem 4 and the subsequent remark we have the following equations,

$$zA\begin{bmatrix} Q_n'/w \\ -(Q_n^*)'/w \end{bmatrix} = (\mathcal{B}_n + \frac{\tilde{B}}{2}\, I)\begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix} \tag{36}$$

and

$$zA\begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}' = (\mathcal{B}_n - \frac{\tilde{B}}{2}\, I)\begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix} \tag{37}$$

with $\mathcal{B}_n = \begin{bmatrix} G_n & H_n \\ -T_n & S_n \end{bmatrix}$ and I the identity matrix of order two.

On the other hand, since $w'(z)/w(z) = \tilde{B}(z)/(zA(z))$, (see [18]) we obtain

$$zA \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}' = zA \begin{bmatrix} Q_n'/w \\ -(Q_n^*)'/w \end{bmatrix} - \tilde{B} \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix} \tag{38}$$

Substituting (36) in (38) we get

$$zA \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}' = (\mathcal{B}_n - \frac{\tilde{B}}{2} I) \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix} \tag{39}$$

Finally, from (37) and (39) we obtain the required differential system. □

Acknowledgments: This work was supported by CMUC, Department of Mathematics, University of Coimbra, and Intas Network Constructive Complex Approximation ref. 03-51-6637. The second author was supported by FCT, Fundação para a Ciência e Tecnologia, with grant ref. SFRH/BD/25426/2005.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.2,243-252,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Weighted Composition Operator between the Little α-Bloch Spaces and the Logarithmic Bloch

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Abstract: We characterize the boundedness and compactness of the weighted composition operator uC_φ between the logarithmic Bloch space β_L and the little α-Bloch spaces β_α^0 on the unit disk. Some necessary and sufficient conditions are given for which uC_φ is a bounded or compact operator between β_L and β_α^0.

MSC 2000: 47B38; 30H05; 30D05.

Keywords: Weighted composition operators, Boundedness, Compactness, Weighted Bloch spaces.

1 Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbf{C}$, and $H(D)$ denote the set of all analytic functions on D. For $f \in H(D)$, let

$$\|f\|_{\beta_\alpha} = \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in D\}, \quad 0 < \alpha < +\infty,$$

$$\|f\|_{\beta_L} = \sup\{(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| : z \in D\}.$$

As in [7, 10], the α-Bloch space β_α consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_\alpha} < +\infty$ and the little α-Bloch space β_α^0 consists of all $f \in H(D)$ satisfying $\lim\limits_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0$; the logarithmic Bloch space β_L consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_L} < +\infty$ and the little logarithmic Bloch space β_L^0 consists of all $f \in H(D)$ satisfying $\lim\limits_{|z|\to 1}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0$. The β_L is a Banach space under the norm

$$\|f\|_L = |f(0)| + \|f\|_{\beta_L}. \tag{1}$$

In [7], the author proved that β_L^0 is a closed subspace and coincides with the closure of polynomials under the norm and also studied the multiplication operator on the β_L space and β_L^0 space. It is well known that with the norm

$\|f\|_\alpha = |f(0)| + \|f\|_{\beta_\alpha}$, β_α is a Banach space and β_α^0 is a closed subspace of β_α. It is easily proved that for $0 < \alpha < 1$, $\beta_\alpha \subsetneqq \beta_L \subsetneqq \beta_1$.

Let φ be a holomorphic self-map of D and $u \in H(D)$. The weighted composition operator uC_φ is defined by:

$$uC_\varphi f = uf \circ \varphi, \quad \text{for } f \in H(D).$$

It is easy to see that this operator is linear. We can regard the operator as a generalization of a multiplication operator M_u, which denote by $M_u f = uf$ for $f \in H(D)$, and a composition operator C_φ, which denote by $C_\varphi f = f \circ \varphi$ for $f \in H(D)$. It is interesting to provide a function theoretic characterization of when φ and u induce a bounded or compact weighted composition operator between different function spaces. The weighted composition operator may be first studied on the Bloch space and the little Bloch space in [4]. In [5], Ohno, Stroethoff and Zhao got the characterization on φ and u for the weighted composition operator is bounded or compact between the α-Bloch spaces. Later, the author [7, 8] study the multiplication operator M_u in β_L, β_L^0, between $\beta_{\alpha_1}^0$ and β_{α_2} for $\alpha_1 \neq \alpha_2$. In this paper, we will consider the boundedness and the compactness of the weighted composition operator uC_φ between the little α-Bloch spaces β_α^0 and logarithmic Bloch space β_L on the unit disk.

Recall that if X, Y are Banach spaces and $T : X \longrightarrow Y$ is a linear operator, T is said to be compact if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a convergent subsequence. T is weakly compact, if for every bounded sequence $\{x_n\}$ in X, $\{T(x_n)\}$ has a weakly convergent subsequence. Every compact operator is weakly compact operator. A useful characterization of weak compactness is Gantmacher's Theorem [1]: T is weakly compact if and only if $T^{**}(X^{**}) \subset Y$ where T^{**} is the second adjoint of T and X is identified with its image under the natural embedding into its second dual X^{**}. In this paper, C denotes the positive constant depending only on the index α,, the C may differ at different places.

2 uC_φ from β_α^0 to β_L

Lemma 2.1 (see [7]) *If $f \in \beta_L$, then $|f(z)| \leq (2 + \ln(\ln \frac{2}{1-|z|}))\|f\|_L$.*

Lemma 2.2 (see [8, 10]) *Let $\alpha > 0$ and $f \in \beta_\alpha$. Then*
(1) $\|f_t\|_\alpha \leq \|f\|_\alpha$, $0 < t < 1$, where $f_t(z) = f(tz)$;
(2) $|f(z)| \leq C\|f\|_\alpha$, where $\alpha < 1$;
(3) $|f(z)| \leq C \ln \dfrac{2}{1-|z|^2}\|f\|_\alpha$, where $\alpha = 1$;
(4) $|f(z)| \leq \dfrac{C}{(\alpha-1)(1-|z|)^{\alpha-1}}\|f\|_\alpha$, where $\alpha > 1$.

Lemma 2.3 *Let $\alpha > 0$ and X be a Banach space. Then $uC_\varphi : \beta_\alpha^0 \to X$ is a weakly compact operator if and only if uC_φ is compact.*

Proof By Theorem 14 and 15 in [10] we know that $(L_a^1)^* \cong \beta_\alpha$ and $(\beta_\alpha^0)^* \cong L_a^1$, where L_a^1 is the Bergman space of analytic function on D. We also have $L_a^1 \cong l^1$, and l^1 has the Schur property[6]. Then we easily complete the proof.

Lemma 2.4 (see [8]) *Let $t > 0$ and $f \in H(D)$. Then*

(1) $\sup_{z\in D}(1-|z|)^t|f(z)| < +\infty$ if and only if $\sup_{z\in D}(1-|z|)^{t+1}|f'(z)| < +\infty$.

(2) $\lim_{|z|\to 1}(1-|z|)^t|f(z)| = 0$ if and only if $\lim_{|z|\to 1}(1-|z|)^{t+1}|f'(z)| = 0$.

Theorem 2.1 *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Let $\alpha > 0$. Then the following statements are equivalent:*

(a) $uC_\varphi : \beta_\alpha^0 \to \beta_L$ is bounded;

(b) $uC_\varphi : \beta_\alpha \to \beta_L$ is bounded;

(c) $uC_\varphi : \beta_\alpha^0 \to \beta_L$ is weakly compact;

(d) $uC_\varphi : \beta_\alpha^0 \to \beta_L$ is compact;

(e) (i) If $\alpha > 1$, $\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| < +\infty$ and

$$\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)| < +\infty.$$

(ii) If $\alpha = 1$, $\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})\ln(\frac{2}{1-|\varphi(z)|^2})|u'(z)| < +\infty$ and

$$\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{1-|\varphi(z)|^2}|\varphi'(z)u(z)| < +\infty.$$

(iii) If $0 < \alpha < 1$, $u \in \beta_L$ and $\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)| < +\infty$.

Proof $(e) \Longrightarrow (a)$. First we consider the case that $\alpha > 1$. Suppose that u and φ satisfy the condition (i) in (e). For $f \in \beta_\alpha^0$, we have $\sup_{z\in D}(1-|z|)^{\alpha-1}|f(z)| \leq C\|f\|_\alpha < +\infty$ by Lemma 2.4. It follows that

$$\begin{aligned}
\|uC_\varphi f\|_{\beta_L} &\leq \sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|u'(z)f(\varphi(z))| \\
&+ \sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|u(z)||f'(\varphi(z))\varphi'(z)| \\
&= \sup_{z\in D}(1-|\varphi(z)|^2)^{\alpha-1}|f(\varphi(z))|\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| \\
&+ \sup_{z\in D}(1-|\varphi(z)|^2)^{\alpha}|f'(\varphi(z))|\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^{\alpha}}|\varphi'(z)u(z)| \leq C\|f\|_\alpha,
\end{aligned}$$

and by Lemma 2.3, $|u(0)f(\varphi(0))| \le \dfrac{C|u(0)|}{(\alpha-1)(1-|\varphi(0)|)^{\alpha-1}}\|f\|_\alpha$. Hence uC_φ is bounded from β_α^0 to β_L.

Secondly, let $\alpha = 1$. Assume that $f \in \beta_1^0$ and that u and φ satisfy the condition (ii) in (e). From Lemma 2.2, we have

$$\|uC_\varphi f\|_{\beta_L} \le C\|f\|_1 \sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}\ln(\frac{2}{1-|\varphi(z)|^2})|u'(z)|$$

$$+\|f\|_{\beta_1}\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{1-|\varphi(z)|^2}|\varphi'(z)u(z)| \le C\|f\|_1,$$

and by Lemma 2.3, $|u(0)f(\varphi(0))| \le C|u(0)|\ln(\dfrac{2}{1-|\varphi(0)|^2})\|f\|_1$. Hence uC_φ is bounded from β_1^0 to β_L.

Finally, let $0 < \alpha < 1$. Similarly, Suppose that u and φ satisfy the condition (iii) in (e) . By Lemma 2.1 we have $\sup_{z\in D}|f(z)| \le C\|f\|_\alpha$ for every $f \in \beta_\alpha^0$. It follows that

$$\|uC_\varphi f\|_{\beta_L} \le \sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|u'(z)f(\varphi(z))|$$

$$+\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|u(z)||f'(\varphi(z))\varphi'(z)|$$

$$\le C\|u\|_L + \|f\|_{\beta_\alpha}\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)u(z)| \le C\|f\|_\alpha.$$

Hence uC_φ is bounded from β_α^0 to β_L.

$(a) \Longrightarrow (b)$. Suppose that $uC_\varphi : \beta_\alpha^0 \to \beta_L$ is bounded. It is clear that for any $f \in \beta_\alpha$, we have $f_t \in \beta_\alpha^0$ for all $0 < t < 1$. Then, according Lemma 2.2,

$$\|uC_\varphi(f_t)\|_L \le \|uC_\varphi\|\|f_t\|_\alpha \le \|uC_\varphi\|\|f\|_\alpha < +\infty.$$

Hence $\|uC_\varphi(f)\|_L \le \|uC_\varphi\|\|f\|_\alpha < +\infty$, which shows uC_φ is bounded from β_α to β_L.

$(b) \Longrightarrow (c)$. Since $(\beta_\alpha^0)^{**} \cong \beta_\alpha$, by Gantmacher's Theorem, we know that $uC_\varphi : \beta_\alpha^0 \to \beta_L$ is weakly compact if and only if $uC_\varphi : \beta_\alpha \to \beta_L$ is bounded. It follows uC_φ is weakly compact from β_α^0 into β_L.

$(c) \Longrightarrow (d)$. By Lemma 2.3, it is obvious.

$(d) \Longrightarrow (e)$. Suppose that uC_φ is compact from β_α^0 to β_L. It is obvious that uC_φ is bounded from β_α^0 to β_L. Taking the constant function and $f(z) = z$ in β_α^0 respectively, we get $u \in \beta_L$ and

$$\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|\varphi'(z)u(z)| < +\infty. \tag{2}$$

First, let $\alpha > 1$. Fixed $w \in D$. Let the test function

$$f_w(z) = \frac{\alpha}{(1-\overline{\varphi(w)}z)^{\alpha-1}} - \frac{(\alpha-1)(1-|\varphi(w)|^2)}{(1-\overline{\varphi(w)}z)^{\alpha}}$$

for $z \in D$. It is obvious that $f_w \in \beta_\alpha^0$ for every $w \in D$. Since

$$f_w'(z) = \frac{\alpha(\alpha-1)\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)^{\alpha}} - \frac{\alpha(\alpha-1)(1-|\varphi(w)|^2)\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)^{\alpha+1}},$$

We get $\sup_{w\in D} \|f_w\|_\alpha \leq \alpha + 3(\alpha-1)\alpha 2^\alpha$ by a direct calculation. Noting that $f_w'(\varphi(w)) = 0$ and $f_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha-1}}$, we have

$$\begin{aligned}\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^{\alpha-1}}\quad|u'(w)| &= (1-|w|^2)\ln(\frac{2}{1-|w|})|u'(w)f_w(\varphi(w))|\\ &= (1-|w|^2)\ln\frac{2}{1-|w|}|(uC_\varphi f_w)'(w)|\\ &\leq \|uC_\varphi f_w\|_L \leq \|uC_\varphi\|\|f_w\|_\alpha \leq C\|uC_\varphi\| < +\infty,\end{aligned}$$

which showing that the first condition in (i) holds.

Next, fix $w \in D$ for $\varphi(w) \neq 0$. We take the test function

$$g_w(z) = \frac{1}{\overline{\varphi(w)}}(\frac{1-|\varphi(w)|^2}{(1-\overline{\varphi(w)}z)^{\alpha}} - \frac{1}{(1-\overline{\varphi(w)}z)^{\alpha-1}})$$

for $z \in D$. It is clear that $g_w \in \beta_\alpha^0$ for every $w \in D$ such that $\varphi(w) \neq 0$. By a direct calculation, we get $\sup_{w\in D} \|g_w\|_\alpha \leq 1 + \alpha 2^{\alpha+2}$. Since $g_w(\varphi(w)) = 0$ and $g_w'(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha}}$, it follows that

$$\begin{aligned}&(1-|w|^2)\ln(\frac{2}{1-|w|})|u(w)g_w'(\varphi(w))\varphi'(w)| = (1-|w|^2)\ln\frac{2}{1-|w|}|(uC_\varphi g_w)'(w)|\\ &\leq \|uC_\varphi g_w\|_L \leq \|uC_\varphi\|\|g_w\|_\alpha \leq C\|uC_\varphi\| < +\infty.\end{aligned}$$

So,

$$\sup_{w\in D}\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)^{\alpha}}|\varphi'(w)u(w)| < +\infty.$$

For $w \in D$ with $\varphi(w) = 0$, by (2), we have

$$\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)\ln\frac{2}{1-|\varphi(w)|}}|u(w)\varphi'(w)| = \frac{1}{\ln 2}(1-|w|^2)\ln(\frac{2}{1-|w|})|u(w)\varphi'(w)| < +\infty.$$

This proves that the second condition in (i) is also necessary.

Secondly, let $\alpha = 1$. Fix $w \in D$, we take the test function

$$h_w(z) = 2\ln\frac{2}{1-\overline{\varphi(w)}z} - \frac{1}{\ln\frac{2}{1-|\varphi(w)|^2}}(\ln\frac{2}{1-\overline{\varphi(w)}z})^2$$

for $z \in D$. Since

$$h'_w(z) = \frac{2\overline{\varphi(w)}}{1-\overline{\varphi(w)}z} - 2\ln(\frac{2}{1-\overline{\varphi(w)}z})\frac{\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)}\frac{1}{\ln\frac{2}{1-|\varphi(w)|^2}},$$

then $h_w \in \beta_1^0$ for every $w \in D$ and $\sup_{w\in D}\|h_w\|_1 \le 13 < +\infty$. Since $h'_w(\varphi(w)) = 0$ and $h_w(\varphi(w)) = \ln\dfrac{2}{1-|\varphi(w)|^2}$, it follows that

$$\begin{aligned}&\sup_{w\in D}(1-|w|^2)\ln\frac{2}{1-|w|}\ln(\frac{2}{1-|\varphi(w)|^2})|u'(w)|\\ &= \sup_{w\in D}(1-|w|^2)\ln(\frac{2}{1-|w|})|(uC_\varphi h_w)'(w)|\\ &\le \|uC_\varphi h_w\|_{\beta_L} \le \|uC_\varphi\|\|h_w\|_1 \le C\|uC_\varphi\| < +\infty,\end{aligned}$$

which proves that the first condition in (ii) holds. That the second condition in (ii) holds is also proved as before.

Finally, let $0 < \alpha < 1$. The proof is the similar to that in (i). The details are omitted. This completes the proof of Theorem 2.1.

In the following corollary, we need some notion. Suppose X and Y are two Banach spaces of analytic functions on D. A complex valued function u in D is called a multiplier from X into Y, if $uf \in Y$ for every $f \in X$. The set of all multipliers from X into Y will be denoted by $M(X,Y)$. Using the closed graph theorem, it is easily seen that every $u \in M(X,Y)$ defines a bounded multiplication operator $M_u : f \to uf$ from X into Y. We also define the spaces of functions $WM(X,Y)$ and $CM(X,Y)$ as follows:

$$WM(X,Y) = \{u \in M(X,Y) : M_u \text{ is weakly compact}\},$$

$$CM(X,Y) = \{u \in M(X,Y) : M_u \text{ is compact}\}.$$

Corollary 2.1 *(1) If $\alpha \ge 1$, then $M(\beta_\alpha^0,\beta_L) = CM(\beta_\alpha^0,\beta_L) = WM(\beta_\alpha^0,\beta_L) = M(\beta_\alpha,\beta_L) = \{u : u = 0\}$;*

(2) If $0 < \alpha < 1$, then $M(\beta_\alpha^0,\beta_L) = CM(\beta_\alpha^0,\beta_L) = WM(\beta_\alpha^0,\beta_L) = M(\beta_\alpha,\beta_L) = \beta_L$.

Proof Assume that $\alpha \ge 1$. Then $u \in M(\beta_\alpha^0,\beta_L)$ implies that $\sup_{z\in D}|u(z)|(1-|z|^2)^{1-\alpha}\ln\dfrac{2}{1-|z|} < +\infty$ if $\alpha > 1$ and $\sup_{z\in D}|u(z)|\ln\dfrac{2}{1-|z|} < \infty$ if $\alpha = 1$ by Theorem 2.1. They both show that $\lim_{|z|\to 1}|u(z)| = 0$.

Next, suppose that $0 < \alpha < 1$. Then $u \in M(\beta_\alpha^0, \beta_L)$ if and only if $u \in \beta_L$ and $\sup_{z\in D}(1-|z|^2)^{1-\alpha}\ln\frac{2}{1-|z|}|u(z)| < \infty$ by Theorem 2.1. Using Lemma 2.1, one may easily prove that $u \in \beta_L$ implies $\sup_{z\in D}(1-|z|^2)^{1-\alpha}\ln(\frac{2}{1-|z|})|u(z)| < \infty$.

Corollary 2.2 *Let $\alpha > 0$ and φ an analytic self-map of D. Then the following statements are equivalent:*

(1) $C_\varphi : \beta_\alpha^0 \to \beta_L$ is bounded;
(2) $C_\varphi : \beta_\alpha^0 \to \beta_L$ is compact;
(3) $C_\varphi : \beta_\alpha^0 \to \beta_L$ is weakly compact;
(4) $C_\varphi : \beta_\alpha \to \beta_L$ is compact;
(5) $\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)^\alpha}|\varphi'(z)| < +\infty$.

3 uC_φ from β_L to β_α^0

Now we study the boundedness and compactness of the weighted composition operator uC_φ from the logarithmic Bloch-type spaces β_L to the little α-Bloch spaces β_α^0 on the unit disk.

Lemma 3.1 *Let $\alpha > 0$ and $U \subset \beta_\alpha^0$. Then U is compact if and only if it is closed, bounded and satisfies $\lim_{|z|\to 1}\sup_{f\in U}(1-|z|^2)^\alpha|f'(z)| = 0$.*

The proof is similar to Lemma 1 in [3], we omitted it. The following two lemmas can be found in [7] or [9].

Lemma 3.2 *Let $f(z) = \frac{(1-|z|)\ln\frac{2}{1-|z|}}{|1-z|\ln\frac{4}{|1-z|}}$, $z \in D$. Then $|f(z)| < 2$.*

Lemma 3.3 *Let $0 \le t \le 1$, $f(z) = \frac{(1-|z|)\ln\frac{2}{1-|z|}}{(1-|tz|)\ln\frac{2}{1-|tz|}}$, $z \in D$. Then $|f(z)| < 2$.*

Theorem 3.1 *Let u be an analytic function on the unit disk D and φ an analytic self-map of D. Let $\alpha > 0$. Then the following statements are equivalent:*

(a) $uC_\varphi : \beta_L \to \beta_\alpha^0$ is compact;
(b) $uC_\varphi : \beta_L^0 \to \beta_\alpha^0$ is compact;
(c) $\lim_{|z|\to 1}(1-|z|^2)^\alpha\ln(\ln\frac{2}{1-|\varphi(z)|})|u'(z)| = 0$
and $\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)| = 0.$

Proof $(c) \Longrightarrow (a)$. Suppose that $f \in \beta_L$. By Lemma 2.1, we get

$$
\begin{aligned}
&(1-|z|^2)^\alpha |(uC_\varphi(f))'(z)| \\
&\le (1-|z|^2)^\alpha |u'(z)f(\varphi(z))| + (1-|z|^2)^\alpha |u(z)f'(\varphi(z))\varphi'(z)| \\
&\le (1-|z|^2)^\alpha (2+\ln(\ln\frac{2}{1-|\varphi(z)|}))|u'(z)|\|f\|_L \\
&+\|f\|_{\beta_L}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)u(z)| \longrightarrow 0 \quad (|z| \longrightarrow 1^-).
\end{aligned}
$$

So $uC_\varphi(f) \in \beta_\alpha^0$, thus uC_φ is bounded from β_L to β_α^0. Then, by Lemma 3.1, we enough show that

$$\lim_{|z|\to 1}\sup\{(1-|z|^2)^\alpha |(uC_\varphi f)'(z)| : f \in \beta_L, \|f\|_L \le 1\} = 0.$$

However, it has just been proved above. Thus uC_φ is compact from β_L to β_α^0.

$(a) \Longrightarrow (b)$. It is obvious.

$(b) \Longrightarrow (c)$. Assume that uC_φ is compact from β_L^0 to β_α^0. Then $u = uC_\varphi 1 \in \beta_\alpha^0$. Also $u\varphi = uC_\varphi z \in \beta_\alpha^0$, thus $(1-|z|^2)^\alpha |u'(z)\varphi(z) + u(z)\varphi'(z)| \longrightarrow 0$ as $|z| \to 1$. Since $|\varphi| \le 1$ and $u \in \beta_\alpha^0$, we have

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\varphi'(z)u(z)| = 0. \tag{3}$$

On the other hand, by Lemma 3.1 we have

$$\lim_{|z|\to 1}\sup\{(1-|z|^2)^\alpha |(uC_\varphi f)'(z)| : f \in \beta_L^0, \|f\|_L \le M\} = 0 \tag{4}$$

for some $M > 0$. Fix $w \in D$, we take the test functions

$$f_w(z) = 2\ln\ln\frac{4}{1-\overline{\varphi(w)}z} - \frac{1}{\ln\ln\frac{4}{1-|\varphi(w)|^2}}(\ln\ln\frac{4}{1-\overline{\varphi(w)}z})^2$$

for $z \in D$. It is clear that $f_w \in \beta_L^0$ for every $w \in D$. By Lemma 3.2 and 3.3 we know that $\sup_{w\in D}\|f_w\|_L \le 16$. Since $f_w'(\varphi(w)) = 0$ and $f_w(\varphi(w)) = \ln\ln\frac{4}{1-|\varphi(w)|^2}$, we get

$$\lim_{|w|\to 1}(1-|w|^2)^\alpha \ln(\ln\frac{2}{1-|\varphi(w)|})|u'(w)| = 0 \tag{5}$$

by (4). Similarly, we take another functions

$$g_w(z) = \int_0^z (1-\frac{\overline{w}^2}{|w|^2}z^2)^{-1}(\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2})^{-1}\,dz.$$

It is clear that $g_w \in \beta_L^0$ for every $w \in D \setminus \{0\}$. By Lemma 3.2, we have

$$\sup_{z_1 \in D}(1-|z_1|^2)(\ln\frac{2}{1-|z_1|^2})|1-z_1^2|^{-1}|\ln\frac{4}{1-z_1^2}|^{-1} < 2 < +\infty,$$

applying $z_1 = \frac{\overline{w}}{|w|}z$, we have

$$\sup_{z \in D}(1-|z|^2)(\ln\frac{2}{1-|z|^2})|1-\frac{\overline{w}^2}{|w|^2}z^2||\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2}|^{-1} < 2 < +\infty.$$

Hence we have $\sup_{w \in D\setminus\{0\}} \|g_w\|_L < 4$. Then by Lemma 2.1 we get

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)|$$

$$\leq 2(1-|z|^2)^\alpha|(uC_\varphi f_w)'(z)| + 2(1-|z|^2)^\alpha(2+\ln(\ln\frac{2}{1-|\varphi(z)|}))|u'(z)|$$

for $\varphi(z) \neq 0$. Thus by (4) and (5) it follows that

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)| = 0$$

for $\varphi(z) \neq 0$. However, if $\varphi(z) = 0$, by (3), we get $\lim_{|z|\to 1}(1-|z|^2)^\alpha|u(z)\varphi'(z)| = 0$. This completes the proof of Theorem 3.1.

Remark 1 *If the β_L with the norm (1) is isometric to the second dual $(\beta_L^0)^{**}$, we can prove that $uC_\varphi : \beta_L \to \beta_\alpha^0$ is bounded if and only if $uC_\varphi : \beta_L \to \beta_\alpha^0$ is compact.*

However, we have the following result.

Theorem 3.2 *Let $\alpha > 0$ and φ an analytic self-map of D. Then the following statements are equivalent:*

(a) $C_\varphi : \beta_L \to \beta_\alpha^0$ is bounded;
(b) $C_\varphi : \beta_L^0 \to \beta_\alpha^0$ is compact;
(c) $C_\varphi : \beta_L \to \beta_\alpha^0$ is compact;
(d) $\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)| = 0$.

For the proof of this theorem, we need another lemma.

Lemma 3.4 (see [2]) *There exist two functions, f, $g \in \beta_L$ such that*

$$|f'(z)| + |g'(z)| \geq \frac{C}{(1-|z|)\ln\frac{2}{1-|z|}}.$$

The proof of Theorem 3.2 It is obvious that $(b) \Longleftrightarrow (c) \Longleftrightarrow (d)$ by Theorem 2.2. We only prove $(a) \Longleftrightarrow (d)$. Assume that C_φ is bounded from β_L to β^0_α. Then using the functions of Lemma 3.4, we get

$$\begin{aligned}
&(1-|z|^2)^\alpha|(C_\varphi(f))'(z)| + (1-|z|^2)^\alpha|(C_\varphi(g))'(z)| \\
&\geq (1-|z|^2)^\alpha(|f'(\varphi(z))| + |g'(\varphi(z))|)|\varphi'(z)| \\
&\geq C\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)|.
\end{aligned}$$

On the other hand, one may easily prove $(d) \Longrightarrow (a)$. This completes the proof of Theorem 3.2.

Acknowledgments

The author is partially supported by National Science Foundation of Fujian Province (No: 2006J0201).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.2,253-258,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

DERIVATIVE-FREE CHARACTERIZATIONS OF BLOCH SPACES

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Abstract: We obtain some new characterizations for Bloch spaces on the unit ball of $\mathbb{C}^n$ in terms of the function $|f(z) - f(w)|/|z - w|$.

MSC 2000: 32A18; 32A37.

Keywords: Bloch space, little Bloch space, derivative-free.

1 Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball in the complex space $\mathbb{C}^n$ of dimension n, $d\nu$ be the normalized Lebesgue measure of B. The class of all holomorphic functions on B is denote by $H(B)$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in the complex vector space $\mathbb{C}^n$ and $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. The ball centered at z with radius r will be denoted by $B(z, r)$. Let $Aut(B)$ be the group of all biholomorphic selfmaps of B. It is known that $Aut(B)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle},$$

where $s_a = (1 - |a|^2)^{1/2}$, P_a is the orthogonal projection into the space spanned by a, i.e., $P_a z = (\langle z, a\rangle a)/|a|^2$, $|a|^2 = \langle a, a\rangle$, $P_0 z = 0$, and $Q_a = I - P_a$ (see [8], [11]).

Let $d\lambda(z) = (1 - |z|^2)^{-n-1} d\nu(z)$. Then $d\lambda(z)$ is a Möbius invariant measure, that is, for any $\psi \in Aut(B)$ and $f \in L^1(B)$,

$$\int_B f(z) d\lambda(z) = \int_B f \circ \psi(z) d\lambda(z).$$

Let $E(z, r) = \{w \in B : |\varphi_z(w)| < r\}$. Then $E(z, r)$ is an ellipsoid and its

volume is given by

$$\nu(E(z,r)) = \frac{r^{2n}(1-|z|^2)^{n+1}}{(1-r^2|z|^2)^{n+1}}. \tag{1}$$

Set $|E(z,r)| = \nu(E(z,r))$. For $r > 0$ and $w \in E(z,r)$, we have (see [4, 8])

$$\frac{1}{1+r} \leq \frac{|1-\langle z,w\rangle|}{1-|w|^2} \leq \frac{2}{1-r^2}, \qquad \frac{1}{2} \leq \frac{|1-\langle z,w\rangle|}{1-|z|^2} \leq \frac{2}{1-r}. \tag{2}$$

It follows from (1) and (2) that

$$(1-|z|^2)^{n+1} \asymp (1-|w|^2)^{n+1} \asymp |1-\langle z,w\rangle|^{n+1} \asymp |E(z,r)|, \tag{3}$$

when $w \in E(z,r)$.

For $f \in H(B)$, let ∇f denote the complex gradient of f, i.e.

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right).$$

For $f \in H(B)$ and $z \in B$, set

$$Q_f(z) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|\langle \nabla f(z), \bar{w}\rangle|}{(H_z(w,w))^{1/2}},$$

where $H_z(w,w)$ is the Bergman metric on B, i.e.

$$H_z(w,w) = \frac{n+1}{2} \frac{(1-|z|^2)|w|^2 + |\langle w,z\rangle|^2}{(1-|z|^2)^2}.$$

The Bloch space $\mathcal{B}(B)$, which was introduced by Timoney (see [9]), is the space of all $f \in H(B)$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in B} Q_f(z) < \infty.$$

The little Bloch space $\mathcal{B}_0(B)$ is the space of all $f \in H(B)$ such that $\lim_{|z|\to 1} Q_f(z) = 0$.

For $f \in H(B^n)$, Nowak [4] proved that $f \in \mathcal{B}$ if and only if

$$M_1 = \sup_{\substack{z,w \in B^n \\ z \neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \cdot \frac{|f(z)-f(w)|}{|w - P_w z - (1-|w|^2)^{1/2} Q_w z|} < \infty. \tag{4}$$

In [7], Ren and Tu replaced the above condition by

$$M_2 = \sup_{\substack{z,w \in B^n \\ z \neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \cdot \frac{|f(z)-f(w)|}{|z-w|} < \infty. \tag{5}$$

Moreover, they proved that $f \in \mathcal{B}_0$ if and only if

$$\lim_{|z|\to 1} \sup_{\substack{w \in B^n \\ z \neq w}} (1-|z|^2)^{1/2}(1-|w|^2)^{1/2} \cdot \frac{|f(z)-f(w)|}{|z-w|} = 0. \tag{6}$$

If $n = 1$, the conditions (4) and (5) become the same thing which was established by Holland and Walsh [2]. Since

$$|w - P_w z - (1-|w|^2)^{1/2} Q_w z| = |P_w(w-z) + (1-|w|^2)^{1/2} Q_w(w-z)| \leq |w-z|,$$

it is easy to see that (4) implies (5). Hence (5) is weaker than (4) if $n > 1$. See [1, 3, 4, 5, 6, 7, 9, 10, 11] for more characterizations of the Bloch space of holomorphic functions.

The purpose of this paper is to establish some equivalent derivative-free characterizations of Bloch functions. The following theorems are our main results.

Theorem 1 *Suppose* $0 < r < 1$, $0 < p < \infty$, $\frac{n}{n+1} < \beta < \infty$, $1 < q < \infty$ *and* $f(z) \in H(B)$. *Then the following conditions are equivalent:*

(a) $f \in \mathcal{B}$;

(b) $\sup_{z\in B} \frac{1}{\nu(E(z,r))} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} d\nu(w) < \infty$;

(c) $\sup_{z\in B} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} d\lambda(w) < \infty$;

(d) $\sup_{z\in B} \int_B \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} (1-|\varphi_z(w)|^2)^{nq} d\lambda(w) < \infty$;

(e) $\sup_{z\in B} \int_B \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} (J_R\varphi_z(w))^\beta (1-|w|^2)^{(n+1)\beta} d\lambda(w) < \infty$.

Theorem 2 *Suppose* $0 < r < 1$, $0 < p < \infty$, $\frac{n}{n+1} < \beta < \infty$, $1 < q < \infty$ *and* $f(z) \in H(B)$. *Then the following conditions are equivalent:*

(a) $f \in \mathcal{B}_0$;

(b) $\lim_{|z|\to 1} \frac{1}{\nu(E(z,r))} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} d\nu(w) = 0$;

(c) $\lim_{|z|\to 1} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} d\lambda(w) = 0$;

(d) $\lim_{|z|\to 1} \int_B \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} (1-|\varphi_z(w)|^2)^{nq} d\lambda(w) = 0$;

(e) $\lim_{|z|\to 1} \int_B \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2} (1-|z|^2)^{p/2} (J_R\varphi_z(w))^\beta (1-|w|^2)^{(n+1)\beta} d\lambda(w) = 0$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2 Proof of Main Results

Proof of Theorem 1. (b)⇒(a). For every $\varepsilon \in (0,1/2)$,

$$\frac{1-|w|}{|z-w|} > (1-2\varepsilon)\frac{1-|z|}{|z-w|} > 1, \quad \text{when} \quad w \in B(z, 2\varepsilon(1-|z|)).$$

By the Cauchy's estimate and the subharmonicity we have that

$$\begin{aligned}
(1-|z|)^p|\nabla f(z)|^p &\leq C \sup_{w\in B(z,\varepsilon(1-|z|))} |f(w)-f(z)|^p \\
&\leq \frac{C}{(1-|z|)^{n+1}} \int_{B(z,2\varepsilon(1-|z|))} |f(w)-f(z)|^p d\nu(w) \\
&\leq C \int_{B(z,2\varepsilon(1-|z|))} |f(w)-f(z)|^p (1-|w|^2)^{-n-1} d\nu(w) \\
&\leq C \int_{B(z,2\varepsilon(1-|z|))} |f(w)-f(z)|^p \frac{(1-|w|^2)^{-n-1+p}}{|w-z|^p} d\nu(w).
\end{aligned} \tag{7}$$

For a fixed $r > 0$, choose a ball $B(z, 2\varepsilon(1-|z|)) \subset E(z,r)$. It follows from (3) and (7) that

$$\begin{aligned}
&(1-|z|)^p|\nabla f(z)|^p \\
\leq\ & C \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{-n-1+p} d\nu(w) \\
\leq\ & \frac{C}{\nu(E(z,r))} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2}(1-|z|^2)^{p/2} d\nu(w).
\end{aligned} \tag{8}$$

Taking $\sup_{z\in B}$ at both sides we get the desired result.

(a)⇒(d). By (5) and the Möbius invariance of $d\lambda(z)$, we have

$$\begin{aligned}
&\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq} d\lambda(w) \\
\leq\ & \sup_{\substack{z,w\in B \\ z\neq w}} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2}(1-|z|^2)^{p/2} \int_B (1-|\varphi_z(w)|^2)^{nq} d\lambda(w) \\
\leq\ & C \int_B (1-|\varphi_z(w)|^2)^{nq} d\lambda(w) \leq C \int_B (1-|w|^2)^{nq} d\lambda(w) \leq CB(n, n(q-1)) < \infty,
\end{aligned}$$

where the condition $q > 1$ and the property of Beta function are used.

(d)⇒(b). For a fixed $r > 0$, since $1-|\varphi_z(w)|^2 > 1-r^2$ when $w \in E(z,r)$, we obtain

$$\begin{aligned}
&\frac{1}{\nu(E(z,r))} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2}(1-|z|^2)^{p/2} d\nu(w) \\
\leq\ & \frac{(1-r^2)^{-nq}}{\nu(E(z,r))} \int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq} d\nu(w)
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w)\\
&\le C\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w).
\end{aligned}
$$

(b)⇔(c). For a fixed $r\in(0,1)$, $\nu(E(z,r))\asymp(1-|z|^2)^{n+1}\asymp(1-|w|^2)^{n+1}$ when $w\in E(z,r)$, then the desired result follows.

(e)⇔(d). Let $q=\frac{n+1}{n}\beta$. Then

$$
\begin{aligned}
&\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(J_R\varphi_z(w))^{\beta}(1-|w|^2)^{(n+1)\beta}d\lambda(w)\\
=&\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(J_R\varphi_z(w)(1-|w|^2)^{n+1})^{\beta}d\lambda(w)\\
=&\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w).
\end{aligned}
$$

It follows that *(e)* and *(d)* are equivalent.

Proof of Theorem 2. By the proof of Theorem 1,

$$
\begin{aligned}
&(1-|z|^2)^p|\nabla f(z)|^p\\
\le&\ \frac{C}{\nu(E(z,r))}\int_{E(z,r)} \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}d\nu(w)\\
\le&\ C\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w).
\end{aligned}
$$

Let $|z|\to1$ in above inequality, we obtain *(d)⇒(b)⇒(a)*.

(a)⇒(d). Let $f\in\mathcal{B}_0$. By (6), for arbitrary $\varepsilon>0$ there exists $r_0\in(0,1)$, when $z\in B\setminus B_{r_0}$,

$$
\frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}<\varepsilon \tag{9}
$$

for any $w\in B$ with $w\ne z$, where $B_{r_0}=\{z\in B:|z|\le r_0\}$. Write

$$
\begin{aligned}
&\int_B \frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w)\\
=&\ \Big(\int_{B_{r_0}}+\int_{B\setminus B_{r_0}}\Big)\frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w)\\
=&\ I_1+I_2. \qquad (10)
\end{aligned}
$$

It follows from (3) that

$$
\begin{aligned}
I_1 &\le \sup_{\substack{z,w\in B\\ z\ne w}}\frac{|f(w)-f(z)|^p}{|w-z|^p}(1-|w|^2)^{p/2}(1-|z|^2)^{p/2}\int_{B_{r_0}}(1-|\varphi_z(w)|^2)^{nq}d\lambda(w)\\
&\le C\int_{E(z,r_0)}(1-|w|^2)^{nq-n-1}d\nu(w)\\
&\le C(1-|z|^2)^{nq}. \qquad (11)
\end{aligned}
$$

On the other hand,

$$I_2 \le \varepsilon \int_{B \setminus B_{r_0}} (1-|\varphi_z(w)|^2)^{nq} d\lambda(w) \le \varepsilon \int_B (1-|w|^2)^{nq-n-1} d\nu(w) \le C\varepsilon. \quad (12)$$

Hence, for given ε above, there exists $r_1 \in (0,1)$ such that $(1-|z|^2)^{nq} < \varepsilon$ when $z \in B \setminus B_{r_1}$. Therefore, (10)-(12) give that

$$\lim_{|z|\to 1} \int_B \frac{|f(w)-f(z)|^p}{|w-z|^p} (1-|w|^2)^{p/2}(1-|z|^2)^{p/2}(1-|\varphi_z(w)|^2)^{nq} d\lambda(w) = 0.$$

(b)⇔(c). The desired result follows from (3).

(e)⇔(d). Let $q = \frac{n+1}{n}\beta$. As in the proof of Theorem 1, we know that they are equivalent. The proof of Theorem 2 is finished.

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TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL.10,NO.2,2008

Volume 10,Number 3 July 2008
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

A quarterly international publication of Eudoxus Press, LLC

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,271-281,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

A new interpolatory type quadrature rule for weighted Cauchy principal value integrals

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Abstract. This paper presents a new interpolatory type quadrature rule for approximating the weighted Cauchy principal value integrals $\int_{-1}^{1}(1-t^2)^{\lambda-1/2}f(t)/(t-c)dt$ where $-1/2<\lambda<1$. We prove that the rule has almost optimal stability property behaving in the form $O(K\log n+L)$, where K and L are constants depending only on c. Also, when $f(t)$ possesses continuous derivatives up to order $p\geq 0$ and the derivative $f^{(p)}(t)$ satisfies Hölder continuity of order ρ, we obtain that the rule has the convergence rate of $O((A+B\log n+n^{2\nu})n^{-p-\rho})$, where ν is as small as we like and A and B are constants depending on c.

Key words: Cauchy principal value integral, quadrature rule, trigonometric interpolation, singular integral, interpolatory type

AMS subject classifications: 30E20, 41A05, 42A15, 45E05

1. Introduction and results

The weighted Cauchy principal value integral(CPV) to be considered in this paper is of the form

$$Q(wf;c)=\fint_{-1}^{1}\frac{w(t)f(t)}{t-c}\,dt=\lim_{\epsilon\to 0}\left\{\int_{-1}^{c-\epsilon}+\int_{c+\epsilon}^{1}\right\}\frac{w(t)f(t)}{t-c}\,dt\,,\quad |c|<1\,, \tag{1.1}$$

where

$$w(t)=(1-t^2)^{\lambda-1/2},\quad -1/2<\lambda<1. \tag{1.2}$$

It have been extensively used to solving singular integral equations (SIE) occurring in several boundary value problems, particularly in aerodynamics, fluid and fracture mechanics and many other fields of physics and the engineering sciences(cf. [3, 8, 14] and the literature cited there in).

There exist many bibliographies on quadratures for the CPV integral (1.1). Among those, we specially mention the papers [1, 2, 10, 11, 12] concerning quadrature rules of interpolatory type. Usually, quadrature rules of interpolatory type for the CPV integral (1.1) are of the form

$$Q(wf;c) = \sum_{i=0}^{n} w_{ni}(c) f(t_{ni}) + R_n(f) \tag{1.3}$$

and obtained by interpolating the function $f(t)$ at $n+1$ distinct nodes $\{t_{ni}\}$ with a polynomial of degree n.

In [2], Elliott and Paget considered a rule of type (1.3) with the Jacobi weight $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$, for $\alpha, \beta > -1$, and they set $\{t_{ni}\}$ to be zeros of the Jacobi polynomial of degree $(n+1)$. They have obtained the bound

$$\sum_{i=0}^{n} |w_{ni}(c)| \leq K + L \log n, \tag{1.4}$$

where K and L are constants depending only on c. By using it, they derived a pointwise convergence property of $R_n(f) = O(n^{-s})$ for the function $f(t)$ Hölder continuous of order μ, where s is any positive real number less than μ. As well as a bound similar to (1.4), Monegato obtained the bound

$$\sum_{i=0}^{n} |w_{ni}(c)| \leq C \log^2 n,$$

where C is a constant depending on c in [11], provided that one takes the zeros of the function $T_n(t)$ (classical nodes) or $(1-t^2)U_{n-1}(t)$ (practical nodes) as the nodes $\{t_{ni}\}$. Here, $T_n(t)$ and $U_n(t)$ denote Chebyshev polynomials of first and second kind, respectively. In [1], Dagnino and Palamara Orsi also obtained a bound of type (1.4) with the weight function $w(t) \in H_{\mu}(-1,1) \cap L_p[-1,1]$, $p > 1$, when the nodes $\{t_{ni}\}$ are the zeros of $T_n(t)$ or $(1-t^2)U_{n-1}(t)$.

The main contribution of this article is to show that the inequality (1.4) holds for the practical nodes when the weight function $w(t)$ is given by (1.2). More specifically, by modifying the rule presented in [4, 5, 6, 7] we construct a new rule of interpolatory type such as (1.3).

First we change the variables of the integral in (1.1) by letting $t = \cos y$ and $c = \cos x$, then (1.1) becomes

$$\begin{aligned} Q(w(\cos\cdot)f(\cos\cdot);\cos x) &= \fint_0^{\pi} \frac{\sin^{2\lambda} y f(\cos y)}{\cos y - \cos x} dy \\ &= \int_0^{\pi} \frac{\sin^{2\lambda} y\, h(y) - \sin^{2\lambda} x\, h(x)}{\cos y - \cos x} dy \\ &:= H(h;x), \text{ say,} \end{aligned}$$

where $h(x) = f(\cos x)$. In the second equation(see [4]) we will use the identity

$$\fint_0^{\pi} \frac{1}{\cos y - \cos x} = 0, \quad x \in (0,\pi). \tag{1.5}$$

Define a trigonometric quadrature rule of interpolatory type for approximating $H(h;x)$ by

$$H_n(h;x) = H(p_n^h;x) = \sum_{k=0}^{n}{}'' b_{nk}J_k(x) = \sum_{k=0}^{n}{}'' h(x_{nk})\omega_{nk}(x), \tag{1.6}$$

where

$$\begin{aligned} &p_n^h(x) = \sum_{k=0}^{n}{}'' b_{nk}\cos kx, \quad b_{nk} = \frac{2}{n}\sum_{j=0}^{n}{}'' h(x_{nj})\cos kx_{nj}, \quad x_{nk} = \frac{\pi k}{n}, \\ &J_k(x) = H(\cos(k\cdot);x), \quad \omega_{nk}(x) = H(l_k^n;x), \\ &l_k^n(x) = \frac{2}{n}\sum_{j=0}^{n}{}'' \cos jx_{nk}\cos jx = \frac{(-1)^k}{n}\frac{\sin nx\sin x}{\cos x_{nk} - \cos x}, \quad x \neq x_{nk}. \end{aligned} \tag{1.7}$$

A summation symbol with double primes denotes a sum with first and last terms halved. The last identity of (1.7) yields that

$$\omega_{nk}(x) = \frac{(-1)^{k+1}}{n}\begin{cases} \dfrac{q_n(x) - q_n(x_{nk})}{\cos x - \cos x_{nk}}, & x \neq x_{nk}, \\ q_n^*(x), & x = x_{nk}, \end{cases} \tag{1.8}$$

where

$$q_n(x) = ⨍_0^\pi \frac{\sin^{2\lambda+1} y\sin ny}{\cos y - \cos x}dy, \quad q_n^*(x) = ⨍_0^\pi \frac{\sin^{2\lambda+1} y\sin ny}{(\cos y - \cos x)^2}dy. \tag{1.9}$$

The quadrature rule (1.6) is exact when the density function $h(x)$ is a trigonometric polynomial of degree $\leq n$, so it is interpolatory type. The reader can find the details of above in [4, 6].

Numerically, $\Lambda_n(x)$ which is defined by

$$\Lambda_n(x) = \sum_{k=0}^{n}{}'' |\omega_{nk}(x)| \tag{1.10}$$

is an important factor. Indeed, if h_1 and h_2 are two functions such that $\sup_{x\in[0,\pi]} |h_1(x) - h_2(x)| < \epsilon$, then (1.6) gives

$$|H_n(h_1;x) - H_n(h_2;x)| \leq \epsilon\Lambda_n(x).$$

Thus we can say that $\Lambda_n(x)$ gives the stability of the quadrature rule. In other words, when n increases, the magnitude of the left hand side is subject to $\Lambda_n(x)$. For this reason, we call $\Lambda_n(x)$ the stability factor.

In this article, we show that the stability factor $\Lambda_n(x)$ has the bound such as

$$\Lambda_n(x) = O(K\log n + L), \tag{1.11}$$

where K and L are constants depending only on fixed $x \in (0,\pi)$. The result of (1.11) is used to prove that the error $R_n(h;x)$ of the quadrature rule has the following behavior:

$$|R_n(h;x)| = O((A + B\log n + n^{2\nu})n^{-p-\rho}),$$

where $\nu > 0$ is as small as we like and constants A and B depend only on x, when the function $f(t)$ has continuous derivatives up to order $p \geq 0$ and its derivative $f^{(p)}(t)$ satisfies Hölder continuity of order ρ for $0 < \rho \leq 1$.

2. Recurrence relations for the quadrature weights

If we use the identity (1.5), then $J_k(x)$ in (1.7) can be written as follows:

$$J_k(x) = I_k(x) + \cos kx q_0(x), \tag{2.1}$$

where

$$I_k(x) = \int_0^\pi \frac{\cos ky - \cos kx}{\cos y - \cos x} \sin^{2\lambda} y dy, \quad q_0(x) = \fint_0^\pi \frac{\sin^{2\lambda} y}{\cos y - \cos x} dy. \tag{2.2}$$

For $\lambda > -1/2$, and $\lambda \neq 0, 1/2$, $q_0(x)$ satisfies

$$\begin{aligned} q_0(x) &= -\pi \tan(\lambda\pi) \sin^{2\lambda-1} x \\ &- 2^{2\lambda-1} \frac{\Gamma(\lambda - 1/2)\Gamma(\lambda + 1/2)}{\Gamma(2\lambda)} F(1, 1 - 2\lambda; \frac{3}{2} - \lambda; \sin^2 \frac{x}{2}), \end{aligned}$$

where $F(\cdot, \cdot; \cdot; \cdot)$ is the hypergeometric function and $\Gamma(\cdot)$ the gamma function (see, e.g., [15]). Also if $\lambda = 0$ or $\lambda = 1/2$, then from (1.5) and the definition of $q_0(x)$, we directly get

$$q_0(x) = \begin{cases} 0, & \lambda = 0, \\ \log \tan^2 \frac{x}{2}, & \lambda = 1/2. \end{cases}$$

The terms $I_k(x)$ in (2.2) can be calculated by the three-term recurrence relation(see [5])

$$I_{k+2}(x) - 2\cos x I_{k+1}(x) + I_k(x) = 2d_{k+1}, \quad k \geq 0$$

with initial values $I_0(x) = 0,\ I_1(x) = d_0$. Further, $I_n(x)$ has the explicit expression

$$I_n(x) = 2 \sum_{j=0}^{n-1}{}' d_j \frac{\sin(n-j)x}{\sin x}. \tag{2.3}$$

Here, d_k are defined by $d_k = \displaystyle\int_0^\pi \cos ky \sin^{2\lambda} y dy$ and satisfy the identities(see [5, (3.3), (3.4)])

$$d_{2k} = \frac{(-1)^k \Gamma(\lambda + 1/2)\Gamma(\lambda + 1)\sqrt{\pi}}{\Gamma(\lambda + k + 1)\Gamma(\lambda - k + 1)}, \quad (\lambda \neq 0), \quad \text{and} \quad d_{2k+1} = 0. \tag{2.4}$$

It also satisfy that

$$d_k = 2^{2\lambda} \frac{\Gamma(\lambda + 1/2)^2}{\Gamma(2\lambda + 1)} g_k,$$

for g_k satisfying

$$(2\lambda + 1 + k)g_{k+1} + (2\lambda + 1 - k)g_{k-1} = 0$$

and having initials $g_0 = 1$ and $g_1 = 0$. For details, we refer to [13].

We devote the rest of this section to find the asymptotic behavior of d_{2k} and a bound of $I_n(x)$. By some manipulation of gamma function, we can write d_{2k} as

$$d_{2k} = -\frac{\sqrt{\pi}\Gamma(\lambda + 1/2)\Gamma(\lambda + 1)}{\Gamma(\lambda)\Gamma(1 - \lambda)} \frac{\Gamma(k - \lambda)}{\Gamma(\lambda + k - 1)} \frac{1}{(\lambda + k - 1)(\lambda + k)}.$$

Using the asymptotic behavior (see [9, p. 15])

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\Big(1+\frac{(\alpha-\beta)(\alpha+\beta-1)}{2z}+O(|z|^{-2})\Big),$$

for $z>0$ and arbitrary constants α and β, we see that for $k \geq 2$

$$d_{2k} = -\frac{\sqrt{\pi}\Gamma(\lambda+1/2)\Gamma(\lambda+1)}{\Gamma(\lambda)\Gamma(1-\lambda)}c_k(\lambda)(1+O((k-1)^{-2})), \tag{2.5}$$

where

$$c_k(\lambda) = \frac{(k-1)^{1-2\lambda}}{(\lambda+k-1)(\lambda+k)}.$$

Lemma 2.1. *Assume that $\{a_k\}$ is a bounded sequence. Then*

$$\Big|\sum_{k=2}^{m} d_{2k}a_k\Big| \leq C \max_{k=2,\cdots,m}\Big|\sum_{i=2}^{k} a_i\Big|,$$

where C is a constant depending only on λ.

Proof. Consider

$$A = \sum_{k=2}^{m} c_k(\lambda)a_k,$$

where $c_k(\lambda)$ is in (2.5). And let $e_1 = 0$ and $e_k = \sum_{i=2}^{k} a_i$, for $k \geq 2$, then we know that $e_k - e_{k-1} = a_k$, $k \geq 2$. Thus A becomes

$$A = \sum_{k=2}^{m} c_k(e_k - e_{k-1}) = c_m e_m + \sum_{k=2}^{m-1}(c_k - c_{k+1})e_k.$$

By using the monotonicity of c_k, we can complete the proof. □

Using this lemma, we can obtain that $I_n(x)$ given in (2.3) is bounded in the case of $x = x_{nk}$.

Corollary 2.2. *For $x = x_{nk}$, $0<k<n$, $I_n(x)$ given in* (2.3) *has the bounds*

$$|I_n(x)| \leq C \begin{cases} \dfrac{1}{\sin^2 x}, & -1/2<\lambda<0\,, \\ \dfrac{1}{\sin x}, & 0<\lambda<\dfrac{1}{2}, \\ 1, & \dfrac{1}{2}<\lambda<1 \end{cases}$$

for some constant C depending only on λ.

Proof. From (2.3) and the fact $x = x_{nk}$,

$$I_n(x) = 2(-1)^{k+1}\sum_{j=1}^{n-1} d_j \frac{\sin jx}{\sin x}.$$

Hence, (2.5) and Lemma 1 yield immediately

$$|I_n(x)| \le C_1 \begin{cases} \Big| \displaystyle\sum_{j=2}^{[n/2]-1} \frac{\sin 2jx}{\sin x} \Big|, & -\dfrac{1}{2} < \lambda < 0, \\ \dfrac{1}{\sin x} + \dfrac{1}{\sin x} \displaystyle\sum_{j=2}^{[n/2]-1} c_j(\lambda), & 0 < \lambda < \dfrac{1}{2}, \\ 1 + \displaystyle\sum_{j=2}^{[n/2]-1} c_j(\lambda) j, & \dfrac{1}{2} < \lambda, \end{cases}$$

where $c_j(\lambda)$ is in (2.5) and C_1 is a constant depending only on λ. Since $2\sin\frac{x}{2}\sum_{j=1}^{n-1}\sin jx = \cos\frac{x}{2} - \cos(n-\frac{1}{2})x$, we can derive the first bound $C/\sin^2 x$ when $-1/2 < \lambda < 0$.

On the other hand, the inequality

$$c_j(\lambda) \le \begin{cases} (j-1+\lambda)^{-1-2\lambda}, & 0 < \lambda < 1/2, \\ (j-1)^{-1-2\lambda}, & 1/2 < \lambda < 1, \end{cases}$$

and the Riemann lower summation rule give

$$\sum_{j=2}^{[n/2]-1} c_j(\lambda) \le \int_1^{[n/2]} \frac{1}{(x-1+\lambda)^{1+2\lambda}} dx < C_2, \quad 0 < \lambda < 1/2.$$

Here, C_2 is a constant depending only on λ. The argument similar to this gives a constant bound for the case $\lambda > 1/2$. The proof is complete. □

3. Asymptotic behavior of $\Lambda_n(x)$

In this section, we shall estimate the asymptotic behavior of the stability factor $\Lambda_n(x)$ of (1.10) using the expression (1.8). Note that if p_n^1 denotes the trigonometric interpolation polynomial of degree n for the constant function 1, then

$$\Lambda_n(x) \ge \Big| {\sum_{k=0}^{n}}'' \omega_{nk}(x) \Big| = |H(p_n^1; x)| = |q_0(x)|,$$

where $q_0(x)$ is defined in (2.2).

Before estimating the upper bound of $\Lambda_n(x)$, we consider an explicit expression of $q_n(x)$ in (1.9). Note that $\frac{2}{\pi}{\sum_{j=0}^{n}}'' d_j \cos jx$ is a truncated cosine series of $\sin^{2\lambda} x$ and it converges uniformly to $\sin^{2\lambda} x$ as $n \to \infty$. Thus (2.1) and (2.3) yield

$$\begin{aligned} q_n(x) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2} &= \sin nx \sin x q_0(x) - 2{\sum_{j=0}^{n}}'' d_j \cos(n-j)x \\ &\sim \sin nx \sin x\Big(q_0(x) - 2{\sum_{j=0}^{n}}'' d_j \frac{\sin jx}{\sin x}\Big) - \pi \cos nx \sin^{2\lambda} x \end{aligned} \tag{3.1}$$

for sufficiently large n. Using this relation and the first equation of (1.8), the weights $\omega_{nk}(x)$ have the bounds

$$|\omega_{nk}(x)| \le C\Big(|\omega^1_{nk}(x)||S^1_n(x)| + |\omega^2_{nk}(x)|\sin^{2\lambda} x + |\omega^3_{nk}(x)|\Big),\ 0 < k < n, \tag{3.2}$$

where C is a constant independent of n, and

$$\omega^1_{nk}(x) = \frac{(-1)^{k+1}}{n}\frac{\sin nx \sin x}{\cos x - \cos x_{nk}}, \quad \omega^2_{nk}(x) = \frac{2(-1)^k}{n}\frac{\cos nx - (-1)^k}{\cos x - \cos x_{nk}},$$
$$\omega^3_{nk}(x) = \frac{2}{n}\frac{\sin^{2\lambda} x - \sin^{2\lambda} x_{nk}}{\cos x - \cos x_{nk}}, \quad S^1_n(x) = q_0(x) - 2\sum_{j=0}^{n}{}'' d_j \frac{\sin jx}{\sin x}.$$

Theorem 3.1. *The stability factor $\Lambda_n(x)$ defined in* (1.10) *has the asymptotic behavior*

$$\Lambda_n(x) = O(K\log n + L), \tag{3.3}$$

where K and L are constants depending only on x.

Proof. Assume that $x \in (x_{nl}, x_{n\,l+1})$ for some fixed $0 \le l < n/2$. Also set n to be even for convenience sake.

Now consider the estimation of $A = |\omega_{n0}(x)| + |\omega_{nn}(x)|$. By (1.8), (3.1), and (2.4), we obtain a upper bound of A in the following way:

$$|A| \le \frac{1}{n}\Big(|q_0(x)||\sin nx|\sin x + 2\Big|\sum_{j=0}^{n/2}{}'' d_{2j}(1-\cos(n-2j)x)\Big|\Big)\frac{2}{\sin^2 x}$$
$$\le 2|q_0(x)| + \frac{C_1}{n}\Big(\frac{2-\cos nx - \cos(n-2)x}{\sin^2 x} + \Big|\sum_{j=2}^{n/2-1} d_{2j}\frac{\sin^2(\frac{n}{2}-j)x}{\sin^2 x}\Big|\Big)$$

for some constant C_1. Since

$$2 - \cos nx - \cos(n-2)x = 2\sin^2\frac{n}{2}x + \sin^2 x\cos nx + \sin x \sin nx \cos x$$

and $|\sin kx/\sin x| \le k$, Lemma 1 gives

$$\begin{aligned} |A| &\le 2|q_0(x)| + C_1\Big(\frac{1}{\sin x} + 1 + \frac{1}{n}\sum_{j=1}^{n/2-1}\frac{\sin^2(\frac{n}{2}-j)x}{\sin^2 x}\Big) \\ &\le 2|q_0(x)| + C_1\sin^{-2} x. \end{aligned} \tag{3.4}$$

The remainder $\Lambda_n(x) - A/2$ is estimated by using (3.2). In order to do this, let

$$\sum_{k=1}^{n-1}|\omega^1_{nk}(x)| = A_1(x) + A_2(x),$$

where

$$A_1(x) = |\omega^1_{nl}(x)| + |\omega^1_{nl+1}(x)|,\ A_2(x) = \sum_{\substack{k=1\\ k\ne l,\ l+1}}^{n-1}|\omega^1_{nk}(x)|.$$

The inequalities

$$|\cos y - \cos x| \geq \begin{cases} \dfrac{2x}{\pi^2}|x-y|, & y \in [0,\pi/2), \\ \\ \dfrac{\sqrt{2}}{\pi}|x-y|, & y \in [\pi/2,\pi] \end{cases} \tag{3.5}$$

yield

$$|\cos x - \cos x_{nk}| \geq \begin{cases} \dfrac{2x}{n\pi}(l-k), \ k = 1,2,\cdots,l-1, \\ \\ \dfrac{2x}{n\pi}(k-l-1), \ k = l+2,\cdots,\dfrac{n}{2}, \\ \\ \dfrac{\sqrt{2}}{n}(k-l-1), \ x_{nl} < x < x_{n\,l+1} \leq \dfrac{\pi}{2} < x_{nk}. \end{cases}$$

Thus,

$$\begin{aligned} A_2(x) &\leq \frac{\pi \sin x}{2x}\Big[\sum_{k=1}^{l-1}\frac{1}{l-k} + \sum_{k=l+2}^{n/2}\frac{1}{k-l-1}\Big] + \frac{1}{\sqrt{2}}\sum_{k=n/2+1}^{n-1}\frac{1}{k-l-1} \\ &\leq (\pi+\sqrt{2})\sum_{k=1}^{n}\frac{1}{k} \leq (\pi+\sqrt{2})\log(en). \end{aligned}$$

To estimate $A_1(x)$, we use the generalized law of mean and the fact that $\sin x$ is increasing on $(0,\pi/2)$. So the term $|\omega_{nl}^1(x)|$ can be bounded as

$$\begin{aligned} |\omega_{nl}^1(x)| &= \frac{\sin x}{n}\,\frac{x - x_{nl}}{\cos x_{nl} - \cos x}\,\frac{|\sin nx|}{x - x_{nl}} \leq \frac{\sin x|\cos n\bar{x}|}{\sin x_{nl}}, \quad \bar{x} \in (x_{nl}, \frac{x_{nl} + x_{n\,l+1}}{2}) \\ &\leq \frac{\sin x_{n\,l+1}}{\sin x_{nl}} = \cos\frac{\pi}{n} + \cot x_{nl}\sin\frac{\pi}{n} \leq 2\cos\frac{\pi}{n} \leq 2. \end{aligned}$$

Through the similar way, we can obtain the same bound for $|\omega_{nl+1}^1(x)|$. In other words, we can get

$$\sum_{k=1}^{n-1}|\omega_{nk}^1(x)| \leq C_2\log(en) \tag{3.6}$$

for some constant C_2 independent of n. Also the analogous argument leads us to the bound of $\omega_{nk}^2(x)$:

$$\sum_{k=1}^{n-1}|\omega_{nk}^2(x)| \leq \frac{C_3}{\sin x}\log(en) \tag{3.7}$$

for some constant C_3 independent of n.

Furthermore, the inequalities in (3.5) and the Riemann lower summation rule give

$$\begin{aligned} \sum_{k=1}^{n-1}|\omega_{nk}^3(x)| &\leq \frac{\pi^2}{xn}\sum_{k=1}^{n-1}\Big|\frac{\sin^{2\lambda}x - \sin^{2\lambda}x_{nk}}{x - x_{nk}}\Big| \\ &\leq \frac{C_4}{x}\int_0^{\pi}\Big|\frac{\sin^{2\lambda}x - \sin^{2\lambda}y}{x-y}\Big|dy = C_5 < \infty, \end{aligned} \tag{3.8}$$

where C_4 and C_5 are independent of n. The bound of $S_n^1(x)$ is immediately derived from the bound of $|q_0(x)|$ and the proof of Corollary 2. Hence, by combining (3.4) and (3.6)-(3.8), we can complete the proof for the case $x \in (x_{nl}, x_{n\,l+1})$.

Finally consider the case of $x = x_{nl}$ for some $0 < l < n$. We can easily verify that, for $q_n(x)$ and $q_n^*(x)$ in (1.9),

$$\lim_{x \to x_{nk}} \frac{q_n(x) - q_n(x_{nk})}{\cos x - \cos x_{nk}} = q_n^*(x_{nk}).$$

Thus the proof can be done by using some limiting arguments in the proof for the case $x \in (x_{nl}, x_{n\,l+1})$. The details left to the reader. □

TABLE 1. Evaluation of $\Lambda_n(x)$ according to (1.8) and (1.9) when $\lambda = 1$.

x	stability						
	$\Lambda_4(x)$	$\Lambda_8(x)$	$\Lambda_{16}(x)$	$\Lambda_{32}(x)$	$\Lambda_{64}(x)$	$\Lambda_{128}(x)$	$\Lambda_{256}(x)$
$4\pi/7$	4.62154	5.15211	7.32288	8.31478	9.26716	11.289	12.3456
$5\pi/7$	3.16745	4.54696	5.7359	7.19066	8.11555	8.93716	10.5329
$6\pi/7$	3.1627	3.51303	4.10565	4.83477	5.61028	6.1445	6.60688

In Table 1, we list the stability factor $\Lambda_n(x)$'s for the node number n varying from 4 to 256 and fixed pole values $\pi i/7$, $i = 4, 5, 6$. In it, we can observe that the stability factor $\Lambda_n(x)$ increases about by 1 when the node number n varies as $n = 2^k$, $k = 2, 3, \cdots, 8$, for each fixed pole value x. Therefore, we can say that the estimation of (3.3) substantiates the actual growth presented in Table 1.

4. Convergence analysis

In this section, we derive a bound for the error $R_n(h; x) = H(h; x) - H_n(h; x)$ when the function $f(\cos x)$ is Hölder continuous.

Using Theorem 4 and Lemma 3.2 of [6], we obtain the following convergence theorem for the quadrature rule (1.6).

Theorem 4.1. *Let us consider the quadrature rule* (1.6). *Suppose the function $f(\tau)$ possesses continuous derivatives up to order $p \geq 0$ and the derivative $f^{(p)}(\tau)$ satisfies Hölder continuity of order ρ. Then the remainder term $R_n(h; x)$ satisfies*

$$|R_n(h; x)| = O((A + B \log n + n^{2\nu}) n^{-p-\rho}),$$

where A and B are constants depending only on x, and $\nu > 0$ is as small as we like.

Proof. Let p_n be any trigonometric polynomial of degree $\leq n$. Then, since $H_n(h; x)$ is an interpolatory type rule, we find that

$$|R_n(h; x)| \leq |H(r_n; x)| + |H_n(r_n; x)|, \tag{4.1}$$

where $r_n(x) = h(x) - p_n(x)$. The quadrature rule (1.6) shows that

$$|H_n(r_n; x)| \leq \max_{\tau \in [-1,1]} |f(\tau) - p_n(\cos^{-1} \tau)| \Lambda_n(x),$$

where $\Lambda_n(x)$ is the stability factor defined in (1.10). Now the behavior of (3.3) and Lemma 3.2 of [6] give

$$|H_n(r_n;x)| = O\Big((K_1 + L_1 \log n)n^{-p-\rho}\Big), \tag{4.2}$$

where K_1 and L_1 are constants depending on x. For estimating $|H(r_n;x)|$, we make use of (1.5) and the change of variables $y = \cos^{-1}\tau$, $x = \cos^{-1} t$. If we let $\bar{r}_n(\tau) = f(\tau) - p_n(\cos^{-1}\tau)$, then

$$|H(r_n;x)| \le |\bar{r}_n(t)||q_0(x)| + \int_{-1}^{1} \frac{(1-\tau^2)^{\lambda-1/2}}{|\tau - t|^{1-\nu}} \frac{|\bar{r}_n(\tau) - \bar{r}_n(t)|}{|\tau - t|^{\nu}} d\tau,$$

where $q_0(x)$ are defined in (2.2). Since for each $t \in (-1,1)$,

$$\int_{-1}^{1} \frac{(1-\tau^2)^{\lambda-1/2}}{|\tau - t|^{1-\nu}} d\tau < \infty, \quad \nu > 0,$$

Lemma 3.2 of [6] shows that

$$|H(r_n;x)| = O\Big((K_2 + L_2 n^{\nu})n^{-p-\rho}\Big), \tag{4.3}$$

where K_2 and L_2 are constants depending on x and $\nu > 0$ is as small as we like. Finally, by substituting (4.2) and (4.3) for (4.1), we can complete the proof. □

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,283-300,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Some Characteristics of Logistic and Bessel Random Variables

by

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Abstract: Products, ratios and sums of random variables arise explicitly in many areas of the sciences, engineering and medicine. This has increased the need to have available the widest possible range of statistical results on products, ratios and sums of random variables. In this note, the exact distributions of XY, X/Y and $X+Y$ are derived when X and Y are logistic and Bessel random variables distributed independently of each other. Tabulations of the associated percentage points obtained by inverting the derived distributions are also provided.

AMS 1990 Subject Classification: 62E15 (Primary) and 33C90 (Secondary).

Keywords and Phrases: Bessel function distribution; Logistic distribution; Products of random variables; Ratios of random variables; Sums of random variables.

1 Introduction

For given random variables X and Y, the distributions of the product XY, the ratio X/Y and the sum $X+Y$ arise explicitly in many areas of the sciences, engineering and medicine. The distributions of XY, X/Y and $X+Y$ have been studied by several authors especially when X and Y are independent random variables and come from the same family. With respect to products of random variables, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, Abu-Salih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). With respect to ratios of random variables, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's t family, Basu and Lochner (1971) for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for non-central chi-squared family, Provost (1989a) for gamma family, and Pham-Gia (2000) for beta family. With respect to sums of random variables, see Fisher (1935) and Chapman (1950) for Student's t family, Christopeit and Helmes (1979) for normal family, Davies (1980) and Farebrother (1984) for chi-squared family, Ali and Obaidullah (1982) for exponential family, Moschopoulos (1985) and Provost (1989b) for gamma family, Dobson *et al.* (1991) for Poisson family, Pham-Gia and Turkkan (1993) and Pham and Turkkan (1994) for beta family, Kamgar-Parsi *et al.* (1995) and Albert (2002) for uniform family, Hitczenko (1998) and Hu and Lin (2001) for Rayleigh family, and Witkovský (2001) for

inverted gamma family.

However, there is relatively little work of the above kind when X and Y belong to different families. In practical applications, it is quite possible that X and Y could arise from different but similar distributions. Nadarajah (2005) considered the exact distributions of XY and X/Y when X and Y are independent random variables having the Laplace and Bessel function distributions. In this note, we consider the exact distributions of XY, X/Y and $X+Y$ when X and Y are independent random variables having the logistic and Bessel function distributions specified by the probability density functions (pdfs)

$$f_X(x) = \frac{\lambda \exp(-\lambda x)}{\{1+\exp(-\lambda x)\}^2} \tag{1}$$

and

$$f_Y(y) = \frac{\left|1-c^2\right|^{m+1/2} \mid y \mid^m}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right), \tag{2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\lambda > 0$, $b > 0$, $\mid c \mid < 1$ and $m > 1$, where

$$K_m(x) = \frac{\sqrt{\pi} x^m}{2^m \Gamma(m+1/2)} \int_1^\infty \left(t^2-1\right)^{m-1/2} \exp(-xt)\, dt$$

is the modified Bessel function of the third kind. Note that (1) can be reexpressed as the mixture of Laplace pdfs:

$$\begin{aligned} f_X(x) &= \sum_{k=0}^\infty \frac{2}{k+1} \binom{-2}{k} \frac{\lambda(k+1)}{2} \exp\{-\lambda(k+1) \mid x \mid\} \\ &= \sum_{k=0}^\infty (-1)^k (k+1) \lambda \exp\{-\lambda(k+1) \mid x \mid\}. \end{aligned} \tag{3}$$

This representation will be crucial for the calculations of this note. Note that (3) holds for all $x \neq 0$. When $x=0$, the right hand side of (3), $\lambda \sum_{k=0}^\infty (-1)^k (k+1)$, is a diverging series. This will not affect the subsequent results because (3) can be applied over all sets $A \backslash (-\delta, \delta)$ and the limit taken as $\delta \downarrow 0$. However, since both sides of (3) should integrate to 1, i.e.

$$\int_{-\infty}^\infty f_X(x) dx = \sum_{k=0}^\infty (-1)^k (k+1) \lambda \int_{-\infty}^\infty \exp\{-\lambda(k+1) \mid x \mid\}\, dx,$$

we will make the convention that $\sum_{k=0}^\infty (-1)^k = 1/2$.

Logistic and Bessel function distributions have found applications in a variety of areas that range from image and speech recognition and ocean engineering to finance. Both are rapidly becoming distributions of first choice whenever "something" with heavier than Gaussian tails is observed in the data. Some examples where these two distributions could arise simultaneously are:

1. in communication theory, X and Y could represent the random noise corresponding to two different signals.

2. in ocean engineering, X and Y could represent distributions of navigation errors.

3. in finance, X and Y could represent distributions of log-returns of two different commodities.
4. in image and speech recognition, X and Y could represent "input" distributions.

For further examples, the readers are referred to Balakrishnan (1992) and Kotz *et al.* (2001).

The results of this note are organized as follows: exact expressions for the pdf and the cumulative distribution function (cdf) of XY are given in Section 2; the same for X/Y and $X+Y$ are given in Sections 3 and 4, respectively; moment properties of XY, X/Y and $X+Y$ including characteristic functions, moments, factorial moments, skewness and kurtosis are considered in Section 5; finally, tabulations of the percentile points of XY, X/Y and $X+Y$ obtained by inverting the derived cdfs are provided in Section 6.

The calculations of this note involve several special functions, including the Bessel function of the first kind defined by

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(-\frac{x^2}{4}\right)^k,$$

the modified Bessel function of the first kind defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4}\right)^k,$$

the ${}_0F_3$ hypergeometric function defined by

$${}_0F_3(a,b,c;x) = \sum_{k=0}^{\infty} \frac{1}{(a)_k (b)_k (c)_k} \frac{x^k}{k!},$$

and the Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Exact Distribution of XY

Theorem 1 derives an explicit expression for the cdf of $\mid XY \mid$ in terms of the ${}_0F_3$ hypergeometric function. The corresponding expression for the pdf of $\mid XY \mid$ is given by Theorem 2.

Theorem 1 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively, with $c = 0$. The cdf of $Z = \mid XY \mid$ can be expressed as*

$$\begin{aligned} F_Z(z) = 1 - \sum_{k=0}^{\infty} (-1)^k \Bigg\{ & \sqrt{\pi} 2^m b^{m+1} \Gamma\left(m+\frac{1}{2}\right) {}_0F_3\left(\frac{1}{2}, \frac{1}{2}-m, \frac{1}{2}; \frac{(k+1)^2\lambda^2 z^2}{16b^2}\right) \\ & + \frac{\{(k+1)\lambda z\}^{2m+1}}{(2b)^m} \Gamma(-m)\Gamma(-2m-1) \, {}_0F_3\left(1+m, \frac{3}{2}+m, 1+m; \frac{(k+1)^2\lambda^2 z^2}{16b^2}\right) \end{aligned}$$

$$
+\left(\frac{3C}{2}-1\right)(2b)^m(k+1)\lambda z\Gamma(m)\ {}_0F_3\left(1-m,\frac{3}{2},1;\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\Bigg\}
\Bigg/\left\{\sqrt{\pi}2^{m-1}b^{m+1}\Gamma\left(m+\frac{1}{2}\right)\right\}, \tag{4}
$$

where C denotes the Euler's constant.

Proof: Using the relationship (3), one can write

$$
\Pr\left(\mid XY\mid\leq z\right) = 2\sum_{k=0}^{\infty}(-1)^k\Pr\left(\mid X_kY\mid\leq z\right), \tag{5}
$$

where X_k are Laplace random variables with parameter $\lambda(k+1)$. The cdf $F_{m,k}(z)=\Pr(\mid X_kY\mid\leq z)$ can be expressed as

$$
\begin{aligned}
F_{m,k}(z) &= \Pr\left(\mid X_k\mid\leq z/\mid Y\mid\right)\\
&= \frac{1}{\sqrt{\pi}2^mb^{m+1}\Gamma\left(m+1/2\right)}\int_{-\infty}^{\infty}\left[1-\exp\left\{-\frac{(k+1)\lambda z}{\mid y\mid}\right\}\right]\mid y\mid^m K_m\left(\left|\frac{y}{b}\right|\right)dy\\
&= 1-\frac{1}{\sqrt{\pi}2^mb^{m+1}\Gamma\left(m+1/2\right)}\int_{-\infty}^{\infty}\exp\left\{-\frac{(k+1)\lambda z}{\mid y\mid}\right\}\mid y\mid^m K_m\left(\left|\frac{y}{b}\right|\right)dy\\
&= 1-\frac{1}{\sqrt{\pi}2^{m-1}b^{m+1}\Gamma\left(m+1/2\right)}\int_{0}^{\infty}\exp\left\{-\frac{(k+1)\lambda z}{y}\right\}y^m K_m\left(\frac{y}{b}\right)dy\\
&= 1-\Bigg\{\sqrt{\pi}2^mb^{m+1}\Gamma\left(m+\frac{1}{2}\right)\ {}_0F_3\left(\frac{1}{2},\frac{1}{2}-m,\frac{1}{2};\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\\
&\quad+\frac{\{(k+1)\lambda z\}^{2m+1}}{(2b)^m}\Gamma(-m)\Gamma(-2m-1)\ {}_0F_3\left(1+m,\frac{3}{2}+m,1+m;\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\\
&\quad+\left(\frac{3C}{2}-1\right)(2b)^m(k+1)\lambda z\Gamma(m)\ {}_0F_3\left(1-m,\frac{3}{2},1;\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\Bigg\}\\
&\quad\Bigg/\left\{\sqrt{\pi}2^mb^{m+1}\Gamma\left(m+\frac{1}{2}\right)\right\},
\end{aligned} \tag{6}
$$

where the last step follows by direct application of equation (2.16.8.9) in Prudnikov *et al.* (1986, volume 2). The result of the theorem follows by using the convention $\sum_{k=0}^{\infty}(-1)^k=1/2$ (see Section 1 below equation (3)), after combining (5) and (6). ■

Theorem 2 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively, with $c=0$. The pdf of $Z=\mid XY\mid$ can be expressed as*

$$
\begin{aligned}
f_Z(z) &= \lambda\sum_{k=0}^{\infty}(-1)^k(k+1)\Bigg\{\sqrt{\pi}2^{m-1}b^{m-1}(k+1)\lambda z\Gamma\left(m-\frac{1}{2}\right)\ {}_0F_3\left(\frac{3}{2},\frac{3}{2}-m,\frac{3}{2};\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\\
&\quad+2^{-m}b^{-m}\left\{(k+1)\lambda z\right\}^{2m}\Gamma(-m)\Gamma(-2m)\ {}_0F_3\left(1+m,1+m,\frac{1}{2}+m;\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\\
&\quad-3C2^{m-1}b^m\Gamma(m)\ {}_0F_3\left(1-m,\frac{1}{2},1;\frac{(k+1)^2\lambda^2z^2}{16b^2}\right)\Bigg\}\\
&\quad\Bigg/\left\{\sqrt{\pi}2^{m-1}b^{m+1}\Gamma\left(m+\frac{1}{2}\right)\right\},
\end{aligned} \tag{7}
$$

where C denotes the Euler's constant.

Proof: Follows by differentiating (4) with respect to z and using properties of the ${}_0F_3$ hypergeometric function. ■

Using special properties of the ${}_0F_3$ hypergeometric function, one can derive simpler forms for the distribution of $\mid XY \mid$ when m takes half integer values. This is illustrated in the corollary below.

Corollary 1 *If $m = 3/2, 5/2, 7/2, 9/2, 11/2$ then (4) can be expressed as*

$$F_Z(z) = 2\sum_{k=0}^{\infty}(-1)^k F_{m,k}(z),$$

where

$$\begin{aligned}
F_{3/2,k}(z) &= -1/(8y)\Big\{-8y - 4I_0(2y)\,y - 3I_0(2y)\,y^3C + 2I_0(2y)\,y^3 - 4J_0(2y)\,y - 3J_0(y)\,y^3C \\
&\quad +2J_0(2y)\,y^3 + 8J_1(2y) + 6I_1(2y)\,y^2C - 4I_1(2y)\,y^2 + 8J_1(2y) \\
&\quad +6J_1(2y)\,y^2C - 4J_1(2y)\,y^2\Big\}, \\
F_{5/2,k}(z) &= -1/(96y)\Big\{-96y - 80I_0(2y)\,y - 45I_0(2y)\,y^3C + 30I_0(2y)\,y^3 - 80J_0(2y)\,y \\
&\quad -45J_0(2y)\,y^3C + 30J_0(2y)\,y^3 + 128I_1(2y) + 72I_1(2y)\,y^2C - 32I_1(2y)\,y^2 \\
&\quad +9I_1(2y)\,y^4C - 6I_1(2y)\,y^4 + 128J_1(2y) + 72J_1(2y)\,y^2C - 64J_1(2y)\,y^2 \\
&\quad -9J_1(2y)\,y^4C + 6J_1(2y)\,y^4\Big\}, \\
F_{7/2,k}(z) &= -1/(960y)\Big\{-960y - 1056I_0(2y)\,y - 495I_0(2y)\,y^3C + 298I_0(2y)\,y^3 \\
&\quad -15I_0(2y)\,y^5C + 10I_0(2y)\,y^5 - 1056J_0(2y)\,y - 495J_0(2y)\,y^3C + 362J_0(2y)\,y^3 \\
&\quad +15J_0(2y)\,y^5C - 10J_0(2y)\,y^5 + 1536I_1(2y) + 720I_1(2y)\,y^2C - 160I_1(2y)\,y^2 \\
&\quad +150I_1(2y)\,y^4C - 100I_1(2y)\,y^4 + 1536J_1(2y) + 720J_1(2y)\,y^2C - 800J_1(2y)\,y^2 \\
&\quad -150J_1(2y)\,y^4C + 100J_1(2y)\,y^4\Big\}, \\
F_{9/2,k}(z) &= -1/(53760y)\Big\{-1120J_0(2y)\,y^5 + 23626J_0(2y)\,y^3 + 15434I_0(2y)\,y^3 \\
&\quad -512I_1(2y)\,y^2 - 6954I_1(2y)\,y^2 - 53248J_1(2y)\,y^2 + 7466J_1(2y)\,y^4 \\
&\quad -70I_1(2y)\,y^6 - 70J_1(2y)\,y^6 - 71424I_0(2y)\,y - 71424J_0(2y)\,y \\
&\quad -29295I_0(2y)\,y^3 - 1680I_0(2y)\,y^5C - 29295J_0(2y)\,y^3C + 1680J_0(2y)\,y^5C \\
&\quad +105I_1(2y)\,y^6C - 10815J_1(2y)\,y^4C + 105J_1(2y)\,y^6C + 40320I_1(2y)\,y^2C \\
&\quad +40320J_1(2y)\,y^2C + 10815I_1(2y)\,y^4C + 1120I_0(2y)\,y^5 + 98304I_1(2y) \\
&\quad -53760y + 98304J_1(2y)\Big\}, \\
F_{11/2,k}(z) &= -1/(967680y)\Big\{1966080I_1(2y) + 1966080J_1(2y) - 967680y + 483042J_0(2y)\,y^3 \\
&\quad -29618J_0(2y)\,y^5 + 126J_0(2y)\,y^7 + 227934I_0(2y)\,y^4C + 4536I_1(2y)\,y^6C \\
&\quad -227934J_1(2y)\,y^4C + 4536J_1(2y)\,y^6C + 725760I_1(2y)\,y^2C \\
&\quad +725760J_1(2y)\,y^2C - 547155I_0(2y)\,y^3C - 43659I_0(2y)\,y^5C - 189I_0(2y)\,y^7C \\
&\quad -547155J_0(2y)\,y^3C + 43659J_0(2y)\,y^5C - 189J_0(2y)\,y^7C + 28594I_0(2y)\,y^5
\end{aligned}$$

$$
\begin{aligned}
&-1101312 J_1(2y)\, y^2 + 164244 J_1(2y)\, y^4 + 246498 I_0(2y)\, y^3 + 126 I_0(2y)\, y^7 \\
&-3024 I_1(2y)\, y^6 - 1482240 I_0(2y)\, y - 3024 J_1(2y)\, y^6 - 1482240 J_0(2y)\, y \\
&+133632 I_1(2y)\, y^2 - 139668 I(2y)\, y^4 \Big\},
\end{aligned}
$$

where $y = \sqrt{(k+1)\lambda z/b}$ *and* C *denotes the Euler's constant.*

3 Exact Distribution of X/Y

Theorem 3 derives an explicit expression for the cdf of $\mid X/Y \mid$ in terms of the Gauss hypergeometric function. The corresponding expression for the pdf of $\mid X/Y \mid$ is given by Theorem 4.

Theorem 3 *Suppose* X *and* Y *are independent random variables distributed according to (1) and (2), respectively. The cdf of* $Z =\mid X/Y \mid$ *can be expressed as:*

$$
\begin{aligned}
F_Z(z) &= 1 - \frac{\left|1-c^2\right|^{m+1/2} \Gamma(2m+1)}{\sqrt{\pi} 2^{2m+1} \Gamma(m+1/2)\, \Gamma(m+3/2)} \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{-2}{k} \\
&\quad \times \left\{ \frac{1}{(k+1)\lambda b z - c}\, {}_2F_1\left(\frac{1}{2}, 1; m+\frac{3}{2}; 1 - \frac{1}{\{(k+1)\lambda b z - c\}^2}\right) \right. \\
&\quad \left. + \frac{1}{(k+1)\lambda b z + c}\, {}_2F_1\left(\frac{1}{2}, 1; m+\frac{3}{2}; 1 - \frac{1}{\{(k+1)\lambda b z + c\}^2}\right) \right\}. \qquad (8)
\end{aligned}
$$

Proof: Using the relationship (3), one can write

$$
\Pr\left(\mid X/Y \mid \leq z\right) = \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{-2}{k} \Pr\left(\mid X_k/Y \mid \leq z\right), \qquad (9)
$$

where X_k are Laplace random variables with parameter $\lambda(k+1)$. The cdf $F_{m,k}(z) = \Pr(\mid X_k/Y \mid \leq z)$ can be expressed as

$$
\begin{aligned}
F_{m,k}(z) &= \frac{\left|1-c^2\right|^{m+1/2}}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \\
&\quad \times \int_{-\infty}^{\infty} \left\{F\left(z \mid y \mid\right) - F\left(-z \mid y \mid\right)\right\} \mid y \mid^m \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right) dy, \qquad (10)
\end{aligned}
$$

where $F(\cdot)$ inside the integral denotes the cdf – corresponding to a Laplace random variable with parameters $(\lambda(k+1), \theta)$ – given by

$$
F(x) = \begin{cases} \frac{1}{2} \exp\left\{(k+1)\lambda x\right\}, & \text{if } x \leq 0, \\ 1 - \frac{1}{2} \exp\left\{-(k+1)\lambda x\right\}, & \text{if } x > 0. \end{cases} \qquad (11)
$$

Substituting (11) for $F(\cdot)$, one can rewrite (10) as

$$
F_{m,k}(z) = \frac{\left|1-c^2\right|^{m+1/2}}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \left\{ \int_{-\infty}^{0} \left[1 - \exp\left\{(k+1)\lambda z y\right\}\right] \mid y \mid^m \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right) dy \right.
$$

$$
\begin{aligned}
& \left. + \int_0^\infty \left[1 - \exp\{-(k+1)\lambda z y\}\right] \mid y \mid^m \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right) dy \right\} \\
= \; & 1 - \frac{\left|1-c^2\right|^{m+1/2}}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \left[\int_0^\infty y^m \exp\left\{-(k+1)\lambda z y + \frac{cy}{b}\right\} K_m\left(\frac{y}{b}\right) dy \right. \\
& \left. + \int_0^\infty y^m \exp\left\{-(k+1)\lambda z y - \frac{cy}{b}\right\} K_m\left(\frac{y}{b}\right) dy \right] \\
= \; & 1 - \frac{\left|1-c^2\right|^{m+1/2} \Gamma(2m+1)}{\sqrt{\pi} 2^{2m+1} \Gamma(m+1/2) \Gamma(m+3/2)} \\
& \times \left\{ \frac{1}{(k+1)\lambda b z - c} \, {}_2F_1\left(\frac{1}{2}, 1; m+\frac{3}{2}; 1 - \frac{1}{\{(k+1)\lambda b z - c\}^2}\right) \right. \\
& \left. + \frac{1}{(k+1)\lambda b z + c} \, {}_2F_1\left(\frac{1}{2}, 1; m+\frac{3}{2}; 1 - \frac{1}{\{(k+1)\lambda b z + c\}^2}\right) \right\}, \qquad (12)
\end{aligned}
$$

where the last step follows by direct application of equation (2.16.6.3) in Prudnikov *et al.* (1986, volume 2). The result of the theorem follows by using the convention $\sum_{k=0}^\infty (-1)^k = 1/2$ after combining (9) and (12). ■

Theorem 4 *Suppose X and Y are independent random variables distributed according to (1) and (2), respectively. The pdf of $Z = \mid X/Y \mid$ can be expressed as*

$$
\begin{aligned}
f_Z(z) \; = \; & 1 - \frac{b\lambda \left|1-c^2\right|^{m+1/2} \Gamma(2m+2)}{\sqrt{\pi} 2^{2m+1} \Gamma(m+1/2) \Gamma(m+5/2)} \sum_{k=0}^\infty \binom{-2}{k} \\
& \times \left\{ \frac{1}{\{(k+1)\lambda b z - c\}^2} \, {}_2F_1\left(1, \frac{3}{2}; m+\frac{5}{2}; 1 - \frac{1}{\{(k+1)\lambda b z - c\}^2}\right) \right. \\
& \left. + \frac{1}{\{(k+1)\lambda b z + c\}^2} \, {}_2F_1\left(1, \frac{3}{2}; m+\frac{5}{2}; 1 - \frac{1}{\{(k+1)\lambda b z + c\}^2}\right) \right\}. \qquad (13)
\end{aligned}
$$

Proof: Follows by differentiating (8) with respect to z and using properties of the Gauss hypergeometric function. ■

Using special properties of the Gauss hypergeometric function (see, for example, Section 7.3 of Prudnikov *et al.* (1986, volume 3)), one can derive elementary forms for the distribution of $\mid X/Y \mid$ when m takes integer or half integer values. This is illustrated in the corollaries below.

Corollary 2 *If $m \geq 2$ is an integer then (8) reduces to*

$$
\begin{aligned}
F_Z(z) \; = \; & 1 - \frac{\left|1-c^2\right|^{m+1/2} \Gamma(2m+1)}{\sqrt{\pi} 2^{2m+1} \Gamma(m+1/2) \Gamma(m+3/2)} \sum_{k=0}^\infty \frac{2}{k+1} \binom{-2}{k} \\
& \times \left[\frac{1}{(k+1)\lambda b z - c} h\left(1 - \frac{1}{\{(k+1)\lambda b z - c\}^2}\right) \right. \\
& \left. + \frac{1}{(k+1)\lambda b z + c} h\left(1 - \frac{1}{\{(k+1)\lambda b z + c\}^2}\right) \right],
\end{aligned}
$$

where

$$h(z) = \frac{(1/2)_{m+1}(z-1)^m}{m! z^{m+1}} \left\{ 2\sqrt{z} \mathrm{arctanh}\left(\sqrt{z}\right) + \sum_{k=1}^{m} \frac{(k-1)!}{(1/2)_k} \left(\frac{z}{z-1}\right)^k \right\}.$$

Corollary 3 *If $m \geq 3/2$ is a half-integer then (8) reduces to*

$$\begin{aligned} F_Z(z) &= 1 - \frac{\left|1-c^2\right|^{m+1/2} \Gamma(2m+1)}{\sqrt{\pi} 2^{2m+1} \Gamma(m+1/2) \Gamma(m+3/2)} \sum_{k=0}^{\infty} \frac{2}{k+1} \binom{-2}{k} \\ &\quad \times \left[\frac{1}{(k+1)\lambda bz - c} h\left(1 - \frac{1}{\{(k+1)\lambda bz - c\}^2}\right) \right. \\ &\quad \left. + \frac{1}{(k+1)\lambda bz + c} h\left(1 - \frac{1}{\{(k+1)\lambda bz + c\}^2}\right) \right], \end{aligned}$$

where

$$h(z) = \frac{(z-1)^{m+1/2}}{(-1/2)_{m+3/2} z^{m+1/2}} \left\{ \sqrt{\pi}(1-z)^{-1/2} + \frac{1}{z} \sum_{k=1}^{m+1/2} (-1/2)_k \left(\frac{z-1}{z}\right)^{-k} \right\}.$$

4 Exact Distribution of $X+Y$

If $Z = X+Y$ then its pdf can be written as

$$\begin{aligned} f_Z(z) &= \frac{\left|1-c^2\right|^{m+1/2} \lambda}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \int_{-\infty}^{\infty} |\, y \,|^m \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right) \frac{\exp\{-\lambda(z-y)\}}{\left[1+\exp\{-\lambda(z-y)\}\right]^2} dy \\ &= \frac{\left|1-c^2\right|^{m+1/2} \lambda}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \int_{-\infty}^{\infty} |\, y \,|^m \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right) \sum_{k=0}^{\infty} (-1)^k (k+1) \\ &\quad \times \exp\{-\lambda(k+1) \mid z-y \mid\} \, dy \\ &= \frac{\left|1-c^2\right|^{m+1/2} \lambda}{\sqrt{\pi} 2^m b^{m+1} \Gamma(m+1/2)} \sum_{k=0}^{\infty} (-1)^k (k+1) \int_{-\infty}^{\infty} |\, y \,|^m \exp\left(-\frac{cy}{b}\right) K_m\left(\left|\frac{y}{b}\right|\right) \\ &\quad \times \exp\{-\lambda(k+1) \mid z-y \mid\} \, dy. \end{aligned} \tag{14}$$

Unfortunately, the integral in (14) cannot be reduced to an analytical form even for the particular case $c = 0$. Thus, the pdf and the cdf of $Z = X+Y$ will have to be obtained by numerical means.

5 Moment Properties of XY, X/Y and $X+Y$

The moment properties of XY, X/Y and $X+Y$ can be derived by knowing the same for X and Y. It is well known (see, for example, Johnson *et al.* (1995)) that

$$E\left(\mid X \mid^n\right) = 2\lambda^{-n} n! \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+k)^n}$$

and

$$E(\mid Y\mid^n) = \frac{(2b)^n\Gamma(m+(n+1)/2)\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(m+1/2)}$$

provided that $c=0$ in (2). Thus, the nth moment of $Z=\mid XY\mid$ is

$$E(Z^n) = \frac{n!2^{n+1}b^n\Gamma(m+(n+1)/2)\Gamma((n+1)/2)}{\lambda^n\sqrt{\pi}\Gamma(m+1/2)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^n}.$$

In particular,

$$E(Z) = \frac{4b\Gamma(m+1)}{\lambda\sqrt{\pi}\Gamma(m+1/2)}\sum_{k=0}^{\infty}\frac{(-1)^k}{1+k},$$

$$E\left(Z^2\right) = 4b^2\lambda^{-2}(2m+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^2},$$

$$E\left(Z^3\right) = \frac{96b^3\Gamma(m+2)}{\lambda^3\sqrt{\pi}\Gamma(3+1/2)}\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^3}$$

and

$$E\left(Z^4\right) = 144b^4\lambda^{-4}(2m+3)(2m+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^4}.$$

Moments of $\mid X/Y\mid$ do not exist because moments of negative order are undefined for Y. The nth moment of $Z=X+Y$ is

$$E(Z^n) = \sum_{k=0}^{n}\binom{n}{k}E\left(X^k\right)E\left(Y^{n-k}\right),$$

where

$$E\left(X^k\right) = \begin{cases} 0, & \text{if } k \text{ odd,}\\ E\left(\mid X\mid^k\right), & \text{if } k \text{ even}\end{cases}$$

and

$$E\left(Y^{n-k}\right) = \begin{cases} 0, & \text{if } n-k \text{ odd,}\\ E\left(\mid Y\mid^{n-k}\right), & \text{if } n-k \text{ even.}\end{cases}$$

In particular,

$$E(Z) = 0,$$

$$E\left(Z^2\right) = b^2(2m+1)+4\lambda^{-2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^2},$$

$$E\left(Z^3\right) = 0$$

and

$$E\left(Z^4\right) = 48\lambda^{-4}\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^4} + 3b^4(2m+3)(2m+1) + 24b^2\lambda^{-2}(2m+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+k)^2}.$$

The factorial moments, skewness and the kurtosis can be calculated by using the relationships that

$$E\left[(Z)_n\right] = E\left[Z(Z-1)\cdots(Z-n+1)\right],$$

$$Skewness(Z) = \frac{E\left(Z^3\right) - 3E(Z)E\left(Z^2\right) + 2E^3(Z)}{\left\{E\left(Z^2\right) - E^2(Z)\right\}^{3/2}},$$

and

$$Kurtosis(Z) = \frac{E\left(Z^4\right) - 4E(Z)E\left(Z^3\right) + 6E\left(Z^2\right)E^2(Z) - 3E^4(Z)}{\left\{E\left(Z^2\right) - E^2(Z)\right\}^2}.$$

Finally, using the facts that the characteristic functions (chfs) of X and Y are

$$E\left[\exp(itX)\right] = B\left(1+\frac{it}{\lambda}, 1-\frac{it}{\lambda}\right)$$

and

$$E\left[\exp(itY)\right] = \left(1+b^2t^2\right)^{-m-1/2},$$

where $i = \sqrt{-1}$ denotes the complex unit and $B(a,b)$ denotes the beta function defined by

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,$$

the chfs of XY, X/Y and $X+Y$ can be expressed as

$$E\left[\exp(itXY)\right] = \frac{1}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)}\int_{-\infty}^{\infty} B\left(1+\frac{ity}{\lambda}, 1-\frac{ity}{\lambda}\right) \mid y \mid^m K_m\left(\left|\frac{y}{b}\right|\right)dy,$$

$$E\left[\exp(itX/Y)\right] = \frac{1}{\sqrt{\pi}2^m b^{m+1}\Gamma(m+1/2)}\int_{-\infty}^{\infty} B\left(1+\frac{it}{\lambda y}, 1-\frac{it}{\lambda y}\right) \mid y \mid^m K_m\left(\left|\frac{y}{b}\right|\right)dy$$

and

$$E\left[\exp(it(X+Y))\right] = B\left(1+\frac{it}{\lambda}, 1-\frac{it}{\lambda}\right)\left(1+b^2t^2\right)^{-m-1/2},$$

respectively. The two integrals do not appear to have tractable analytical solutions.

6 Percentiles of XY, X/Y and $X+Y$

In this section, we provide tabulations of percentage points associated with the distributions of $\mid XY \mid$, $\mid X/Y \mid$ and $X+Y$. These values are obtained numerically by solving the equation $\int_{-\infty}^{z_p} f_Z(w)dw = p$ when $f_Z(\cdot)$ is given by (7), (13) and (14), respectively. Evidently, this involves computation of the hypergeometric functions and routines for this are widely available. We used the function hypergeom $(\cdots)$ in the algebraic manipulation package, MAPLE. Tables 1, 2 and 3 provide the numerical values of z_p for $m = 2, 3, \ldots, 50$. We have assumed that $b = 1$ and $\lambda = 1$.

Table 1. Percentage points of $Z = \mid XY \mid$.

m	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
2	1.723	2.474	3.648	5.937	8.549	15.768
3	2.103	3.005	4.407	7.109	10.161	18.489
4	2.426	3.458	5.055	8.116	11.552	20.852
5	2.711	3.860	5.630	9.013	12.794	22.971
6	2.969	4.223	6.153	9.829	13.926	24.910
7	3.207	4.558	6.635	10.582	14.973	26.707
8	3.428	4.870	7.084	11.286	15.952	28.391
9	3.636	5.163	7.506	11.948	16.874	29.980
10	3.833	5.441	7.906	12.576	17.749	31.489
11	4.020	5.705	8.287	13.174	18.582	32.929
12	4.199	5.957	8.651	13.746	19.380	34.308
13	4.370	6.200	9.001	14.295	20.146	35.634
14	4.536	6.433	9.337	14.824	20.884	36.912
15	4.695	6.658	9.662	15.335	21.597	38.147
16	4.849	6.875	9.976	15.829	22.287	39.344
17	4.998	7.086	10.280	16.308	22.956	40.505
18	5.143	7.291	10.576	16.774	23.607	41.634
19	5.284	7.490	10.864	17.227	24.240	42.733
20	5.421	7.684	11.144	17.668	24.856	43.805
21	5.555	7.874	11.418	18.099	25.458	44.851
22	5.686	8.059	11.685	18.520	26.046	45.873
23	5.814	8.239	11.946	18.931	26.622	46.873
24	5.939	8.416	12.201	19.334	27.184	47.851
25	6.062	8.589	12.452	19.728	27.736	48.811
26	6.182	8.759	12.697	20.115	28.277	49.754
27	6.299	8.926	12.938	20.494	28.778	50.675
28	6.415	9.089	13.174	20.866	29.328	51.582
29	6.529	9.250	13.406	21.232	29.840	52.473
30	6.640	9.408	13.634	21.592	30.344	53.349
31	6.750	9.563	13.859	21.946	30.839	54.211
32	6.858	9.716	14.080	22.294	31.326	55.059
33	6.965	9.866	14.297	22.637	31.806	55.895
34	7.069	10.014	14.511	22.975	32.278	56.718
35	7.173	10.160	14.723	23.308	32.744	57.530
36	7.274	10.304	14.931	23.636	33.203	58.330
37	7.375	10.446	15.136	23.959	33.656	59.120
38	7.474	10.586	15.338	24.279	34.104	59.899
39	7.571	10.725	15.538	24.594	34.545	60.668
40	7.668	10.861	15.735	24.905	34.980	61.427
41	7.763	10.996	15.930	25.213	35.411	62.177
42	7.857	11.129	16.123	25.516	35.836	62.919
43	7.950	11.261	16.313	25.816	36.256	63.651
44	8.042	11.391	16.501	26.113	36.671	64.375
45	8.133	11.519	16.687	26.406	37.082	65.092
46	8.223	11.646	16.871	26.697	37.488	65.800

47	8.312	11.772	17.053	26.984	37.890	66.501
48	8.400	11.897	17.233	27.267	38.288	67.194
49	8.487	12.020	17.411	27.549	38.681	67.881
50	8.573	12.142	17.587	27.827	39.071	68.560

Table 1. Percentage points of $Z =\mid XY \mid$.

m	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
2	1.723	2.474	3.648	5.937	8.549	15.768
3	2.103	3.005	4.407	7.109	10.161	18.489
4	2.426	3.458	5.055	8.116	11.552	20.852
5	2.711	3.860	5.630	9.013	12.794	22.971
6	2.969	4.223	6.153	9.829	13.926	24.910
7	3.207	4.558	6.635	10.582	14.973	26.707
8	3.428	4.870	7.084	11.286	15.952	28.391
9	3.636	5.163	7.506	11.948	16.874	29.980
10	3.833	5.441	7.906	12.576	17.749	31.489
11	4.020	5.705	8.287	13.174	18.582	32.929
12	4.199	5.957	8.651	13.746	19.380	34.308
13	4.370	6.200	9.001	14.295	20.146	35.634
14	4.536	6.433	9.337	14.824	20.884	36.912
15	4.695	6.658	9.662	15.335	21.597	38.147
16	4.849	6.875	9.976	15.829	22.287	39.344
17	4.998	7.086	10.280	16.308	22.956	40.505
18	5.143	7.291	10.576	16.774	23.607	41.634
19	5.284	7.490	10.864	17.227	24.240	42.733
20	5.421	7.684	11.144	17.668	24.856	43.805
21	5.555	7.874	11.418	18.099	25.458	44.851
22	5.686	8.059	11.685	18.520	26.046	45.873
23	5.814	8.239	11.946	18.931	26.622	46.873
24	5.939	8.416	12.201	19.334	27.184	47.851
25	6.062	8.589	12.452	19.728	27.736	48.811
26	6.182	8.759	12.697	20.115	28.277	49.754
27	6.299	8.926	12.938	20.494	28.778	50.675
28	6.415	9.089	13.174	20.866	29.328	51.582
29	6.529	9.250	13.406	21.232	29.840	52.473
30	6.640	9.408	13.634	21.592	30.344	53.349
31	6.750	9.563	13.859	21.946	30.839	54.211
32	6.858	9.716	14.080	22.294	31.326	55.059
33	6.965	9.866	14.297	22.637	31.806	55.895
34	7.069	10.014	14.511	22.975	32.278	56.718
35	7.173	10.160	14.723	23.308	32.744	57.530
36	7.274	10.304	14.931	23.636	33.203	58.330
37	7.375	10.446	15.136	23.959	33.656	59.120
38	7.474	10.586	15.338	24.279	34.104	59.899
39	7.571	10.725	15.538	24.594	34.545	60.668

40	7.668	10.861	15.735	24.905	34.980	61.427
41	7.763	10.996	15.930	25.213	35.411	62.177
42	7.857	11.129	16.123	25.516	35.836	62.919
43	7.950	11.261	16.313	25.816	36.256	63.651
44	8.042	11.391	16.501	26.113	36.671	64.375
45	8.133	11.519	16.687	26.406	37.082	65.092
46	8.223	11.646	16.871	26.697	37.488	65.800
47	8.312	11.772	17.053	26.984	37.890	66.501
48	8.400	11.897	17.233	27.267	38.288	67.194
49	8.487	12.020	17.411	27.549	38.681	67.881
50	8.573	12.142	17.587	27.827	39.071	68.560

Table 2. Percentage points of $Z = \mid X/Y \mid$.

m	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
2	1.169	1.715	2.761	5.783	11.714	58.825
3	0.954	1.392	2.228	4.640	9.380	47.062
4	0.825	1.201	1.916	3.982	8.042	40.339
5	0.737	1.071	1.707	3.542	7.150	35.858
6	0.673	0.976	1.554	3.221	6.500	32.598
7	0.623	0.902	1.435	2.974	6.001	30.090
8	0.582	0.843	1.340	2.776	5.601	28.084
9	0.549	0.794	1.262	2.613	5.272	26.432
10	0.520	0.753	1.196	2.476	4.994	25.041
11	0.496	0.718	1.140	2.358	4.757	23.849
12	0.475	0.687	1.090	2.256	4.550	22.812
13	0.456	0.660	1.047	2.166	4.368	21.900
14	0.439	0.635	1.008	2.086	4.206	21.088
15	0.424	0.614	0.974	2.014	4.061	20.361
16	0.411	0.594	0.942	1.949	3.930	19.704
17	0.399	0.576	0.914	1.890	3.811	19.107
18	0.387	0.560	0.888	1.836	3.702	18.561
19	0.377	0.545	0.864	1.786	3.602	18.060
20	0.367	0.531	0.842	1.741	3.510	17.597
21	0.358	0.518	0.821	1.698	3.424	17.168
22	0.350	0.506	0.802	1.659	3.345	16.768
23	0.342	0.495	0.785	1.622	3.270	16.396
24	0.335	0.484	0.768	1.587	3.201	16.049
25	0.328	0.475	0.752	1.555	3.136	15.719
26	0.322	0.465	0.738	1.525	3.074	15.411
27	0.316	0.457	0.724	1.496	3.016	15.120
28	0.310	0.448	0.711	1.469	2.961	14.845
29	0.305	0.440	0.698	1.443	2.909	14.585
30	0.300	0.433	0.686	1.418	2.860	14.338
31	0.295	0.426	0.675	1.395	2.813	14.103
32	0.290	0.419	0.664	1.373	2.769	13.879

33	0.286	0.413	0.654	1.352	2.726	13.665
34	0.282	0.407	0.644	1.332	2.685	13.461
35	0.278	0.401	0.635	1.312	2.646	13.266
36	0.274	0.395	0.626	1.294	2.609	13.079
37	0.270	0.390	0.618	1.276	2.573	12.900
38	0.266	0.385	0.609	1.259	2.539	12.728
39	0.263	0.380	0.602	1.243	2.506	12.563
40	0.260	0.375	0.594	1.227	2.474	12.404
41	0.256	0.370	0.587	1.212	2.444	12.251
42	0.253	0.366	0.580	1.197	2.414	12.103
43	0.250	0.361	0.573	1.183	2.386	11.961
44	0.247	0.357	0.566	1.170	2.359	11.823
45	0.245	0.353	0.560	1.157	2.332	11.691
46	0.242	0.349	0.554	1.144	2.306	11.562
47	0.239	0.346	0.548	1.132	2.282	11.438
48	0.237	0.342	0.542	1.120	2.258	11.317
49	0.234	0.339	0.536	1.108	2.234	11.201
50	0.232	0.335	0.531	1.097	2.212	11.088

Table 3. Percentage points of $Z = X + Y$.

m	$p = 0.6$	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$	$p = 0.99$
2	0.675	1.407	2.288	3.575	4.712	7.068
3	0.762	1.586	2.572	4.002	5.254	7.814
4	0.840	1.747	2.830	4.390	5.745	8.489
5	0.913	1.896	3.068	4.747	6.197	9.112
6	0.980	2.035	3.288	5.079	6.619	9.693
7	1.043	2.165	3.496	5.391	7.015	10.240
8	1.102	2.288	3.692	5.687	7.390	10.759
9	1.159	2.404	3.878	5.968	7.747	11.253
10	1.213	2.515	4.056	6.236	8.088	11.726
11	1.264	2.622	4.226	6.494	8.416	12.181
12	1.314	2.725	4.390	6.741	8.731	12.618
13	1.362	2.824	4.549	6.980	9.035	13.041
14	1.408	2.919	4.701	7.211	9.330	13.451
15	1.453	3.012	4.849	7.435	9.615	13.848
16	1.496	3.102	4.993	7.653	9.892	14.234
17	1.538	3.189	5.133	7.864	10.162	14.610
18	1.580	3.274	5.269	8.070	10.425	14.976
19	1.620	3.357	5.401	8.271	10.681	15.334
20	1.659	3.438	5.531	8.467	10.931	15.683
21	1.697	3.517	5.657	8.659	11.176	16.025
22	1.734	3.594	5.781	8.846	11.415	16.360
23	1.771	3.670	5.902	9.030	11.650	16.687
24	1.807	3.744	6.021	9.210	11.880	17.009
25	1.842	3.816	6.137	9.386	12.105	17.324

26	1.876	3.888	6.252	9.559	12.327	17.634
27	1.910	3.958	6.364	9.730	12.544	17.939
28	1.944	4.027	6.474	9.897	12.758	18.238
29	1.976	4.094	6.583	10.061	12.968	18.533
30	2.009	4.161	6.689	10.223	13.175	18.823
31	2.040	4.226	6.794	10.383	13.379	19.108
32	2.071	4.291	6.898	10.540	13.580	19.389
33	2.102	4.355	7.000	10.694	13.777	19.667
34	2.132	4.417	7.100	10.847	13.972	19.940
35	2.162	4.479	7.199	10.997	14.165	20.210
36	2.192	4.540	7.297	11.145	14.354	20.476
37	2.221	4.600	7.393	11.292	14.542	20.739
38	2.250	4.660	7.488	11.436	14.726	20.998
39	2.278	4.718	7.582	11.579	14.909	21.254
40	2.306	4.776	7.675	11.720	15.089	21.507
41	2.334	4.833	7.767	11.859	15.268	21.758
42	2.361	4.890	7.858	11.997	15.444	22.005
43	2.388	4.946	7.947	12.133	15.618	22.250
44	2.415	5.001	8.036	12.267	15.790	22.492
45	2.441	5.056	8.123	12.401	15.961	22.731
46	2.467	5.110	8.210	12.532	16.129	22.968
47	2.493	5.163	8.296	12.663	16.296	23.202
48	2.519	5.216	8.381	12.792	16.461	23.435
49	2.544	5.269	8.465	12.919	16.625	23.664
50	2.569	5.321	8.548	13.046	16.787	23.892

We expect these numbers could be of use to the practitioners mentioned in Section 1. Similar tabulations could be easily derived for other values of m, b, λ and p by using the hypergeom $(\cdots)$ function in MAPLE.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,301-303,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

On the Moments of the $\mathcal{B}$ Distribution

by

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Abstract: Explicit expressions are derived for the moments and the moment generating function of the $\mathcal{B}$ distribution introduced by Bousquet *et al.* (2006).

Keywords and Phrases: $\mathcal{B}$ distribution, Generalized hypergeometric function, Moment generating function, Moments.

1 Introduction

The recent paper by Bousquet *et al.* (2006) introduced the $\mathcal{B}$ distribution specified by the cumulative distribution function (cdf)

$$F_X(x) = 1 - \exp\left(-ax^r - sx\right) \tag{1}$$

for $x > 0$, $a > 0$, $-\infty < r < \infty$ and $s > 0$. The paper discussed various properties of (1) with applications. However, little was presented in terms of basic mathematical properties. The only mathematical properties of substance discussed in Bousquet *et al.* (2006) were the expected value ($E(X)$), variance ($Var(X)$) and the moment generating function (mgf) ($M(t) = E[\exp(tX)]$) all for the particular case $r = 2$. Here, I would like to point out that much more general expressions can be derived for the moments and the mgf associated with (1). The main results are presented in Section 2. The calculations involve the generalized hypergeometric function defined by

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; x\right) = \sum_{k=0}^{\infty} \frac{\left(a_1\right)_k \left(a_2\right)_k \cdots \left(a_p\right)_k}{\left(b_1\right)_k \left(b_2\right)_k \cdots \left(b_q\right)_k} \frac{x^k}{k!},$$

where $(c)_k = c(c+1) \cdots (c+k-1)$ denotes the ascending factorial. The properties of the generalized hypergeometric function can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Main Results

Theorem 1 derives explicit expressions for $E(X^\gamma)$ associated with (1), where r and γ can be any real numbers. The corresponding mgf is given by Theorem 2. An essential assumption of the theorems is that the parameter r is a rational number.

Theorem 1 *Suppose a random variable X has the cdf (1). Then, the γth moment of X is given by*

$$E\left(X^{\gamma}\right) = \begin{cases} \gamma I(\gamma, a, r, s), & \text{if } r > 0 \text{ and } \gamma > 0, \\ \gamma J(\gamma, a, -r, s), & \text{if } r < 0 \end{cases}$$

for $a > 0$ and $s > 0$, where $I(\cdots)$ and $J(\cdots)$ are given by Lemmas 1 and 2, respectively.

Proof: The result follows by writing

$$E\left(X^{\gamma}\right) = \gamma \int_0^{\infty} x^{\gamma-1} \exp\left(-ax^r - sx\right) dx$$

and using the results of Lemmas 1 and 2. ■

Theorem 2 *Suppose a random variable X has the cdf (1). Then, the mgf of X is given by*

$$M(t) = \begin{cases} arI(r, a, r, s-t) + sI(1, a, r, s-t), & \text{if } r > 0, \\ arJ(r, a, -r, s-t) + sJ(1, a, -r, s-t), & \text{if } r < 0 \end{cases}$$

for $a > 0$ and $s > t$, where $I(\cdots)$ and $J(\cdots)$ are given by Lemmas 1 and 2, respectively.

Proof: The probability density function (pdf) corresponding to (1) is:

$$f_X(x) = \left(arx^{r-1} + s\right) \exp\left(-ax^r - sx\right).$$

Thus, the mgf can be expressed as

$$M(t) = ar \int_0^{\infty} x^{r-1} \exp\left\{-ax^r - (s-t)x\right\} dx + s \int_0^{\infty} \exp\left\{-ax^r - (s-t)x\right\} dx.$$

The result follows by Lemmas 1 and 2. ■

Appendix

The proofs of Theorems 1 and 2 require the following technical lemmas.

Lemma 1 *(Equation (2.3.1.13), Prudnikov et al., 1986, volume 1) For $r > 0$, $\gamma > 0$, $a > 0$ and $s > 0$,*

$$\int_0^{\infty} x^{\gamma-1} \exp\left(-ax^r - sx\right) dx = I(\gamma, a, r, s),$$

where

$$I = \begin{cases} \displaystyle\sum_{j=0}^{q-1} \frac{(-a)^j}{j! s^{\gamma+rj}} \Gamma\left(\gamma + rj\right) {}_{p+1}F_q\left(1, \Delta\left(p, \gamma + rj\right); \Delta(q, 1+j); (-1)^q z\right), & \text{if } 0 < r < 1, \\ \displaystyle\sum_{h=0}^{p-1} \frac{(-s)^h}{rh! a^{(\gamma+h)/r}} \Gamma\left(\frac{\gamma+h}{r}\right) {}_{q+1}F_p\left(1, \Delta\left(q, \frac{\gamma+h}{r}\right); \Delta(p, 1+h); \frac{(-1)^p}{z}\right), & \text{if } r > 1, \\ \dfrac{\Gamma(\gamma)}{(a+s)^{\gamma}}, & \text{if } r = 1 \end{cases}$$

provided that $r = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers, where $z = p^p a^q / \{s^p q^q\}$ and $\Delta(k, a) = (a/k, (a+1)/k, \ldots, (a+k-1)/k)$.

Lemma 2 *(Equation (2.3.1.14), Prudnikov et al., 1986, volume 1) For* $r > 0$*,* $a > 0$ *and* $s > 0$*,*

$$\int_0^\infty x^{\gamma-1} \exp\left(-ax^{-r} - sx\right) dx \quad = \quad J(\gamma, a, r, s),$$

where

$$\begin{aligned} J \quad = \quad & \sum_{j=0}^{q-1} \frac{(-a)^j}{j! s^{\gamma - rj}} \Gamma\left(\gamma - rj\right) \ {}_1F_{p+q}\left(1; \Delta\left(p, 1 - \gamma + rj\right), \Delta(q, 1 + j); z\right) \\ & + \sum_{h=0}^{p-1} \frac{(-s)^h a^{(\gamma+h)/r}}{rh!} \Gamma\left(-\frac{\gamma+h}{r}\right) \ {}_1F_{p+q}\left(1; \Delta\left(q, 1 + \frac{\gamma+h}{r}\right), \Delta(p, 1+h); z\right) \end{aligned}$$

provided that $r = p/q$*, where* $p \geq 1$ *and* $q \geq 1$ *are co-prime integers, where* $z = (-1)^{p+q} s^p a^q / \{p^p q^q\}$ *and* $\Delta(k, a) = (a/k, (a+1)/k, \ldots, (a+k-1)/k)$*.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,305-318,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

The Beta-Laplace Distribution

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Abstract. Motivated by the recent work of Eugene et al. [3] and Gupta and Nadarajah [5], we introduce the beta Laplace distribution generated from the logit of a beta random variable. Our focus are the basic theoretical properties of this distribution, including modality and concavity of the density, moments and related parameters, and stochastic representations that aid in random variate generation from this model.

Keywords: Beta-normal distribution, Gauss hypergeometric exponential distribution, Generalized exponential distribution, Mixture representation, Unimodality

AMS 2000 Subject Classifications: 60E05, 62E10

1 Introduction

Every cumulative distribution function (c.d.f.) G generates a generalized class of distributions with c.d.f.'s

$$F_G(x) = \frac{B_{G(x)}(a,b)}{B(a,b)}, \quad x \in R, \quad a, b > 0, \tag{1}$$

where

$$B_y(a,b) = \int_0^y w^{a-1}(1-w)^{b-1} dw \tag{2}$$

is the incomplete beta function and

$$B(a,b) = B_1(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Note that if a random variable X has the c.d.f. given by (1) then it admits the representation

$$X \stackrel{d}{=} G^{-1}(W), \tag{3}$$

where W has the beta distribution with parameters $a > 0$, $b > 0$, denoted by $Be(a, b)$. In the special case $a = b = 1$ the variable W is standard uniform, and (3) becomes the probability integral transformation.

Eugene et al. [3] introduced what is known as the *beta normal* distribution by taking G to be the c.d.f. of the normal distribution with parameters μ and σ. The properties of this distribution have been studied in more detail in [5]. In this paper, we introduce the *beta Laplace* (BL) distribution by taking G in (1) to be the c.d.f. of the Laplace distribution. Thus, the c.d.f. of the BL distribution is given by (1), where

$$G(x) \quad = \quad \begin{cases} \frac{1}{2} \exp\left(\frac{x-\theta}{\sigma}\right), & \text{if } x < \theta, \\ 1 - \frac{1}{2} \exp\left(\frac{\theta - x}{\sigma}\right), & \text{if } x \geq \theta, \end{cases} \tag{4}$$

and $-\infty < \theta < \infty$ and $\sigma > 0$. The corresponding probability density function (p.d.f.) and the hazard rate function are

$$f_{a,b,\theta,\sigma}(x) \quad = \quad \frac{1}{2\sigma B(a,b)} \exp\left(-\frac{\mid x-\theta \mid}{\sigma}\right) G^{a-1}(x) \left\{1 - G(x)\right\}^{b-1} \tag{5}$$

and

$$\lambda_{a,b,\theta,\sigma}(x) \quad = \quad \frac{1}{2\sigma[B(a,b)]^2} \exp\left(-\frac{\mid x-\theta \mid}{\phi}\right) \frac{G^{a-1}(x) \left\{1 - G(x)\right\}^{b-1}}{B_{1-G(x)}(b,a)}, \tag{6}$$

respectively. We shall denote this distribution by $BL(a, b, \theta, \sigma)$. Since θ and σ are location and scale parameters, we shall restrict attention to the standard case with $\theta = 0$ and $\sigma = 1$, denoted by $BL(a, b)$. In this case the p.d.f. (5) takes the form

$$f_{a,b}(x) \quad = \quad \left(\frac{1}{2}\right)^{a+b-1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \begin{cases} e^{ax}(2 - e^{x})^{b-1}, & \text{if } x \leq 0, \\ e^{-bx}(2 - e^{-x})^{a-1}, & \text{if } x > 0. \end{cases} \tag{7}$$

Note that $X \sim BL(a, b, \theta, \sigma)$ if and only if $-X \sim BL(b, a, -\theta, \sigma)$. In particular, for the standard variable we have $X \sim BL(a, b)$ if and only if $-X \sim BL(b, a)$.

The standard Laplace distribution is contained as the particular case of (1) for $a = 1$ and $b = 1$. Another special case is

$$F_G(x) = \sum_{i=a}^{n} \binom{n}{i} G^i(x) \{1 - G(x)\}^{n-i}$$

for $b = n - a + 1$ and integer values of a. This is the distribution of the ath order statistic connected with a random sample of size $n = a + b - 1$ from the Laplace distribution with c.d.f. G, see [2]. Basic properties of this distribution can be found in Section 2.5 of [7]. Other special cases include

$$F_G(x) = 1 - \frac{\{1 - G(x)\}^b}{\Gamma(b)} \sum_{i=1}^{a} \frac{\Gamma(b + i - 1)}{\Gamma(i)} G^{i-1}(x)$$

for integer values of a,

$$F_G(x) = \frac{G^a(x)}{\Gamma(a)} \sum_{i=1}^{b} \frac{\Gamma(a + i - 1)}{\Gamma(i)} \{1 - G(x)\}^{i-1}$$

for integer values of b, and

$$F_G(x) = \frac{2}{\pi} \arctan \sqrt{\frac{G(x)}{1 - G(x)}}$$

for $a = 1/2$ and $b = 1/2$.

In this paper we derive basic theoretical properties of (5), deferring practical issues of estimation and testing to future work. In particular, in Section 2 we derive stochastic representations of the corresponding random variables, which aid in simulation from this distribution, characterize the corresponding densities in terms of modality and concavity in Section 3, and provide expressions for the moments and related characteristics in Section 4. Proofs and technical lemmas are collected in Section 5.

2 Representations

Note that if $X \sim BL(a, b)$, then the distribution of $X|X > 0$ (the distribution of X truncated below at zero) is given by the p.d.f.

$$g_{a,b}(x) = \frac{(1/2)^{a+b-1}}{B_{1/2}(b, a)} e^{-bx} (2 - e^{-x})^{a-1}, \quad x \geq 0, a, b > 0, \tag{8}$$

where $B_{1/2}(b,a)$ is the incomplete beta function given by (2). This "one-sided" beta-Laplace distribution is a generalization of the standard exponential distribution, similar in spirit to the *generalized exponential* distribution introduced in [4]. The latter can be defined through the representation (3), where W has the power function distribution with the c.d.f. w^α, $0 < w \le 1$, $\alpha > 0$, while $G(x) = 1 - e^{-x}$, $0 < x \le 1$, is the c.d.f. of the standard exponential distribution (so it reduces to the exponential when $\alpha = 1$). Similarly, a variable with p.d.f. (8) admits such a representation with the same G and W having the p.d.f.

$$w_{a,b}(x) = \frac{(1/2)^{a+b-1}}{B_{1/2}(b,a)}(1-x)^{b-1}(1+x)^{a-1}, \;\; 0 \le x \le 1, a, b > 0. \tag{9}$$

The distribution with density (9) is a special case of *Gauss hypergeometric* distribution studied in [1] (see also [6], p. 253). For this reason we shall refer to the distribution with density (8) as the *Gauss hypergeometric exponential* distribution with parameters $a, b > 0$, denoted by $GHE(a,b)$. Note that when $a = b = 1$ this distribution reduces to the standard exponential as in this case W is standard uniform.

Similarly, when we truncate $X \sim BL(a,b)$ above at zero, we obtain a distribution on $(-\infty, 0)$ corresponding to a r.v. $-Y$, where Y has a $GHE(b,a)$ distribution on $(0,\infty)$. This leads to the following representation of X in terms of its "one-sided" counterparts.

Proposition 1 *If $X \sim BL(a,b)$ then we have*

$$X \stackrel{d}{=} IY_1 + (I-1)Y_2, \tag{10}$$

where $Y_1 \sim GHE(a,b)$, $Y_2 \sim GHE(b,a)$, I takes on the values 0 and 1 with probabilities

$$q_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}B_{1/2}(a,b) \;\; \textit{and} \;\; p_{a,b} = 1 - q_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}B_{1/2}(b,a), \tag{11}$$

respectively, and all the variables on the right-hand-side of (10) are mutually independent.

As we stated before, the r.v.'s Y_1 and Y_2 in the above representation have the same distributions as $F^{-1}(W_1)$ and $F^{-1}(W_2)$, respectively, where F is the standard exponential c.d.f. and W_1 and W_2 have densities given by $w_{a,b}$ and $w_{b,a}$, respectively. Further, W_1 and W_2 have the

same distributions as $2V_1 - 1$ and $1 - 2V_2$, respectively, where V_1 and V_2 have truncated beta distributions, that is

$$V_1 \stackrel{d}{=} W|W > 1/2, \text{ and } V_2 \stackrel{d}{=} W|W < 1/2 \tag{12}$$

with $W \sim Be(a,b)$. Moreover, the quantities $q_{a,b}$ and $p_{a,b}$ in (11) are the probabilities $P(W < 1/2)$ and $P(W > 1/2)$, respectively. This result admits a generalization, where X is still given by (3) but G and W are not necessarily the Laplace c.d.f. and the beta variable, respectively. Indeed, let V have a continuous distribution on $(0,\infty)$ with density f and c.d.f. F, and let Y be the corresponding "symmetrization" with p.d.f. $g(x) = f(|x|)$, $x \in R$, and c.d.f.

$$G(x) = \begin{cases} \frac{1}{2}(1 - F(-x)), & \text{if } x < 0, \\ 1 - \frac{1}{2}(1 - F(x)), & \text{if } x \geq 0. \end{cases} \tag{13}$$

[If V is standard exponential then Y is standard Laplace (4) with $\theta = 0$ and $\sigma = 1$.] Further, define a r.v. X via (3) where W has some continuous distribution on $(0,1)$. Then the following representation holds.

Proposition 2 *The r.v.* $X \sim G^{-1}(W)$ *admits the representation (10), where*

$$Y_1 \stackrel{d}{=} F^{-1}(2V_1 - 1) \text{ and } Y_2 \stackrel{d}{=} F^{-1}(1 - 2V_2) \tag{14}$$

with V_1, V_2 *given by (12),* I *takes on the values 0 and 1 with probabilities* $P(W < 1/2)$ *and* $P(W > 1/2)$, *respectively, and all the variables on the right-hand-side of (10) are mutually independent.*

3 Characterization of the density

Before we describe the density (7) of the standard beta-Laplace distribution in terms of modality and concavity, we start with one-sided distributions given by (8). Thus, we consider the function

$$h_{a,b}(x) = e^{-bx}(2 - e^{-x})^{a-1}, \quad x \geq 0, a, b > 0, \tag{15}$$

which appears in the expression for the density. The relevant properties of $h_{a,b}$ are as follows.

Proposition 3 *The function $h_{a,b}$ given by (15) admits the following properties:*
(i) The function $h_{a,b}$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$ with $h_{a,b}(0) = 1$, $\lim_{x\to\infty} h_{a,b}(x) = 0$, and $\lim_{x\to 0^+} h'_{a,b}(x) = a - b - 1$.
(ii) The function $h_{a,b}$ is monotonically decreasing on $(0,\infty)$ whenever $a > 0$ and $b \geq a-1$, and is monotonically increasing on $(0, x_{a,b})$ and decreasing on $(x_{a,b},\infty)$ whenever $a > 1$ and $b < a-1$, where

$$x_{a,b} = \ln\frac{a+b-1}{2b}, \quad 0 < b < a-1. \tag{16}$$

Moreover, in the latter case the maximum value of $h_{a,b}$ is

$$h_{a,b}(x_{a,b}) = 2^{a+b-1}\frac{(a-1)^{a-1}b^b}{(a+b-1)^{a+b-1}}, \quad 0 < b < a-1. \tag{17}$$

(iii) If either $a > 3$ and $b^-(a) \leq b < b^+(a)$ or $1 < a \leq 3$ and $b < b^+(a)$ then $h_{a,b}$ is concave down on $(0, x^+_{a,b})$ and concave up on $(x^+_{a,b},\infty)$. Here,

$$b^{\pm}(a) = a - 1 \pm \sqrt{2(a-1)} \tag{18}$$

and

$$x^{\pm}_{a,b} = \ln\frac{1+t^{\pm}_{a,b}}{2}, \tag{19}$$

where

$$t^{\pm}_{a,b} = \frac{(a-1)(1+2b) \pm \sqrt{(a-1)^2(1+2b)^2 - 4b^2(a-1)(a-2)}}{2b^2}. \tag{20}$$

(iv) If $a > 3$ and $b < b^-(a)$ then $h_{a,b}$ is concave down on $(x^-_{a,b}, x^+_{a,b})$ and concave up on $(0, x^-_{a,b})$ and $(x^+_{a,b},\infty)$.
(v) If either $1 < a$ and $b \geq b^+(a)$ or $a \leq 1$ then $h_{a,b}$ is concave up on $(0,\infty)$.

Using the results above we obtain the following four distinct cases of the one-sided BL density $g_{a,b}$ given by (8):

- If either $0 < a \leq 1$ or $a > 1$ but $b \geq b^+(a)$ then $g_{a,b}(x)$ is concave up and decreasing in x on $(0,\infty)$.

- If $a > 1$ and $a - 1 \leq b < b^+(a)$ [note that $b^+(a) > a - 1$ whenever $a > 1$] then $g_{a,b}(x)$ is decreasing in x on $(0, \infty)$. Moreover, it is concave down on $(0, x^+_{a,b})$ and concave up on $(x^+_{a,b}, \infty)$ with $x^+_{a,b}$ as in (19).

- If either $1 < a \leq 3$ and $0 < b < a - 1$ or $a > 3$ and $b^-(a) \leq b < a - 1$ [note that $b^-(a) < a - 1$ whenever $a > 3$] then $g_{a,b}(x)$ is increasing in x on $(0, x_{a,b})$, decreasing in x on $(x_{a,b}, \infty)$, concave down on $(0, x^+_{a,b})$, and concave up on $(x^+_{a,b}, \infty)$, with $0 < x_{a,b} < x^+_{a,b} < \infty$.

- If $3 < a$ and $b < b^-(a)$ then $g_{a,b}(x)$ is increasing in x on $(0, x_{a,b})$, decreasing in x on $(x_{a,b}, \infty)$, concave down on $(x^-_{a,b}, x^+_{a,b})$, and concave up on $(0, x^-_{a,b})$ and $(x^+_{a,b}, \infty)$, with $0 < x^-_{a,b} < x_{a,b} < x^+_{a,b} < \infty$.

Using the above proposition and the fact that the $BL(a, b)$ density is related to the function (15) via

$$f_{a,b}(x) = \left(\frac{1}{2}\right)^{a+b-1} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \cdot \begin{cases} h_{b,a}(|x|), & \text{if } x \leq 0, \\ h_{a,b}(x), & \text{if } x > 0. \end{cases} \tag{21}$$

we obtain the following result.

Proposition 4 *For any $a, b > 0$ the $BL(a, b)$ distribution is unimodal and we have the following three distinct cases:*
(i) If $a - 1 \leq b \leq a + 1$ then the density (21) is monotonically increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ (so that the mode occurs at $x = 0$).
(ii) If $0 < b < a - 1$ then the density (21) is monotonically increasing on $(-\infty, x_{a,b})$ and decreasing on $(x_{a,b}, \infty)$ (so that the mode occurs at $x = x_{a,b} > 0$ given by (16)).
(iii) If $0 < a < b - 1$ then the density (21) is monotonically increasing on $(-\infty, -x_{b,a})$ and decreasing on $(-x_{b,a}, \infty)$ (so that the mode occurs at $x = -x_{b,a} < 0$), where

$$x_{b,a} = \ln \frac{a+b-1}{2a}, \quad 0 < a < b - 1. \tag{22}$$

Moreover, in cases (ii) and (iii) the maximum values of $f_{a,b}$ are

$$f_{a,b}(x_{a,b}) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{(a-1)^{a-1}b^b}{(a+b-1)^{a+b-1}}, \quad \textit{and} \;\; f_{a,b}(-x_{b,a}) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{(b-1)^{b-1}a^a}{(a+b-1)^{a+b-1}}, \quad (23)$$

respectively.

Remark 1. Note that by Part (i) of Proposition 3 the BL density is continuous on R but differentiable only at $x \neq 0$ (regardless of whether the mode is at zero or not).

Remark 2. Note that, in contrast, the beta normal distribution introduced in [3] is not always unimodal.

Remark 3. Combining Propositions 3 and 4 and using the representation (21) one can derive a number of distinct cases of the density $f_{a,b}$ in terms of its modality and concavity.

4 Moments and related measures

We start with the moment generating function (m.g.f.) corresponding to the $BL(a,b)$ distribution. By (7) we have

$$M(t) = E(e^{tX}) = C \left\{ \int_0^\infty \frac{e^{-(a+t)x}}{2^a} \left(1 - \frac{e^{-x}}{2}\right)^{b-1} dx + \int_0^\infty \frac{e^{-(b-t)x}}{2^b} \left(1 - \frac{e^{-x}}{2}\right)^{a-1} dx \right\}, \quad (24)$$

where $C = \Gamma(a+b)/[\Gamma(a)\Gamma(b)]$. Both integrals above are convergent whenever $-a < t < b$. Since $e^{-x}/2 \in (0, 1/2)$ when $x \in (0, \infty)$, using the binomial expansion

$$(1+z)^p = \sum_{j=0}^{\infty} pjz^j, \quad |z| < 1, p \in R, \quad (25)$$

we can write (24) as

$$C \left\{ \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^j}{2^{a+j}} b - 1je^{-(a+t+j)x} dx + \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^j}{2^{b+j}} a - 1je^{-(b-t+j)x} dx \right\}. \quad (26)$$

Since the quantities under the summations are absolutely integrable (when $-a < t < b$), interchanging the order of integration and summation leads to

$$M(t) = C\left\{\sum_{j=0}^{\infty} \frac{(-1)^j}{2^{a+j}} b-1j \frac{1}{a+t+j} + \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{b+j}} a-1j \frac{1}{b-t+j}\right\}, \quad -a < t < b. \tag{27}$$

If $a = k+1$ and $b = m+1$, where k and m are nonnegative integers, then

$$b-1j = 0 \text{ for } j > m \text{ and } a-1j = 0 \text{ for } j > k \tag{28}$$

so that the series above have only finite number of terms and we have

$$M(t) = C\left\{\sum_{j=0}^{m} \frac{(-1)^j}{2^{k+1+j}} mj \frac{1}{k+1+t+j} + \sum_{j=0}^{k} \frac{(-1)^j}{2^{m+1+j}} kj \frac{1}{m+1-t+j}\right\} \tag{29}$$

for $-k-1 < t < m+1$. Note that when $k = m = 0$ (so that $a = b = 1$) the above expression simplifies to $(1-t^2)^{-1}$, which is the m.g.f. corresponding to the standard Laplace distribution.

Similar derivations lead to the characteristic function (ch.f.), which is of the form

$$\phi(t) = Ee^{itX} = C\left\{\sum_{j=0}^{\infty} \frac{(-1)^j}{2^{a+j}} b-1j \frac{1}{a+it+j} + \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{b+j}} a-1j \frac{1}{b-it+j}\right\}, \quad t \in R. \tag{30}$$

Differentiating either the m.g.f. or the ch.f. n times and evaluating at $t = 0$ we obtain the following expression for the nth moment $\mu_n = EX^n$ of $X \sim BL(a,b)$, where $n = 1, 2, \ldots$,

$$\mu_n = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} n! \left\{\sum_{j=0}^{\infty} \frac{(-1)^{j+n}}{2^{a+j}} b-1j \frac{1}{(a+j)^{n+1}} + \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{b+j}} a-1j \frac{1}{(b+j)^{n+1}}\right\}. \tag{31}$$

As before, if $a = k+1$ and $b = m+1$, where k and m are nonnegative integers, the series above have only finite number of terms and we have

$$\mu_n = \left(\frac{1}{2}\right)^r \frac{r!n!}{k!m!} \left\{\sum_{j=0}^{m} mj \frac{(-1)^{n+m-j} 2^j}{(r-j)^{n+1}} + \sum_{j=0}^{k} kj \frac{(-1)^{k-j} 2^j}{(r-j)^{n+1}}\right\}, \tag{32}$$

where $r = k + m + 1$.

5 Proofs

Proof of Proposition 1. This is a special case of Proposition 2.

Proof of Proposition 2. We proceed by showing that for each $x \in R$ the c.d.f. of $G^{-1}(W)$ coincides with the c.d.f. of the right-hand-side of (10) with Y_1, Y_2, and I as stated above. Assume first that $x > 0$. Since

$$G^{-1}(p) \quad = \quad \begin{cases} -F^{-1}(1-2p), & \text{if } p \in (0, 1/2), \\ F^{-1}(2p-1), & \text{if } p \in [1/2, 1), \end{cases} \tag{33}$$

and $G(0) = 1/2$, we have

$$P(G^{-1}(W) > x) = P(W > 1/2 \text{ and } F^{-1}(2W-1) > x) = P(W > 1/2 \text{ and } 2W - 1 > F(x))$$

$$= P(W > 1/2 \text{ and } W > (1 + F(x))/2) = P(W > (1 + F(x))/2)$$

as $W > (1 + F(x))/2$ implies $W > 1/2$. On the other hand, we have

$$P(IY_1 + (I-1)Y_2 > x) = P(IF^{-1}(2V_1 - 1) > x \text{ and } I = 1) = P(2V_1 - 1 > F(x))P(I = 1)$$

$$= P(V_1 > (1 + F(x))/2)P(I = 1) = P(W > (1 + F(x))/2|W > 1/2)P(W > 1/2),$$

which also simplifies to $P(W > (1+F(x))/2)$. We thus established the equality of the c.d.f.'s when $x > 0$. The case $x \leq 0$ is analogous. This concludes the proof.

The following lemma is needed to prove Proposition 3.

Lemma 1 *Consider the function*

$$u(t) = b^2t^2 - (a-1)(1+2b)t + (a-1)(a-2), \ \ t \in R, \tag{34}$$

where $a > 1$ and $b > 0$. Then

(i) The function u is decreasing on $(-\infty, v_{ab})$ and increasing on $(v_{a,b}, \infty)$, where

$$v_{a,b} = \frac{(a-1)(1+2b)}{2b^2} > 0. \tag{35}$$

Moreover, the minimum value of u *on* $(-\infty, \infty)$ *is equal to*

$$u(v_{a,b}) = -(a-1)\left\{(a-1)\left[\left(\frac{1+2b}{2b}\right)^2 - 1\right] + 1\right\} < 0. \tag{36}$$

(ii) We have $u(0) = (a-1)(a-2)$*, so that* $u(0) < 0$ *when* $a < 2$*,* $u(0) = 0$ *when* $a = 2$*, and* $u(0) > 0$ *when* $a > 2$*.*

(iii) The equation $u(t) = 0$ *admits the roots* $t^{\pm}_{a,b}$ *given by (20).*

(iv) For any fixed $a > 1$*, as a function of* b *the larger root,* $t^{+}_{a,b}$*, is continuous and decreasing on* $(0, \infty)$ *with* $\lim_{b\to 0^+} t^{+}_{a,b} = \infty$ *and* $\lim_{b\to\infty} t^{+}_{a,b} = 0$*. Moreover,* $t^{+}_{a,b} = 1$ *when* $b = b^{+}(a)$ *with* $b^{+}(a)$ *given by (18).*

(v) For any fixed $a > 2$*, as a function of* b *the smaller root,* $t^{-}_{a,b}$*, is continuous and decreasing on* $(0, \infty)$ *with* $\lim_{b\to 0^+} t^{-}_{a,b} = t^{-}_{a,0} = a - 2$ *and* $\lim_{b\to\infty} t^{-}_{a,b} = 0$*. Moreover,* $t^{-}_{a,b} = 1$ *when* $b = b^{-}(a)$ *with* $b^{-}(a)$ *given by (18).*

Proof. Calculations needed to prove Parts (i) - (iii) are elementary. To establish Part (iv), write

$$t^{+}_{a,b} = (a-1)\left(\frac{1}{2b^2} + \frac{1}{b}\right) + \frac{1}{2b}\sqrt{(a-1)^2\left(\frac{1}{b} + 2\right)^2 - 4(a-1)(a-2)} \tag{37}$$

to see that it is decreasing in b on $(0, \infty)$ and the limits are as stated. With some tedious albeit routine algebra one can check that the solution of the equation $t^{+}_{a,b} = 1$ is given by $b^{+}(a)$ in (18). Similarly, straightforward algebra leads to

$$t^{-}_{a,b} = \frac{2(a-1)(a-2)}{(a-1)(1+2b) + \sqrt{(a-1)\left\{(a-1)(1+4b) + 4b^2\right\}}}, \tag{38}$$

which shows that this is a decreasing function of b on $(0, \infty)$ with limits as stated. Further routine calculations show that the solution of the equation $t^{-}_{a,b} = 1$ is given by $b^{-}(a)$ in (18). This concludes Part (v).

Proof of Proposition 3. The continuity and differentiability of $h_{a,b}$, as well as the values at zero and infinity, are clear. To check the limit of the derivative, write $h_{a,b}(x) = \exp(s_{a,b}(x))$, where

$$s_{a,b}(x) = \ln h_{a,b}(x) = -bx + (a-1)\ln(2 - e^{-x}), \quad x \in [0, \infty), \tag{39}$$

so that for $x > 0$ we have $h'_{a,b}(x) = h_{a,b}(x)s'_{a,b}(x)$ with

$$s'_{a,b}(x) = -b + \frac{a-1}{2e^x - 1}, \quad x \in (0, \infty). \tag{40}$$

Since $s_{a,b}(x)$ and $s'_{a,b}(x)$ converge to 0 and $a - b - 1$, respectively, as $x \to 0^+$, we conclude that the corresponding limit of $h'_{a,b}(x)$ is $a - b - 1$. This concludes Part (i).

To study the monotonicity of the function $h_{a,b}$ we look at the derivate (40) since the signs of $s'_{a,b}(x)$ and $h'_{a,b}(x)$ coincide. It is clear that $s'_{a,b}(x) < 0$ for all $x > 0$ whenever $\alpha \leq 1$, in which case $h_{a,b}$ is monotonically decreasing on $(0, \infty)$. Further, if $a > 1$, simple algebra shows that the derivative is negative if and only if

$$e^x > \frac{a+b-1}{2b}. \tag{41}$$

This inequality is always true whenever $b \geq a-1$, while for $b < a-1$ (and $a > 1$) its solution is $x > x_{a,b}$, where $x_{a,b}$ given by (16). Moreover, routine calculations show that in the latter case the maximum value of $h_{a,b}$ is given by (17). This concludes Part (ii).

Next, we study the concavity of $h_{a,b}$. Observe that

$$h''_{a,b}(x) = h_{a,b}(x)(s''_{a,b}(x) + [s'_{a,b}(x)]^2), \quad x \in (0, \infty), \tag{42}$$

so that $h''_{a,b}(x) > 0$ if and only if

$$s''_{a,b}(x) + [s'_{a,b}(x)]^2 > 0. \tag{43}$$

Since

$$s''_{a,b}(x) = \frac{-2(a-1)e^x}{(2e^x - 1)^2}, \quad x \in (0, \infty), \tag{44}$$

the inequality (43) holds for all $x \in (0, \infty)$ whenever $a \leq 1$, in which case the function $h_{a,b}$ is concave up on this interval. We thus established the second part of (v). In the sequel, we shall study the inequality (43) assuming that $a > 1$. Utilizing the expressions for the first and the second derivatives of $s_{a,b}$, given by (40) and (44), respectively, after some routine algebra we find that this inequality is equivalent to

$$u(2e^x - 1) > 0, \tag{45}$$

where $u(\cdot)$ is the quadratic function defined in Lemma 1. Since for any $x > 0$ we have $2e^x - 1 > 1$, we need to check how the roots of the equation $u(t) = 0$, which by Part (iii) of Lemma 1 are given by (20), relate to $t = 1$. Observe that by Part (iv) of Lemma 1 the larger root (denoted by $t^+_{a,b}$ in (20)) is less than or equal to 1 whenever $b \geq b^+(a)$, where $b^+(a)$ is given by (18). Consequently, for these values of b we have $u(t) > 0$ for all $t > 1$, so that the inequality (45) is satisfied by all $x > 0$. This shows that whenever $a > 1$ and $b \geq b^+(a)$ the function $h_{a,b}$ is concave up on the interval $(0, \infty)$, which is the first part of (v). To establish Parts (iii) and (iv) we shall proceed by separately considering the cases $1 < a \leq 3$ and $a > 3$.
Case 1: $1 < a \leq 3$. If $a \leq 2$, then by Parts (i) and (ii) of Lemma 1, we conclude that the smaller root of the equation $u(t) = 0$, denoted by $t^-_{a,b}$ in (20), is less than or equal to zero. Thus, when $b < b^+(a)$, in which case we have $t^+_{a,b} > 1$ by Part (iv) of Lemma 1, the function u is negative for $t \in (0, t^+_{a,b})$ and positive for $t \in (t^+_{a,b}, \infty)$. In turn, the inequality (45) holds for all $x > x^+_{a,b}$ with $x^+_{a,b}$ defined in (19). The same conclusion is reached when $2 < a \leq 3$, since in this case $a - 2 \leq 1$ so that by Part (v) of Lemma 1 we have $t^-_{a,b} \leq 1$. We thus obtained the second part of (iii).
Case 2: $3 < a$. We shall restrict attention to the case $b < b^+(a)$ (so that the larger root of the equation $u(t) = 0$ is greater than one), as otherwise we get the case $t^+_{a,b} \leq 1$ already considered (where the function $h_{a,b}$ is concave up on $(0, \infty)$). By Part (v) of Lemma 1 we have $t^-_{a,b} > 1$ if and only if $b < b^-(a)$ with $b^-(a)$ given by (18). Consequently, with these values of b the equation (45) holds whenever $1 < 2e^x - 1 < t^-_{a,b}(a)$ or $t^+_{a,b}(a) < 2e^x - 1 < \infty$. Equivalently, the function $h_{a,b}$ is concave up on the intervals $(0, x^-_{a,b})$ and $(x^+_{a,b}, \infty)$, and is concave down on the interval $(x^-_{a,b}, x^+_{a,b})$, where $x^\pm_{a,b}$ are given by (19). This is Part (iv) of our result. In turn, when $b^-(a) \leq b < b^+(a)$ then $t^-_{a,b} \leq 1$ and we similarly conclude that now the function $h_{a,b}$ is concave up on the interval $(x^+_{a,b}, \infty)$ and concave down on the interval $(0, x^+_{a,b})$, which the first part of (iii). This concludes the proof.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,319-331,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

On the Stability and Asymptotic Behavior of Generalized Quadratic Mappings *

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ABSTRACT

Let E_1 and E_2 be linear spaces. In this paper we extend a classical quadratic functional equation to more general equations of two types. In addition we solve the generalized Hyers-Ulam-Rassias stability problem for the functional equations, and thus obtain asymptotic properties of quadratic mappings as an application.

Keywords: stability, functional equation, generalized quadratic mappings.

2000 Mathematics Subject Classification: 39B82, 39B72.

1 Introduction

In 1960 and in 1964 S.M. Ulam [24] proposed the general Ulam stability problem: "When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?" The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus one can ask the following question for general functional equations: If we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality must be close to the solutions of the given equation? If the answer is affirmative, we would say that a given functional equation is stable. In 1978 P.M. Gruber [8] remarked that Ulam's problem is of particular interest in probability theory and in the case of functional equations of different types. We wish to note that stability properties of different functional equations can have applications to unrelated fields. For instance, Zhou [25] used a stability property of the functional equation $f(x-y)+f(x+y)=2f(x)$ to prove a conjecture of Z. Ditzian about the relationship between the smoothness of a mapping and the degree of its approximation by the associated Bernstein polynomials.
The Ulam's problem for ε-additive mappings $f: E_1 \to E_2$ between Banach spaces i.e., $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all $x, y \in E_1$, was solved by D.H. Hyers [9] and then generalized by D.G. Bourgin [4], Th.M. Rassias [17] and P. Găvruta [7] who permitted the Cauchy difference to become unbounded.
Now, a square norm on an inner product space satisfies the important parallelogram equality $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$ for all vectors x, y. If $\triangle ABC$ is a triangle in a finite

dimensional Euclidean space and I is the center of the side $\overline{BC}$, then the following identity $\|\overrightarrow{AB}\|^2 + \|\overrightarrow{AC}\|^2 = 2(\|\overrightarrow{AI}\|^2 + \|\overrightarrow{CI}\|^2)$ holds for all vectors A, B and C. The following functional equation which was motivated by these equations

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1.1}$$

is called a quadratic functional equation, and every solution of the equation (1.1) is said to be a quadratic mapping. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [1, 2, 18, 22]. A Hyers-Ulam stability theorem for the quadratic functional equation was proved by a lot of authors [5, 19]. C. Borelli and G.L. Forti [3] generalized the stability result as follows: Let G be an abelian group, and E a Banach space. Assume that a mapping $f : G \to E$ satisfies the functional inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to E$ satisfying

$$\|f(x) - \frac{f(0)}{3} - Q(x)\| \le \Phi(x, x)$$

for all $x \in G$. In 1983 F. Skof [23] was the first author to solve the Ulam problem for additive mappings on a restricted domain. In 1998 S. Jung [12] and in 2004 J.M. Rassias [16] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem [10, 11, 14, 15, 20, 21].

Now we are going to extend the equation (1.1) to more generalized equations with $(d+1)$-variables. For this purpose, we employ the operator $\biguplus_{x_2} f(x_1)$, which is defined in [13] as follows

$$\biguplus_{x_2} f(x_1) = f(x_1 + x_2) + f(x_1 - x_2)$$

for a given mapping $f : E_1 \to E_2$ between vector spaces. Similarly, we define $\biguplus^2_{x_2,x_3} f(x_1) = \biguplus_{x_3} \left(\biguplus_{x_2} f(x_1) \right)$ and inductively

$$\biguplus^{d}_{x_2,\cdots,x_{d+1}} f(x_1) = \biguplus_{x_{d+1}} \Big(\biguplus^{d-1}_{x_2,\cdots,x_d} f(x_1) \Big)$$

for all natural number d. Then it is easy to see that the operation $\biguplus^d_{x_2,\cdots,x_{d+1}} f(x_1)$ can be expressed in the form

$$\biguplus^{d}_{x_2,\cdots,x_{d+1}} f(x_1) = \sum_{k=0}^{d} \left(\sum_{2 \le i_1 < i_2 < \cdots < i_k \le d+1} f(-x_{i_1} - x_{i_2} \cdots - x_{i_k} + \sum_{j \ne i_1, i_2, \cdots, i_k} x_j) \right).$$

Thus it follows from definition that

$$\biguplus_{x_2,x_3}^{2} f(x_1) = \biguplus_{x_3,x_2}^{2} f(x_1), \qquad \biguplus_{x_2,\cdots,x_{k+1},\underbrace{0,\cdots,0}_{d-k}}^{d} f(x_1) = 2^{d-k} \biguplus_{x_2,\cdots,x_{k+1}}^{k} f(x_1), \tag{1.2}$$

and
$$\biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1+x_{d+1}) + \biguplus_{x_2,\cdots,x_d}^{d-1} f(x_1-x_{d+1}) = \biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1).$$

In [13], the author introduced and determined the general solution of the equation

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) + 2^d(d-1)\sum_{i=1}^{d+1} f(x_i) = 2^{d-1} \sum_{1\le i<j\le d+1} \left(\biguplus_{x_j} f(x_i)\right),$$

and then investigated the generalized Hyers-Ulam-Rassias stability problem for the equations. Now, we consider the following new functional equation,

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) + \sum_{1\le i<j\le d+1} \left(\biguplus_{x_j} f(x_i)\right) = (2^d+2d)\sum_{i=1}^{d+1} f(x_i), \quad d \ge 1, \tag{1.3}$$

$$\biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) = \sum_{1\le i<j\le d+1} \left(\biguplus_{x_j} f(x_i)\right) + (2^d-2d)\sum_{i=1}^{d+1} f(x_i), \quad d \ge 2 \tag{1.4}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in E_1$, where d is a natural number. As a special case, the equation (1.3) reduces to the equation (1.1) in the case $d=1$. In this paper, it will be verified that the general solutions of the above functional equations (1.3) and (1.4) are quadratic mappings in the class of functions between vector spaces. Besides we establish new theorems about the Ulam stability for the general equations and apply our results on restricted domains to the asymptotic behavior of functional equations.

2 Generalized quadratic mappings

Let E_1 and E_2 be vector spaces. First, we present the general solutions of the functional equations (1.3) and (1.4).

Lemma 2.1. If a mapping $f: E_1 \to E_2$ satisfies the functional equation (1.3) or (1.4), then the mapping f is quadratic.

Proof. Let f be a solution of the functional equation (1.3). Set $x_i := 0$ in (1.3) for all $i = 1, \cdots, d+1$ to get $f(0) = 0$. Putting $x_i := 0$ in (1.3) for all $i = 3, \cdots, d+1$, by (1.2) we get $(2^{d-1}+1)[f(x_1+x_2)+f(x_1-x_2)] = (2^d+2)[f(x_1)+f(x_2)]$ for all $x_1, x_2 \in E_1$. So the mapping f is quadratic.
Now if a mapping f is a solution of the functional equation (1.4), then we get $f(0)=0$ by setting $x_i := 0$ in (1.4) for all $i = 1, \cdots, d+1$. Putting $x_i := 0$ in (1.4) for all $i = 3, \cdots, d+1$, by (1.2) we get $(2^{d-1}-1)[f(x_1+x_2)+f(x_1-x_2)] = (2^d-2)[f(x_1)+f(x_2)]$ for all $x_1, x_2 \in E_1$. So the mapping f is quadratic. □

We now investigate the generalized Hyers-Ulam-Rassias stability problem for the equation (1.3). Thus we give conditions in order for a true mapping near an approximate mapping of the

equation (1.3) to exist. From now on, let X be a normed linear space and Y a Banach space unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers. Now before taking up the main subject, given a mapping $f : X \to Y$, we define the difference operator $Df : X^{d+1} \to Y$ by

$$\begin{aligned} Df(x_1, x_2, \cdots, x_{d+1}) \quad &:= \biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) + \sum_{1\le i<j\le d+1} \left(\biguplus_{x_j} f(x_i) \right) \\ &\quad -(2^d + 2d) \sum_{i=1}^{d+1} f(x_i), \quad d \ge 1 \end{aligned}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, which acts as a perturbation of the equation (1.3).

Theorem 2.2. Suppose that a mapping $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \le \varepsilon(x_1, \cdots, x_{d+1}) \tag{2.1}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, \cdots, 2^i x_{d+1})}{2^{2i}} \tag{2.2}$$

converges for all $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\begin{aligned} &\left\| f(x) + \frac{2^{d-1}(2d-3) + (d^2+d-3)}{3(2^{d-1}+1)} f(0) - Q(x) \right\| \\ &\le \frac{1}{4(2^{d-1}+1)} \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x)}{4^i} \end{aligned} \tag{2.3}$$

for all $x \in X$, where the mapping $\phi : X^2 \to Y$ is given by

$$\phi(x, y) \;\; := \;\; min \left\{ \varepsilon\Big(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0\Big) \;\Big|\; 2 \le i \le d+1 \right\}. \tag{2.4}$$

The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Proof. If we take $\Big(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0\Big)$ instead of $(x_1, \cdots, x_{d+1})$ in (2.1), we obtain by virtue of (1.2)

$$\begin{aligned} &\Big\| (2^{d-1}+1)[f(x+y) + f(x-y)] - 2(2^{d-1}+1)[f(x) + f(y)] \\ &\qquad -(d-1)(2^d+d+2)f(0) \Big\| \\ &\le \varepsilon\Big(x, 0, \cdots, 0, \overbrace{y}^{i}, 0, \cdots, 0\Big) \end{aligned}$$

for all $x, y \in X$, and all i with $2 \leq i \leq d+1$, which can be written in the form

$$\|q(x+y)+q(x-y)-2[q(x)+q(y)]-q(0)\| \leq \frac{1}{2^{d-1}+1}\phi(x,y) \tag{2.5}$$

for all $x, y \in X$, where a mapping $q : X \to Y$ is defined by $q(x) := f(x) + \frac{2^{d-1}(2d-3)+(d^2+d-3)}{3(2^{d-1}+1)}f(0)$ and a mapping $\phi : X^2 \to Y$ is given by (2.4). Taking $y := x$ in (2.5), we get

$$\left\|\frac{q(2x)}{4} - q(x)\right\| \leq \frac{1}{4(2^{d-1}+1)}\phi(x,x) \tag{2.6}$$

for all $x \in X$. Hence

$$\left\|\frac{q(2^n x)}{4^n} - \frac{q(2^m x)}{4^m}\right\| \leq \frac{1}{4(2^{d-1}+1)}\sum_{i=m}^{n-1}\frac{\phi(2^i x, 2^i x)}{4^i} \tag{2.7}$$

for all nonnegative integers m and n with $n > m$ and all $x \in X$. It follows from (2.2) and (2.7) that the sequence $\left\{\frac{q(2^n x)}{4^n}\right\}$ is a Cauchy sequence for all $x \in X$. So one can define a mapping $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{q(2^n x)}{4^n} = \lim_{n\to\infty}\frac{f(2^n x)}{4^n}$$

for all $x \in X$.
Now by (2.1) and (2.2), we have

$$\begin{aligned}\|DQ(x_1, x_2, \cdots, x_{d+1})\| \qquad &= \lim_{n\to\infty}\frac{1}{4^n}\|Df(2^n x_1, \cdots, 2^n x_{d+1})\| \\ &\leq \lim_{n\to\infty}\frac{1}{4^n}\varepsilon(2^n x_1, \cdots, 2^n x_{d+1}) = 0\end{aligned}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$. Hence by Lemma 2.1, the mapping Q is a quadratic mapping satisfying the equation (1.3). Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.7), we get the desired inequality (2.3).
To prove the afore-mentioned uniqueness, let $\varepsilon_1 : X^3 \to \mathbb{R}^+$ be a mapping such that a functional inequality

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon_1(x_1, \cdots, x_{d+1})$$

holds for all $x_1, \cdots, x_{d+1} \in X$ and the series

$$\sum_{i=0}^{\infty}\frac{\varepsilon_1(2^i x_1, \cdots, 2^i x_{d+1})}{4^i}$$

converges for all $x_1, \cdots, x_{d+1} \in X$, and assume that there exists a quadratic mapping $Q_1 : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\begin{aligned}&\left\|f(x) + \frac{2^{d-1}(2d-3)+(d^2+d-3)}{3(2^{d-1}+1)}f(0) - Q_1(x)\right\| \\ &\leq \frac{1}{4(2^{d-1}+1)}\sum_{i=0}^{\infty}\frac{\phi_1(2^i x, 2^i x)}{4^i}\end{aligned} \tag{2.8}$$

for all $x \in X$, where a mapping $\phi_1 : X^2 \to Y$ is given by

$$\phi_1(x,y) := min\left\{ \varepsilon_1\Big(x, 0, \cdots, 0, \overbrace{y}^{i}, 0 \cdots, 0\Big) \;\Big|\; 2 \le i \le d+1 \right\}.$$

Since Q and Q_1 are quadratic, we see the identities $Q(x) = 2^{-2n}Q(2^n x)$, $Q_1(x) = 2^{-2n}Q_1(2^n x)$ hold for all $x \in X$ and all $n \in \mathbb{N}$. Thus it follows from inequalities (2.3) and (2.8) that

$$\begin{aligned}
&\|Q(x) - Q_1(x)\| = \frac{1}{2^{2n}}\|Q(2^n x) - Q_1(2^n x)\| \\
&\le \frac{1}{2^{2n}}\Bigg(\left\| Q(2^n x) - f(2^n x) - \frac{2^{d-1}(2d-3) + (d^2+d-3)}{3(2^{d-1}+1)} f(0) \right\| \\
&\qquad + \left\| f(2^n x) + \frac{2^{d-1}(2d-3) + (d^2+d-3)}{3(2^{d-1}+1)} f(0) - Q_1(2^n x) \right\| \Bigg) \\
&\le \frac{1}{4(2^{d-1}+1)} \sum_{i=n}^{\infty} \frac{\phi(2^i x, 2^i x)}{4^i} + \frac{1}{4(2^{d-1}+1)} \sum_{i=n}^{\infty} \frac{\phi_1(2^i x, 2^i x)}{4^i}
\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore letting $n \to \infty$, one has $Q(x) - Q_1(x) = 0$ for all $x \in X$, which completes the proof of uniqueness.
The last assertion of homogeneous of degree two of Q in the theorem follows by the same reasoning as the proof of [6]. The proof is complete. □

Theorem 2.3. Suppose that a mapping $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \le \varepsilon(x_1, \cdots, x_{d+1})$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 4^i \varepsilon\left(\frac{x_1}{2^i}, \cdots, \frac{x_{d+1}}{2^i}\right)$$

converges for all $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - Q(x)\| \le \frac{1}{4(2^{d-1}+1)} \sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)$$

for all $x \in X$, where the mapping $\phi : X^2 \to Y$ is given by (2.4). The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Note that one has $f(0) = 0$ in the above theorem because $\varepsilon(0, \cdots, 0) = 0$ by the convergence of the series.

Corollary 2.4. Suppose that there exist a real number $\varepsilon \geq 0$ and a positive real $p \neq 2$ such that a mapping $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon(\|x_1\|^p + \cdots + \|x_{d+1}\|^p)$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{2\varepsilon\|x\|^p}{(2^{d-1}+1)|4-2^p|}$$

for all $x \in X$. The mapping Q is defined by

$$Q(x) = \begin{cases} \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, & \text{if } 0 < p < 2, \\ \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right), & \text{if } p > 2, \end{cases}$$

for all $x \in X$.

Corollary 2.5. Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\left\| f(x) + \frac{2^{d-1}(2d-3) + (d^2+d-3)}{3(2^{d-1}+1)} f(0) - Q(x) \right\| \leq \frac{\varepsilon}{3(2^{d-1}+1)}$$

for all $x \in X$.

For a given mapping $f : X \to Y$, we define the difference operator $Ef : X^{d+1} \to Y$ by

$$\begin{aligned} Ef(x_1, x_2, \cdots, x_{d+1}) \quad := & \biguplus_{x_2,\cdots,x_{d+1}}^{d} f(x_1) - \sum_{1\leq i<j\leq d+1} \left(\biguplus_{x_j} f(x_i) \right) \\ & -(2^d - 2d) \sum_{i=1}^{d+1} f(x_i), \quad d \geq 2 \end{aligned}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$.
We are going to investigate the generalized Hyers-Ulam-Rassias stability problem for the equation (1.4). The proof of the following theorems goes through by the same way as that of Theorem 2.2.

Theorem 2.6. Suppose that a mapping $f : X \to Y$ satisfies

$$\|Ef(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon(x_1, \cdots, x_{d+1}) \tag{2.9}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=0}^{\infty} \frac{\varepsilon(2^i x_1, \cdots, 2^i x_{d+1})}{2^{2i}} \tag{2.10}$$

converges for all $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\begin{aligned}&\left\|f(x) + \frac{2^{d-1}(2d-3) - (d^2+d-3)}{3(2^{d-1}-1)} f(0) - Q(x)\right\| \\ &\leq \frac{1}{4(2^{d-1}-1)} \sum_{i=0}^{\infty} \frac{\phi(2^i x, 2^i x)}{4^i}\end{aligned} \tag{2.11}$$

for all $x \in X$, where the mapping $\phi : X^2 \to Y$ is given by (2.4). The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

Theorem 2.7. Suppose that a mapping $f : X \to Y$ satisfies

$$\|Ef(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon(x_1, \cdots, x_{d+1})$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$, and that $\varepsilon : X^{d+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} 4^i \varepsilon\left(\frac{x_1}{2^i}, \cdots, \frac{x_{d+1}}{2^i}\right)$$

converges for all $x_1, \cdots, x_{d+1} \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - Q(x)\| \leq \frac{1}{4(2^{d-1}-1)} \sum_{i=1}^{\infty} 4^i \phi\left(\frac{x}{2^i}, \frac{x}{2^i}\right)$$

for all $x \in X$, where the mapping $\phi : X^2 \to Y$ is given by (2.4). The mapping Q is defined by

$$Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then the mapping Q is homogeneous of degree 2 over $\mathbb{R}$.

3 Approximately quadratic mappings on restricted domains

In this section we are going to investigate the Hyers-Ulam stability problem for the equations (1.3) and (1.4) on an unbounded restricted domain. As results we have corollaries concerning an asymptotic property of the equation (1.3).

Theorem 3.1. Let $r > 0$ be fixed. Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon \tag{3.1}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \geq r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\left\| f(x) + \frac{(d-1)(2^d+d+2)f(0)}{2^d+2} - Q(x) \right\| \leq \frac{3\varepsilon}{2(2^{d-1}+1)} \tag{3.2}$$

for all $x \in X$.

Proof. Taking $(x_1, \cdots, x_{d+1})$ as $(x, y, 0, \cdots, 0)$ in (3.1) with $\|x\| + \|y\| \geq r$, we obtain by (2.5)

$$\|q(x+y) + q(x-y) - 2[q(x) + q(y)]\| \leq \frac{\varepsilon}{2^{d-1}+1} \tag{3.3}$$

where a mapping $q : X \to Y$ is defined by $q(x) := f(x) + \frac{(d-1)(2^d+d+2)f(0)}{2^d+2}$ for all $x, y \in X$ with $\|x\| + \|y\| \geq r$. Specially, we have $\|q(0)\| \leq \frac{\varepsilon}{2(2^{d-1}+1)}$ by setting $y := 0$ and $x := t$ with $\|t\| \geq r$ in (3.3). Now, assume $\|x\| + \|y\| < r$. And choose a $t \in X$ with $\|t\| \geq 2r$. Then it holds clearly

$$\|x \pm t\| \geq r, \ \|y \pm t\| \geq r, \text{ and } \|x \pm t\| + \|y \pm t\| \geq r.$$

Therefore from (3.3) and the following functional identity

$$\begin{aligned}
&2\big[q(x+y) + q(x-y) - 2q(x) - 2q(y) - q(0)\big] \\
&= \big[q(x+y+2t) + q(x-y) - 2q(x+t) - 2q(y+t)\big] \\
&\quad + \big[q(x+y-2t) + q(x-y) - 2q(x-t) - 2q(y-t)\big] \\
&\quad + \big[-q(x+y+2t) - q(x+y-2t) + 2q(x+y) + 2q(2t)\big] \\
&\quad + \big[2q(x+t) + 2q(x-t) - 4q(x) - 4q(t)\big] \\
&\quad + \big[2q(y+t) + 2q(y-t) - 4q(y) - 4q(t)\big] \\
&\quad + \big[-2q(2t) - 2q(0) + 4q(t) + 4q(t)\big],
\end{aligned}$$

we get

$$\|q(x+y) + q(x-y) - 2q(x) - 2q(y) - q(0)\| \leq \frac{9\varepsilon}{2(2^{d-1}+1)} \tag{3.4}$$

for all $x, y \in X$ with $\|x\| + \|y\| < r$. Consequently, the last functional inequality holds for all $x, y \in X$ in view of (3.3) and (3.4) because of $\|q(0)\| \leq \frac{\varepsilon}{2(2^{d-1}+1)}$. Now letting $y := x$ in (3.4), we obtain

$$\|q(2x) - 4q(x)\| \leq \frac{9\varepsilon}{2(2^{d-1}+1)}.$$

Now applying a standard procedure of direct method [9] to the last inequality, we see that there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|q(x) - Q(x)\| \leq \frac{3\varepsilon}{2(2^{d-1}+1)}$$

for all $x \in X$. □

The proof of the following theorem is verified by the same way as that of Theorem 3.1.

Theorem 3.2. Let $r > 0$ and $\varepsilon \geq 0$ be fixed. Suppose that there exists a nonnegative real number ε for which a mapping $f : X \to Y$ satisfies

$$\|Ef(x_1, x_2, \cdots, x_{d+1})\| \leq \varepsilon \tag{3.5}$$

for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \geq r$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\left\| f(x) + \frac{(d-1)(2^d - d - 2)f(0)}{2^d - 2} - Q(x) \right\| \leq \frac{3\varepsilon}{2(2^{d-1} - 1)} \tag{3.6}$$

for all $x \in X$.

We note that if we define $S_{d+1} = \{(x_1, \cdots, x_{d+1}) \in X^{d+1} : \|x_i\| < r, \forall i = 1, \cdots, d+1\}$ for some fixed $r > 0$, then we have

$$\left\{ (x_1, \cdots, x_{d+1}) \in X^{d+1} : \sum_{i=1}^{d+1} \|x_i\| \geq (d+1)r \right\} \subset X^{d+1} \setminus S_{d+1}.$$

Thus the following corollaries are immediate consequences of Theorem 3.1 and Theorem 3.2.

Corollary 3.3. If a mapping $f : X \to Y$ satisfies the functional inequality (3.1) for all vectors $(x_1, \cdots, x_{d+1}) \in X^{d+1} \setminus S_{d+1}$, then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.3) and the inequality (3.2)

Corollary 3.4. If a mapping $f : X \to Y$ satisfies the functional inequality (3.5) for all vectors $(x_1, \cdots, x_{d+1}) \in X^{d+1} \setminus S_{d+1}$, then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies the equation (1.4) and the inequality (3.6)

From Theorem 3.1 and Theorem 3.2, we have the following corollaries concerning asymptotic properties of quadratic mappings.

Corollary 3.5. A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if either

$$\|Df(x_1, \cdots, x_{d+1})\| \to 0$$

or

$$\|Ef(x_1, \cdots, x_{d+1})\| \to 0$$

as $\sum_{i=1}^{d+1} \|x_i\| \to \infty$.

Proof. According to our asymptotic condition, there is a sequence (ε_m) decreasing to zero such that $\|Df(x_1, \cdots, x_{d+1})\| \leq \varepsilon_m$ for all $(d+1)$-variables $x_1, \cdots, x_{d+1} \in X$ with $\sum_{i=1}^{d+1} \|x_i\| \geq m$. Hence, it follows from Theorem 3.1 that there exists a unique quadratic mapping $Q_m : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - Q_m(x)\| \leq \frac{3\varepsilon_m}{2(2^{d-1} + 1)}$$

for all $x \in X$. Let m and l be positive integers with $m > l$. Then, we obtain

$$\|f(x) - Q_m(x)\| \leq \frac{3\varepsilon_m}{2(2^{d-1} + 1)} \leq \frac{3\varepsilon_l}{2(2^{d-1} + 1)}$$

for all $x \in X$. The uniqueness of Q_l implies that $Q_m = Q_l$ for all $m > l$, and so

$$\|f(x) - Q_l(x)\| \leq \frac{3\varepsilon_m}{2(2^{d-1}+1)}$$

for all $x \in X$. By letting $m \to \infty$, we conclude that f is itself quadratic.
The reverse assertion is trivial. □

Corollary 3.6. A mapping $f : X \to Y$ with $f(0) = 0$ is quadratic if and only if there exists a positive real $r > 0$ such that either

$$\sup_{x_1,\cdots,x_{d+1}} \left\{ \|Df(x_1, x_2, \cdots, x_{d+1})\| : \sum_{i=1}^{d+1} \|x_i\| \geq r \right\}$$

or

$$\sup_{x_1,\cdots,x_{d+1}} \left\{ \|Ef(x_1, x_2, \cdots, x_{d+1})\| : \sum_{i=1}^{d+1} \|x_i\| \geq r \right\}$$

is bounded for all $d \geq 1$.

Proof. Let $\sup_{x_1,\cdots,x_{d+1}} \|Df(x_1, x_2, \cdots, x_{d+1})\| \leq M < \infty$ for all $d \geq 1$. Then for each $d \geq 1$, there exists a unique quadratic mapping $Q_d : X \to Y$ which satisfies the equation (1.3) and the inequality

$$\|f(x) - Q_d(x)\| \leq \frac{3M}{2(2^{d-1}+1)}$$

for all $x \in X$ by Theorem 3.1. Let m be a positive integer with $m > d$. Then, we obtain

$$\|f(x) - Q_m(x)\| \leq \frac{3M}{2(2^{m-1}+1)} \leq \frac{3M}{2(2^{d-1}+1)}$$

for all $x \in X$. The uniqueness of Q_d implies that $Q_m = Q_d$ for all m with $m > d$, and so

$$\|f(x) - Q_d(x)\| \leq \frac{M}{3(2^{m-1}+1)}$$

for all $x \in X$. By letting $m \to \infty$, we conclude that f is itself quadratic.
The reverse assertion is trivial. □

Acknowledgment

* This work was supported by the Brain Korea 21 Project in 2006.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,333-344,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

ON STATISTICAL FUZZY TRIGONOMETRIC KOROVKIN THEORY

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Abstract. In this study, we use regular matrix transformations in the approximation by fuzzy positive linear operators, where the test functions are trigonometric. So we prove a trigonometric fuzzy Korovkin theorem by means of A-statistical convergence, where A is a non-negative regular summability matrix. We also study rates of A-statistical convergence of a sequence of fuzzy positive linear operators in the trigonometric environment.

1. Introduction

The study of the Korovkin type approximation theory is an area of active research, which deals with the problem of approximating a function by means of a sequence of positive linear operators. In recent years, this theory has been improved with the help of two different ways. The first one, *statistical approximation*, is to use the notion of statistical convergence in the approximation by operators. Earlier studies show that the statistical approximation enables us to obtain more powerful results than the classical aspects (see, for instance, [1, 2, 3, 4, 5]). The second one is to obtain *fuzzy approximation* by fuzzy positive linear operators via the concept of fuzzy set theory (see, [6, 7, 8, 9, 10]). Our primary interest of the present paper is to combine these ways: statistical approximation and fuzzy approximation. So, we obtain a statistical fuzzy Korovkin-type approximation theorem and compute its statistical rates in the approximation. Here the test functions are trigonometric. We should note that, as a rule, neither limits nor statistical limits can be calculated or measured with absolute precision. To reflect this imprecision several approaches in mathematics have been developed: fuzzy set theory, fuzzy logic, interval analysis, set valued analysis, etc.

We first collect some basic definitions and results used in the paper.

A fuzzy number is a function $\mu : \mathbb{R} \to [0,1]$, which is normal, convex, upper semi-continuous and the closure of the set $\mathrm{supp}(\mu)$ is compact, where $\mathrm{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$. The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathcal{F}}$. Let

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}} \text{ and } [\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\},\ (0 < r \leq 1).$$

Then, it is well-known [11] that, for each $r \in [0,1]$, the set $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, it is possible to define uniquely the sum $u \oplus v$ and the product $u \odot v$ as follows:

$$[u \oplus v]^r = [u]^r + [v]^r \text{ and } [\lambda \odot v]^r = \lambda [u]^r,\ (0 \leq r \leq 1).$$

Key words and phrases. Statistical convergence, statistical rates, fuzzy positive linear operators, trigonometric functions, trigonometric fuzzy Korovkin theory, fuzzy modulus of continuity.

2000 *Mathematics Subject Classification.* 26E50, 41A25, 41A36.

Now denote the interval $[u]^r$ by $[u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$ and $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$ for $r \in [0,1]$. Then, for $u, v \in \mathbb{R}_{\mathcal{F}}$, define

$$u \preceq v \Leftrightarrow u_-^{(r)} \leq v_-^{(r)} \text{ and } u_+^{(r)} \leq v_+^{(r)} \text{ for all } 0 \leq r \leq 1.$$

Define also the following metric $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$ by

$$D(u,v) = \sup_{r\in[0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}.$$

In this case, $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space (see [12]). Let $f, g : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy number valued functions. Then, the distance between f and g on $\mathbb{R}$ is given by

$$D^*(f,g) = \sup_{x\in\mathbb{R}} \sup_{r\in[0,1]} \max\left\{\left|f_-^{(r)} - g_-^{(r)}\right|, \left|f_+^{(r)} - g_+^{(r)}\right|\right\}.$$

In this article we consider that $j \to \infty$. Let K be a subset of $\mathbb{N}$. Then, the (asymptotic) density of K is defined by

$$\delta(K) := \lim_j \frac{\#\{n \leq j : n \in K\}}{j}$$

provided the limit exists, where the symbol $\#\{B\}$ denotes the cardinality of a set B. Using this density Fast [13] introduced the notion of statistical convergence of number sequences as follows: $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to a number L if, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ has density zero, i.e.,

$$\delta(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = \lim_j \frac{\#\{n \leq j : |x_n - L| \geq \varepsilon\}}{j} = 0.$$

Now let $A = (a_{jn})$ be an infinite summability matrix. Then, the A-transform of x, denoted $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn} x_n$, provided the series converges for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$ [14]. Assume now that A is a nonnegative regular summability matrix and K is a subset of $\mathbb{N}$. The A-density of K is defined by

$$\delta_A(K) := \lim_j \sum_{n\in K} a_{jn}$$

provided the limit exists. Observe that if we take $A = C_1 = (c_{jn})$, the Cesáro matrix of order one, defined by

$$c_{jn} = \begin{cases} \dfrac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise} \end{cases}$$

then $\delta_{C_1}(K) = \delta(K)$ for any subset K of $\mathbb{N}$. With the help of the A-density, Freedman and Sember [16] introduced the notion of A-statistical convergence, which is a more general method of statistical convergence. Recall that the sequence $(x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$; or equivalently

$$\lim_j \sum_{n:\,|x_n - L|\geq\varepsilon} a_{jn} = 0.$$

This limit is denoted by $st_A - \lim_n x_n = L$. It is not hard to see that if we take $A = C_1$, then C_1-statistical convergence coincides with the statistical convergence mentioned above. If A is replaced by the identity matrix, then we get the ordinary

convergence of number sequences. We also note that if $A = (a_{jn})$ is any nonnegative regular summability matrix for which $\lim_j \max_n\{a_{jn}\} = 0$, then $A-$statistical convergence is stronger than convergence (see [17]). Actually, every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A, but its converse is not always true. Some other results regarding statistical and A-statistical convergences may be found in the papers [18, 19].

Now let $(\mu_n)_{n\in\mathbb{N}}$ be a fuzzy number valued sequence. Then, Nuray and Savaş [20] introduced the fuzzy analog of statistical convergence by using the fuzzy metric D^* instead of the classical absolute value in the above definition. So, by a similar idea, one can obtain the following definition of A-statistical convergence of fuzzy valued sequences. We say that $(\mu_n)_{n\in\mathbb{N}}$ is A-statistically convergent to $\mu \in \mathbb{R}_{\mathcal{F}}$, which is denoted by $st_A - \lim_n D(\mu_n, \mu) = 0$, if for every $\varepsilon > 0$, $\delta_A\left(\{n \in \mathbb{N} : D(\mu_n, \mu) \geq \varepsilon\}\right) = 0$, i.e.,

$$\lim_j \sum_{n:D(\mu_n,\mu)\geq\varepsilon} a_{jn} = 0$$

holds. Of course, the case of $A = C_1$ immediately reduces to the statistical convergence of fuzzy valued sequences. Also, replacing A with the identity matrix, we get the classical fuzzy convergence introduced by Matloka [21].

2. Statistical Fuzzy Trigonometric Korovkin Theory

In this section we prove a fuzzy trigonometric Korovkin-type approximation theorem by means of A-statistical convergence. In order to show that our result is stronger than its classical case we display an example of fuzzy positive linear operators by using fuzzy Fejer operators.

Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy number valued functions. Then f is said to be fuzzy continuous at $x_0 \in \mathbb{R}$ provided that whenever $x_n \to x_0$, then $D\left(f(x_n), f(x_0)\right) \to \infty$ as $n \to \infty$. Also, we say that f is fuzzy continuous on $\mathbb{R}$ if it is fuzzy continuous at every point $x \in \mathbb{R}$. The set of all fuzzy continuous functions on $\mathbb{R}$ is denoted by $C_{\mathcal{F}}(\mathbb{R})$ (see, for instance, [7, 9]). Notice that $C_{\mathcal{F}}(\mathbb{R})$ is only a cone not a vector space. By $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ we mean the space of all fuzzy continuous and 2π-periodic functions on $\mathbb{R}$. Also the space of all real valued continuous and 2π-periodic functions is denoted by $C_{2\pi}(\mathbb{R})$.

Assume that $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is a fuzzy number valued function. Then, f is said to be fuzzy-Riemann integrable (or, FR-integrable) to $I \in \mathbb{R}_{\mathcal{F}}$ if, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any partition $P = \{[u, v]; \xi\}$ of $[a, b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\bigoplus_P (v - u) \odot f(\xi), I\right) < \varepsilon.$$

In this case, we write

$$I := (FR)\int_a^b f(x)dx.$$

By Corollary 13.2 of [10, p. 644], we conclude that if $f \in C_{\mathcal{F}}[a, b]$ (fuzzy continuous on $[a, b]$), then f is FR-integrable on $[a, b]$.

Now let $L : C_{\mathcal{F}}(\mathbb{R}) \to C_{\mathcal{F}}(\mathbb{R})$ be an operator. Then L is said to be fuzzy linear if, for every $\lambda_1, \lambda_2 \in \mathbb{R}$, $f_1, f_2 \in C_{\mathcal{F}}(\mathbb{R})$, and $x \in \mathbb{R}$,

$$L\left(\lambda_1 \odot f_1 \oplus \lambda_2 \odot f_2; x\right) = \lambda_1 \odot L(f_1; x) \oplus \lambda_2 \odot L(f_2; x)$$

holds. Also L is called fuzzy positive linear operator if it is fuzzy linear and, the condition $L(f;x) \preceq L(g;x)$ is satisfied for any $f, g \in C_{\mathcal{F}}(\mathbb{R})$ and all $x \in \mathbb{R}$ with $f(x) \preceq g(x)$.

Throughout the paper we use the test functions f_i $(i = 0, 1, 2)$ defined by

$$f_0(x) = 1,\ \ f_1(x) = \cos x,\ \ f_2(x) = \sin x.$$

Then, we get the following result.

Theorem 2.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators defined on $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators defined on $C_{2\pi}(\mathbb{R})$ with the property*

$$\{L_n(f;x)\}_{\pm}^{(r)} = \tilde{L}_n\left(f_{\pm}^{(r)}; x\right) \tag{2.1}$$

for all $x \in [a,b]$, $r \in [0,1]$, $n \in \mathbb{N}$ and $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$.Assume further that

$$st_A - \lim_n \left\| \tilde{L}_n(f_i) - f_i \right\| = 0 \ \ \textit{for each } i = 0, 1, 2, \tag{2.2}$$

the symbol $\|g\|$ denotes the usual supremum norm of $g \in C_{2\pi}(\mathbb{R})$. Then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, we have

$$st_A - \lim_n D^*\left(L_n(f), f\right) = 0.$$

Proof. Suppose that I is a closed bounded interval with length 2π of $\mathbb{R}$. Now let $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, $x \in I$ and $r \in [0,1]$. Taking $[f(x)]^{(r)} = \left[f_{-}^{(r)}(x),\ f_{+}^{(r)}(x)\right]$ we get $f_{\pm}^{(r)} \in C_{2\pi}(\mathbb{R})$. Hence, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x) \right| < \varepsilon \tag{2.3}$$

for all y satisfying $|y - x| < \delta$. On the other hand, by the boundedness of $f_{\pm}^{(r)}$,

$$\left| f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x) \right| \leq 2 \left\| f_{\pm}^{(r)} \right\| \tag{2.4}$$

holds for all $y \in \mathbb{R}$. Now consider the subinterval $(x - \delta, 2\pi + x - \delta]$ with length 2π. Then, by (2.3) and (2.4), it is not hard to see that

$$\left| f_{\pm}^{(r)}(y) - f_{\pm}^{(r)}(x) \right| \leq \varepsilon + 2M_{\pm}^{(r)} \frac{\varphi(y)}{\sin^2 \frac{\delta}{2}} \tag{2.5}$$

holds for all $y \in (x - \delta, 2\pi + x - \delta]$, where $\varphi(y) := \sin^2\left(\frac{y-x}{2}\right)$ and $M_{\pm}^{(r)} := \left\| f_{\pm}^{(r)} \right\|$. Observe that inequality (2.5) also holds for all $y \in \mathbb{R}$ because of the periodicity of $f_{\pm}^{(r)}$ (see, for instance, [15]). Now using the linearity and the positivity of the

operators $\tilde{L}_n$ and considering inequality (2.5), we may write, for each $n \in \mathbb{N}$, that

$$\begin{aligned} \left|\tilde{L}_n\left(f_{\pm}^{(r)};x\right)-f_{\pm}^{(r)}(x)\right| &\leq \tilde{L}_n\left(\left|f_{\pm}^{(r)}(y)-f_{\pm}^{(r)}(x)\right|;x\right) \\ &\quad +M_{\pm}^{(r)}\left|\tilde{L}_n\left(f_0;x\right)-f_0(x)\right| \\ &\leq \varepsilon+\left(\varepsilon+M_{\pm}^{(r)}\right)\left|\tilde{L}_n\left(f_0;x\right)-f_0(x)\right| \\ &\quad +\frac{2M_{\pm}^{(r)}}{\sin^2\frac{\delta}{2}}\left|\tilde{L}_n\left(\varphi;x\right)\right|. \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} \left|\tilde{L}_n\left(f_{\pm}^{(r)};x\right)-f_{\pm}^{(r)}(x)\right| &\leq \varepsilon+\left(\varepsilon+M_{\pm}^{(r)}+\frac{2M_{\pm}^{(r)}}{\sin^2\frac{\delta}{2}}\right)\left|\tilde{L}_n\left(f_0;x\right)-f_0(x)\right| \\ &\quad +\frac{2M_{\pm}^{(r)}}{\sin^2\frac{\delta}{2}}\left|\tilde{L}_n\left(f_1;x\right)-f_1(x)\right| \\ &\quad +\frac{2M_{\pm}^{(r)}}{\sin^2\frac{\delta}{2}}\left|\tilde{L}_n\left(f_2;x\right)-f_2(x)\right|. \end{aligned}$$

Setting $K_{\pm}^{(r)}(\varepsilon) := \varepsilon + M_{\pm}^{(r)} + \frac{2M_{\pm}^{(r)}}{\sin^2\frac{\delta}{2}}$ and taking supremum over $x \in \mathbb{R}$, we easily see that

$$\begin{aligned} \left\|\tilde{L}_n\left(f_{\pm}^{(r)}\right)-f_{\pm}^{(r)}\right\| &\leq \varepsilon+K_{\pm}^{(r)}(\varepsilon)\{\left\|\tilde{L}_n\left(f_0\right)-f_0\right\| \\ &\quad +\left\|\tilde{L}_n\left(f_1\right)-f_1\right\|+\left\|\tilde{L}_n\left(f_2\right)-f_2\right\|\}. \end{aligned} \tag{2.6}$$

Now it follows from (2.6) that

$$\begin{aligned} D^*\left(L_n(f),f\right) &= \sup_{x\in\mathbb{R}} D\left(L_n(f;x),f(x)\right) \\ &= \sup_{x\in\mathbb{R}}\sup_{r\in[0,1]}\max\left\{\left|\tilde{L}_n\left(f_{-}^{(r)};x\right)-f_{-}^{(r)}(x)\right|,\left|\tilde{L}_n\left(f_{+}^{(r)};x\right)-f_{+}^{(r)}(x)\right|\right\} \\ &= \sup_{r\in[0,1]}\max\left\{\left\|\tilde{L}_n\left(f_{-}^{(r)}\right)-f_{-}^{(r)}\right\|,\left\|\tilde{L}_n\left(f_{+}^{(r)}\right)-f_{+}^{(r)}\right\|\right\}. \end{aligned}$$

Therefore, combining the above equality with (2.6), we have

$$\begin{aligned} D^*\left(L_n(f),f\right) &\leq \varepsilon+K(\varepsilon)\{\left\|\tilde{L}_n\left(f_0\right)-f_0\right\|+\left\|\tilde{L}_n\left(f_1\right)-f_1\right\| \\ &\quad +\left\|\tilde{L}_n\left(f_2\right)-f_2\right\|\}, \end{aligned} \tag{2.7}$$

where $K(\varepsilon) := \sup_{r\in[0,1]} \max \left\{ K_-^{(r)}(\varepsilon), K_+^{(r)}(\varepsilon) \right\}$. Now, for a given $\varepsilon' > 0$, chose $\varepsilon > 0$ such that $0 < \varepsilon < \varepsilon'$, and consider the following sets

$$\begin{aligned} U &: = \{n \in \mathbb{N} : D^* (L_n(f), f) \geq \varepsilon'\}, \\ U_0 &: = \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n (f_0) - f_0 \right\| \geq \frac{\varepsilon' - \varepsilon}{3K(\varepsilon)} \right\}, \\ U_1 &: = \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n (f_1) - f_1 \right\| \geq \frac{\varepsilon' - \varepsilon}{3K(\varepsilon)} \right\}, \\ U_2 &: = \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n (f_2) - f_2 \right\| \geq \frac{\varepsilon' - \varepsilon}{3K(\varepsilon)} \right\}. \end{aligned}$$

Then inequality (2.4) gives

$$U \subseteq U_0 \cup U_1 \cup U_2,$$

which guarantees that, for each $j \in \mathbb{N}$,

$$\sum_{n\in U} a_{jn} \leq \sum_{n\in U_0} a_{jn} + \sum_{n\in U_1} a_{jn} + \sum_{n\in U_2} a_{jn}.$$

If we take limit as $j \to \infty$ on the both sides of inequality (2.6) and use the hypothesis (2.2), we immediately see that

$$\lim_j \sum_{n\in U} a_{jn} = 0,$$

whence the result. □

Concluding Remarks.

1. If we replace the matrix A in Theorem 2.1 by the Cesáro matrix C_1, we immediately get the following statistical fuzzy Korovkin result in the trigonometric case.

Corollary 2.2. *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators defined on $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, and let $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ be a corresponding sequence of positive linear operators defined on $C_{2\pi}(\mathbb{R})$ with the property* (2.1). *Assume that*

$$st - \lim_n \left\| \tilde{L}_n(f_i) - f_i \right\| = 0 \quad \textit{for each } i = 0, 1, 2.$$

Then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, we have

$$st - \lim_n D^* (L_n(f), f) = 0.$$

2. Replacing the matrix A by the identity matrix, one can obtain the classical fuzzy Korovkin result which was introduced by Anastassiou and Gal [9].

Corollary 2.3 ([9])**.** *Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators defined on $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, and let $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ be a corresponding sequence of positive linear operators defined on $C_{2\pi}(\mathbb{R})$ with the property* (2.1)*. Assume that the sequence $\{\tilde{L}_n(f_i)\}_{n\in\mathbb{N}}$ is uniformly convergent to f_i on the whole real line (in the ordinary sense). Then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, the sequence $\{L_n(f)\}_{n\in\mathbb{N}}$ is uniformly convergent to f on the whole real line (in the fuzzy sense).*

3. Now the following application shows that our A-statistical fuzzy Korovkin-type approximation theorem in the trigonometric case (Theorem 2.1) is a non-trivial generalization of its classical case (Corollary 2.3) given by Anastassiou and Gal [9].

Let $A = (a_{jn})$ be any non-negative regular summability matrix. Assume that K is any subset of $\mathbb{N}$ satisfying $\delta_A(K) = 0$. Then define a sequence $(u_n)_{n\in\mathbb{N}}$ by:

$$u_n = \begin{cases} \sqrt{n}, & \text{if } n \in K \\ 0, & \text{if } n \in \mathbb{N}\backslash K. \end{cases} \tag{2.8}$$

In this case, observe that $(u_n)_{n\in\mathbb{N}}$ is non-convergent (in the ordinary sense). However, since for every $\varepsilon > 0$

$$\lim_j \sum_{n:|u_n|\geq\varepsilon} a_{jn} = \lim_j \sum_{n\in K} a_{jn} = \delta_A(K) = 0,$$

we have

$$st_A - \lim_n u_n = 0, \tag{2.9}$$

although the sequence $(u_n)_{n\in\mathbb{N}}$ is unbounded from above. Now define the fuzzy Fejer operators F_n as follows:

$$F_n(f;x) = \frac{1}{n\pi} \odot \left\{ (FR)\int_{-\pi}^{\pi} f(y) \odot \frac{\sin^2\left(\frac{n}{2}(y-x)\right)}{2\sin^2\left[\left(\frac{y-x}{2}\right)\right]} dy \right\}, \tag{2.10}$$

where $n \in \mathbb{N}$, $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ and $x \in \mathbb{R}$. Then observe that the operators F_n are fuzzy positive linear. Also, the corresponding real Fejer operators have the following form:

$$\{F_n(f;x)\}_{\pm}^{(r)} = \tilde{F}_n\left(f_{\pm}^{(r)};x\right) := \frac{1}{n\pi}\int_{-\pi}^{\pi} f_{\pm}(y)\frac{\sin^2\left(\frac{n}{2}(y-x)\right)}{2\sin^2\left[\left(\frac{y-x}{2}\right)\right]}dy$$

where $f_{\pm}^{(r)} \in C_{2\pi}(\mathbb{R})$ and $r \in [0,1]$. Then, we obtain that (see [15])

$$\begin{aligned} \tilde{F}_n(f_0;x) &= 1, \\ \tilde{F}_n(f_1;x) &= \frac{n-1}{n}\cos x, \\ \tilde{F}_n(f_2;x) &= \frac{n-1}{n}\sin x. \end{aligned}$$

Now using the sequence $(u_n)_{n\in\mathbb{N}}$ given by (2.8) we introduce the following fuzzy positive linear operators defined on the space $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$:

$$T_n(f;x) = (1+u_n) \odot F_n(f;x), \tag{2.11}$$

where $n \in \mathbb{N}$, $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$ and $x \in \mathbb{R}$. So, the corresponding real positive linear operators are given by

$$\tilde{T}_n\left(f_{\pm}^{(r)};x\right) := \frac{1+u_n}{n\pi}\int_{-\pi}^{\pi} f_{\pm}(y)\frac{\sin^2\left(\frac{n}{2}(y-x)\right)}{2\sin^2\left[\left(\frac{y-x}{2}\right)\right]}dy,$$

where $f_{\pm}^{(r)} \in C_{2\pi}(\mathbb{R})$. Then we get, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, that

$$\begin{aligned} \left\| \tilde{T}_n(f_0) - f_0 \right\| &= u_n, \\ \left\| \tilde{T}_n(f_1) - f_1 \right\| &\leq u_n + \frac{1+u_n}{n}, \\ \left\| \tilde{T}_n(f_2) - f_2 \right\| &\leq u_n + \frac{1+u_n}{n}. \end{aligned}$$

It follows from (2.9) that

$$st_A - \lim_n \left\| \tilde{T}_n(f_0) - f_0 \right\| = 0. \tag{2.12}$$

Also, by the definition of $(u_n)_{n\in\mathbb{N}}$ we have

$$\lim_n \frac{1+u_n}{n} = 0,$$

which yields, for any non-negative regular matrix $A = (a_{jn})$, that

$$st_A - \lim_n \frac{1+u_n}{n} = 0. \tag{2.13}$$

Now by (2.9) and (2.13) we easily see that, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\left\| \tilde{T}_n(f_1) - f_1 \right\| \geq \varepsilon} a_{jn} \leq \lim_j \sum_{n:|u_n| \geq \frac{\varepsilon}{2}} a_{jn} + \lim_j \sum_{n:\left|\frac{1+u_n}{n}\right| \geq \frac{\varepsilon}{2}} a_{jn} = 0.$$

So we get

$$st_A - \lim_n \left\| \tilde{T}_n(f_1) - f_1 \right\| = 0. \tag{2.14}$$

By a similar idea, one can obtain that

$$st_A - \lim_n \left\| \tilde{T}_n(f_2) - f_2 \right\| = 0. \tag{2.15}$$

Now, with the help of (2.12), (2.14), (2.15), all hypotheses of Theorem 2.1 hold. Then, we conclude, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, that

$$st_A - \lim_n D^*(T_n(f), f) = 0.$$

However, since the sequence $(u_n)_{n\in\mathbb{N}}$ is non-convergent and also unbounded from above, the sequence $\{T_n(f)\}_{n\in\mathbb{N}}$ is not fuzzy convergent to f. Hence, Corollary 2.3 does not work for the operators T_n defined by (2.11).

3. Statistical Fuzzy Rates

In the classical summability settings rates of summation have been introduced in several ways (see for instance, [22, 23, 24]). The concept of statistical rates of convergence, for nonvanishing two null sequences, is studied in [23]. Furthermore, various ways of defining rates of convergence in the A-statistical sense have been introduced in [3] as the following way.

Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $(p_n)_{n\in\mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then

(a) A sequence $x = (x_n)$ is A-statistically convergent to the number L with the rate of $o(p_n)$ if for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

In this case we write $x_n - L = st_A - o(p_n)$ as $n \to \infty$.

(b) If for every $\varepsilon > 0$,

$$\sup_j \frac{1}{p_j} \sum_{n:|x_n| \geq \varepsilon} a_{jn} < \infty,$$

then $(x_n)_{n\in\mathbb{N}}$ is $A-$statistically bounded with the rate of $O(p_n)$ and it is denoted by $x_n = st_A - O(p_n)$ as $n \to \infty$.

(c) $(x_n)_{n\in\mathbb{N}}$ is A-statistically convergent to L with the rate of $o_m(p_n)$, denoted by $x_n - L = st_A - o_m(p_n)$ as $n \to \infty$, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon p_n} a_{jn} = 0.$$

(d) $(x_n)_{n\in\mathbb{N}}$ is A-statistically bounded with the rate of $O_m(p_n)$ provided that there is a positive number M satisfying

$$\lim_j \sum_{n:|x_n| \geq M p_n} a_{jn} = 0,$$

which is denoted by $x_n = st_A - O_m(p_n)$ as $n \to \infty$.

Notice that, in definitions (a) and (b), the "rate" is more controlled by the entries of the summability method rather than the terms of the sequence $(x_n)_{n\in\mathbb{N}}$. But, in order to see the effect on the terms of the sequence we need the definitions (c) and (d), respectively. We should also remark that, for the convergence of fuzzy number valued sequences or fuzzy number valued function sequences, we have to use the metrics D and D^* instead of the absolute value metric in all definitions mentioned above.

Let $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Then the (first) fuzzy modulus of continuity of f, which is introduced by [10] (see also [7, 9]), is defined by

$$w_1^{(\mathcal{F})}(f,\delta) := \sup_{x,y\in\mathbb{R};\ |x-y|\leq\delta} D\left(f(x), f(y)\right)$$

for any $\delta > 0$. With this terminology, we have the following result.

Theorem 3.1. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators defined on $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C_{2\pi}(\mathbb{R})$ with the property* (2.1). *Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are positive non-increasing sequences and also that the operators $\tilde{L}_n$ satisfy the following conditions:*

(i) $\left\|\tilde{L}_n(f_0) - f_0\right\| = st_A - o(a_n)$ *as* $n \to \infty$,

(ii) $w_1^{(\mathcal{F})}(f,\mu_n) = st_A - o(b_n)$ *as* $n \to \infty$, *where* $\mu_n = \sqrt{\left\|\tilde{L}_n(\varphi)\right\|}$ *and* $\varphi(y) = \sin^2\left(\frac{y-x}{2}\right)$ *for each* $x \in \mathbb{R}$.

Then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, we have

$$D^*\left(L_n(f), f\right) = st_A - o(c_n) \text{ as } n \to \infty,$$

where $c_n := \max\{a_n, b_n\}$ for each $n \in \mathbb{N}$. Furthermore, similar results hold when little "o" is replaced by big "O".

Proof. Let $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Then, using the property (2.1) and applying Theorem 4 of [9], we immediately see, for each $n \in \mathbb{N}$, that

$$D^*\left(L_n(f), f\right) \le M \left\| \tilde{L}_n\left(f_0\right) - f_0 \right\| + \left\| \tilde{L}_n\left(f_0\right) + f_0 \right\| w_1^{(\mathcal{F})}(f, \mu_n),$$

where $M := D^*\left(f, \chi_{\{0\}}\right)$ and $\chi_{\{0\}}$ denotes the neutral element for $\oplus$. It follows from the above inequality that

(3.1)

$$D^*\left(L_n(f), f\right) \le M \left\| \tilde{L}_n\left(f_0\right) - f_0 \right\| + \left\| \tilde{L}_n\left(f_0\right) - f_0 \right\| w_1^{(\mathcal{F})}(f, \mu_n) + 2 w_1^{(\mathcal{F})}(f, \mu_n)$$

holds for each $n \in \mathbb{N}$. Now, for a given $\varepsilon > 0$, consider the following sets:

$$\begin{aligned}
V &: = \left\{ n \in \mathbb{N} : D^*\left(L_n(f), f\right) \ge \varepsilon \right\}, \\
V_0 &: = \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n\left(f_0\right) - f_0 \right\| \ge \frac{\varepsilon}{3M} \right\}, \\
V_1 &: = \left\{ n \in \mathbb{N} : \left\| \tilde{L}_n\left(f_0\right) - f_0 \right\| \ge \sqrt{\frac{\varepsilon}{3}} \right\}, \\
V_2 &: = \left\{ n \in \mathbb{N} : w_1^{(\mathcal{F})}(f, \mu_n) \ge \sqrt{\frac{\varepsilon}{3}} \right\}, \\
V_3 &: = \left\{ n \in \mathbb{N} : w_1^{(\mathcal{F})}(f, \mu_n) \ge \frac{\varepsilon}{6} \right\}.
\end{aligned}$$

Hence, inequality (3.1) implies that $V \subseteq V_0 \cup V_1 \cup V_2 \cup V_3$. Then we may write, for each $j \in \mathbb{N}$, that

$$\frac{1}{c_j} \sum_{n \in V} a_{jn} \le \frac{1}{c_j} \sum_{n \in V_0} a_{jn} + \frac{1}{c_j} \sum_{n \in V_1'} a_{jn} + \frac{1}{c_j} \sum_{n \in V_1''} a_{jn} + \frac{1}{c_j} \sum_{n \in V_2} a_{jn}. \tag{3.2}$$

Also using the fact $c_j = \max\{a_j, b_j\}$, we obtain from (3.2) that

$$\frac{1}{c_j} \sum_{n \in V} a_{jn} \le \frac{1}{a_j} \sum_{n \in V_0} a_{jn} + \frac{1}{a_j} \sum_{n \in V_1'} a_{jn} + \frac{1}{b_j} \sum_{n \in V_1''} a_{jn} + \frac{1}{b_j} \sum_{n \in V_2} a_{jn}. \tag{3.3}$$

Therefore, letting $j \to \infty$ on the both sides of inequality (3.3) and using the hypotheses (i) and (ii), we conclude that

$$\lim_j \frac{1}{c_j} \sum_{n \in V} a_{jn} = 0,$$

which means that

$$st_A - \lim_n D^*\left(L_n(f), f\right) = 0$$

for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. □

The above proof can easily be modified to prove the following analog.

Theorem 3.2. *Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of fuzzy positive linear operators on $C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$. Assume that there exists a corresponding sequence $\{\tilde{L}_n\}_{n\in\mathbb{N}}$ of positive linear operators on $C_{2\pi}(\mathbb{R})$ with the property* (2.1). *Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are positive non-increasing sequences and also that the operators $\tilde{L}_n$ satisfy the following conditions:*

(i) $\left\|\tilde{L}_n(f_0) - f_0\right\| = st_A - o_m(a_n)$ *as* $n \to \infty$,

(ii) $w_1^{(\mathcal{F})}(f, \mu_n) = st_A - o_m(b_n)$ *as* $n \to \infty$, *where* μ_n *is given as in Theorem* 3.1.

Then, for all $f \in C_{2\pi}^{(\mathcal{F})}(\mathbb{R})$, *we have*

$$D^* \left(L_n(f), f\right) = st_A - o(d_n) \text{ as } n \to \infty,$$

where $d_n := \max\{a_n, b_n, a_n b_n\}$ *for each* $n \in \mathbb{N}$. *Furthermore, similar results hold when little "o_m" is replaced by big "O_m".*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,345-354,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

THE LAGRANGE - POINCARÉ EQUATIONS FOR A REFINEMENT OF A PRINCIPAL $GP(n;\mathbf{R})$ - BUNDLE

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ABSTRACT. In this paper the geometric structures defined on the refinement of a principal bundle determined by the factorization of the projective group are studied. For this refinement the Poisson structure, the Lagrange - Poincaré and Wong equations are written. [1]

Introduction

The paper consists of three sections. In the first section we present some definitions and results developed in [2] concerning the geometry of the manifold TQ/G. The manifold TQ/G is isomorphic with $T(Q/G)\oplus\widetilde{\mathcal{G}}^G$ by the isomorphism α_{A^G} associated to a connection A^G. The brackets on the vector fields of $\mathcal{X}(Q/G)\oplus Sect(\widetilde{\mathcal{G}}^G)$ and the Poisson structure on $T^*(Q)\oplus\widetilde{\mathcal{G}}^G$ are presented. We associate the reduced of a G - invariant Lagrangian for which the Lagrange - Poincaré equations are written.

The second section deals with refinements of a differentiable principal bundle defined by closed subgroups of the structure group. Also, we define the reduced bundles associated to a refinement of a principal bundle. In the Section 3 is used the theory of reduction described in the first section for the fibre bundles which constitute a refinement of a principal bundle having the projective group as structure group.

Throughout the paper all manifolds are of finite dimension and whitout boundary. All maps are differentiable of C^∞ - class. We use the definitions and results about the fibre bundles and connections defined on manifolds given in [1], [4] and [5]. Notations used are the same as those in [2] and [6].

1. The reduced bundle of a principal G - bundle and the reduced Lagrangian.

We assume that we have the following set up: a manifold Q and an action of the Lie group G on Q, say $\rho: G\times Q\rightarrow Q$.

If $\pi_G: Q\rightarrow Q/G$ is a left principal bundle and the Lie group G acts differentiably on the manifold F on the left, then the *associated fibre bundle* with standard fibre F is, by definition, $Q\times_G F=(Q\times F)/G$, where the action of G on $Q\times F$ is given by $a(q,y)=(aq,ay), (\forall)\ q\in Q, y\in F, a\in G$. Also, $\pi_F: Q\times_G F\rightarrow Q/G$ is a (left) fibre bundle with structure group G.

Let $\mathcal{G}$ the Lie algebra of the Lie group G. The associated bundle with standard fibre $\mathcal{G}$, where the action of G on $\mathcal{G}$ is the adjoint action is called the *adjoint bundle* (here

[1] 2000 Mathematics Subject Classification : 55R05,53C05, 58A30.

Key words and phrases: adjoint bundle, invariant bundle, connection, refinement

$F=\mathcal{G},\ \rho_g = Ad_g, g\in G$) and it is denoted by $\widetilde{\mathcal{G}}^G = Ad_G(Q)$. We let $\widetilde{\pi}_G : \widetilde{\mathcal{G}} \to Q/G$ denote the projection given by $\widetilde{\pi}_G([q,\xi]_G = [q]_G$. Each fibre $\widetilde{\mathcal{G}}^G_x$ of $\widetilde{\mathcal{G}}^G$ carries a natural Lie algebra structure defined by $[\ [q,\xi]_G, [q,\eta]_G\ ] = [\ q, [\xi,\eta]_G\ ]$.

Let TQ be the tangent bundle on Q. An element of T_qQ will be denoted by $v_q, u_q, \ldots$ or by $(q,\dot{q})$. The tangent lift of the action of G on Q defines an action of G on TQ and we can form the quotient $(TQ)/G =: TQ/G$. There is a well defined map $\tau_Q : TQ/G \to Q/G$ induced by the tangent of the projection map $\pi_G : Q \to Q/G$ and given by $[v_q]_G \to [q]_G$. The rules $[v_q]+[u_q]_G = [v_q+u_q]_G$ and $\lambda[v_q]_G = [\lambda v_q]_G$, where $\lambda\in\mathbf{R}, v_q, u_q \in T_qQ$ and $[v_q]_G$ and $[u_q]_G$ are their equivalence classes in the quotient TQ/G define a vector bundle structure on TQ/G having the base Q/G. The fibre $(TQ/G)_x$ is isomorphic, as vector space, to T_qQ, for each $x=[q]_G$.

Let a connection A^G on Q given by one form $A^G : TQ \to \mathcal{G}$ with the properties:

(1) $A^G(\xi q)=\xi$, for all $\xi\in\mathcal{G}$;

(2) $A^G(T_q\rho_g\cdot v) = Ad_g(A^G(v))$, where Ad_g is the adjoint action of G on $\mathcal{G}$.

The restriction of a connection to T_qQ is denoted A^G_q and the vertical and horizontal space defined at $q\in Q$ is $Ver_q = KerT_q\pi_G,\ Hor_q = A^G_q$.

The curvature of A^G, denoted by B^{A^G} is a Lie algebra valued two form on Q given by:

$$B^{A^G}(u_q,v_q) = dA^G(Hor_q(u_q), Hor_q(v_q)).$$

We assume that we have a connection A^G on the bundle $\pi_G : Q \to Q/G$. The map $\alpha_{A^G} : TQ/G \to T(Q/G)\oplus\widetilde{\mathcal{G}}^G$ defined by

$$\alpha_{A^G}([q,\dot{q}]_G) = T\pi_G(q,\dot{q}) \oplus [q, A^G(q,\dot{q})]_G$$

is a well defined vector bundle isomorphism.

Let $TQ = Hor(TQ)\oplus Ver(TQ)$ be the decomposition into horizontal and vertical parts. Since the bundles $Hor(TQ)$ and $Ver(TQ)$ are G- invariant we have $TQ/G = Hor(TQ)/G \oplus Ver(TQ)/G$. We have that

$\alpha_{A^G}(Ver(TQ)/G) = T(Q/G)$ and $\alpha_{A^G}(Ver(TQ)/G) = \widetilde{\mathcal{G}}^G$.

Let $\iota_G(TQ) : I_G(TQ) \to Q/G$ be the vector bundle whose fibre $(\iota_G(TQ))^{-1}(x)$ at an element $x=[q]_G\in Q/G$ is the vector space of all invariant vector fields on Q along π_G^{-1}. That is, an element of $I_G(TQ)$ is a vector field, say, Z, defined only at points $q\in\pi_G^{-1}(x)$, that is $Z(q)\in TQ$ for all $q\in\pi_G^{-1}(x)$ such that $gZ=Z,\ (\forall)\ g\in G$.

We also let $\iota_G(TQ)^V : I^V_G(TQ) \to Q/G$ (resp., $\iota_G(TQ)^H : I^H_G(TQ) \to Q/G$) be the vector bundle whose fibre $(\iota_G(TQ)^V)^{-1}(x)$ (resp., $(\iota_G(TQ)^H)^{-1}(x)$) at an element $x=[q]_G\in Q/G$ is the vector space of all vertical (resp., horizontal) invariant vector fields on π_G^{-1}. That is, an element of $I^V_G(TQ)$ (resp., $I^H_G(TQ)$) is a vector

field, say, Y (resp., X) on the manifold $\pi_G^{-1}(x)$ such that $gY = Y$, $(\forall)\ g \in G$ (resp., $gX = X$, $(\forall)\ g \in G$). We call $\iota_G(TQ)^V : I_G^V(TQ) \to Q/G$ (resp., $\iota_G(TQ)^H : I_G^V(TQ) \to Q/G$) the *vertical* (resp., *horizontal*) *invariant bundle.*

Let $Sect(I_G^H(TQ))$ and $Sect(I_G^V(TQ))$ the Lie algebra of sections of the horizontal and vertical invariant bundle, respectively. The map $T\pi_G$ establishes a well defined isomorphism between $Sect(I_G^H(TQ))$ and $\mathcal{X}(Q/G)$. We can deduce that there are natural identification

$$Sect(T(Q/G) \oplus \widetilde{\mathcal{G}}^G) = \mathcal{X}(Q/G) \oplus Sect(\widetilde{\mathcal{G}}^G).$$

Given the basis $\{\varepsilon_a \mid a = \overline{1,p}\ \}$ for the Lie algebra $\mathcal{G}$ for which $\{C_{bc}^a\ \}$ are the structure constants, we obtain the local basis $\{\ \frac{\partial}{\partial x^i}, e_a\ \}$ for $\pi_G : TQ/G \to Q/G$ and $\{\ e_a\ \}$ for $I_G^V(TQ) \to Q/G$ such that $[e_a, e_b] = C_{ab}^c e_c$.

Let $\{\ A_i^a(x)\ \}$ the local functions on Q/G for the connection A^G defined for $\pi_G : Q \to Q/G$. The corresponding covariant derivative $\widetilde{\nabla}^{A^G}\xi$ of a section $\xi = \xi^a e_a$ of $I_G^V(TQ)$ reads $\widetilde{\nabla}^{A^G}\xi : Q/G \to T^*(Q/G) \oplus I_G^V(TQ)$,

$$\widetilde{\nabla}^{A^G}\xi\ = (\frac{\partial \xi^a}{\partial x^i} + C_{bc}^a A_i^{Gb}\xi^c) dx^i \otimes e_a$$

and if $X \in \mathcal{X}(Q/G)$, then $\widetilde{\nabla}_X^{A^G}\xi$ is given by

$$\widetilde{\nabla}_X^{A^G}\xi\ = X^i(\frac{\partial \xi^a}{\partial x^i} + C_{bc}^a A_i^{Gb}\xi^c) e_a.$$

In particular, we have

$$\widetilde{\nabla}_i^{A^G} e_a = C_{ba}^c A_i^{Gb} e_c. \tag{1.1}$$

The curvature $\widetilde{B}^{A^G}$ is given by $\widetilde{B}^{A^G} = \frac{1}{2}\widetilde{B}_{ij}^{Ga} dx^i \wedge dx^j \otimes e_a$, where

$$\widetilde{B}_{ij}^{Ga}\ =\ \frac{\partial A_j^{Ga}}{\partial x^i} - \frac{A_i^{Ga}}{\partial x^j} + C_{bc}^a A_i^{Gb} A_j^{Gc}. \tag{1.2}$$

Let $X_i \oplus \overline{\xi}_i \in Sect(T(Q/G) \oplus \widetilde{\mathcal{G}}^G)$, $i = 1, 2$ be given two sections. Then $[X_1 \oplus \overline{\xi}_1, X_2 \oplus \overline{\xi}_2] = [X_1, X_2] \oplus \widetilde{\nabla}_{X_1}^{A^G}\xi_2 - \widetilde{\nabla}_{X_2}^{A^G}\xi_1 - \widetilde{B}^{A^G}(X_1, X_2) + [\overline{\xi}_1, \overline{\xi}_2]$.

For $\{\ \frac{\partial}{\partial x^i} \oplus e_a,\ i = \overline{1.n},\ a = \overline{1,p}\ \}$ we have

$$[\frac{\partial}{\partial x^i} \oplus e_a, \frac{\partial}{\partial x^j} \oplus e_b] = (C_{cb}^d A_i^{Gc} - C_{ca}^d A_j^{Gc} - \widetilde{B}_{ij}^{A^G d} + C_{ab}^d) e_d. \tag{1.3}$$

Let $(x^i, \dot{x}^i, \xi^a)$ the local coordinates of $TQ \oplus \widetilde{\mathcal{G}}^G$ and (x^i, p_i, μ_a) the local coordinates of $T^*Q \oplus \widetilde{\mathcal{G}}^*$. The structure Poisson on $T^*Q \oplus \widetilde{G}^*$ is given by

$$(1.4)\qquad \Lambda^G = \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i} - \widetilde{B}^{A^G c}_{ij}\mu_c\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial \mu_j} - $$
$$-C^d_{ca}\mu_d A^{Gc}_i\frac{\partial}{\partial p_i}\wedge\frac{\partial}{\partial \mu_a} + C^c_{ab}\mu_c\frac{\partial}{\partial \mu_a}\wedge\frac{\partial}{\partial \mu_b}.$$

Let $L : TQ \ \to \ \mathbf{R}$ be an invariant Lagrangian, that is $L(g(q,\dot q)) = L(q,\dot q)$, for all $(q,\dot q) \in TQ$, $g \in G$. Because holds this invariance, we get a well defined reduced Lagrangian $l^G : TQ/G \ \to \ \mathbf{R}$ satisfying the relation $l^G([q,\dot q]_G) = L(q,\dot q)$.

We will consider l^G as a function defined on $T(Q/G)\oplus\widetilde{\mathcal{G}}^G$ or TQ/G interchangeably, using the isomorphism α_{A^G}. Also, we will write $l^G(q_G,\dot q_G,\xi)$, to emphasize the dependence of l^G on $(q_G,\dot q_G)\in T(Q/G)$ and $\xi\in\widetilde{\mathcal{G}}^G$.

The vertical resp., horizontal Lagrange-Poincar$\acute{e}$ equation for l^G is given by (1.5.1) resp., (1.5.2):

$$(1.5.1)\qquad \frac{D^{A^G}}{Dt}\frac{\partial l^G}{\partial \xi}(q_G,\dot q_G,\xi) = ad^*_\xi\frac{\partial l^G}{\partial \xi}(q_G,\dot q_G,\xi),\quad \xi\in\widetilde{\mathcal{G}}^G$$

$$(1.5.2)\qquad \frac{\partial l^G}{\partial q_G}(q_g,\dot q_G,\xi) - \frac{D^{A^G}}{Dt}\frac{\partial l^G}{\partial \dot q_G}(q_G,\dot q_G,\xi) =$$
$$=< \frac{\partial l^G}{\partial \xi}(q_G,\dot q_G,\xi, i_{\dot q_G}\widetilde{B}^{A^G}(q_G) >,\quad \xi\in\widetilde{\mathcal{G}}^G.$$

In local coordinates the equation (1.5.1) resp., (1.5.2) becomes :

$$(1.6.1)\qquad \frac{d}{dt}(\frac{\partial l^G}{\partial \xi^a}) = \frac{\partial l^G}{\partial \xi^b}(C^b_{ca}\xi^c - C^b_{ca}A^{Gc}_i\dot x^i)$$

$$(1.6.2)\qquad \frac{\partial l^G}{\partial x^i} - \frac{d}{dt}(\frac{\partial l^G}{\partial \dot x^i}) = \frac{\partial l^G}{\partial \xi^a}(B^{Ga}_{ji}\dot x^j + C^a_{cb}A^{Gb}_i\xi^c)$$

Let μ a metric G - invariant on Q and m^G the metric on Q/G given by :

$$(1.7)\qquad m^G(u_{q_G},v_{q_G}) = \mu(u_q,v_q),\ q_G\in G/G,\ q\in Q,$$

where u_q, v_q are the horizontal vectors for the connection A^G.

Let $\overline{\kappa}^G$ the bi-invariant Riemannian on G and the Lagrangian $L : TQ \ \to \ \mathbf{R}$ given by:

$$(1.8)\qquad L^G(q,\dot q) = \overline{\kappa}^G(A^G(q,\dot q), A^G(q,\dot q) + \frac{1}{2}\mu((q,\dot q),(q,\dot q))).$$

This Lagrangian is G-invariant. An element of $\widetilde{\mathcal{G}}$ has the form $(q,\overline{\xi})_G$, where $q\in Q$ and $\overline{\xi}\in\mathcal{G}$. since $\overline{\kappa}^G$ is bi-invariant follows that its restriction to $\mathcal{G}$ is Ad-invariant and so we can define the fiber metric κ^G on $\widetilde{\mathcal{G}}$ by $\kappa^G((q,\overline{\xi})_G,(q,\overline{\eta})_G) = \overline{\kappa}^G(\overline{\xi},\overline{\eta})$.

The reduced bundle is $T(Q/G)\oplus\widetilde{\mathcal{G}}^G$ and a typical element of it is denoted $(x,\dot x,\xi)$. The reduced Lagrangian is given by:

$$(1.9)\qquad l^G(x,\dot x,\xi) = \frac{1}{2}\kappa^G(\xi,\xi) + \frac{1}{2}\mu(x)(x,\dot x).$$

The vertical resp., horizontal Lagrange- Poincar$\acute{e}$ equation is (1.10.1) resp., (1.10.2):

$$(1.10.1)\qquad \frac{D}{Dt}\kappa^G(\xi,\xi)=0$$

$$(1.10.2)\qquad \frac{\partial l^G}{\partial x}(x,\dot{x},\xi)-\frac{D}{Dt}\frac{\partial l^G}{\partial \dot{x}}(x,\dot{x},\xi)=<\frac{\partial l^G}{\partial \xi}(x,\dot{x},\xi),i_{\dot{x}}\widetilde{B}^G(x)>$$

The equations (1.10.1) and (1.10.2) are the first and the second Wong's equation.
Locally, the expression of the Lagrangian l^G is the following:

$$(1.11)\qquad l^G(x,\dot{x},\xi)=\frac{1}{2}\kappa^G_{ab}\xi^a\xi^b+\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j.$$

The local expression of the vertical resp., horizontal Lagrange-Poincar$\acute{e}$ equation is given by (1.12.1) resp., (1.12.2):

$$(1.12.1)\qquad \frac{dp_b}{dt}=-p_aC^a_{db}A^{Gd}_i\dot{x}^i,\quad \text{where}\quad p_a=\frac{\partial l^G}{\partial \xi^a}=\kappa_{ab}\xi^b$$

$$(1.12.2)\qquad \frac{dp_i}{dt}=-p_aB^{Ga}_{ji}\dot{x}^j-\frac{1}{2}\frac{\partial g^{jk}}{\partial x^i}p_jp_k,\quad \text{where}\quad p_i=g_{ij}x^j.$$

2. The reduced bundles associated to a refinement of a principal G - bundle.

Let G be a Lie group and $\mathcal{N}_q=(G=H_0\supset H_1\supset\ldots\supset H_{q-1}\supset H_q=\{e\})$ (e is the identity element of G) a sequence of Lie groups such that H_j is a closed subgroup of H_{j-1} for $1\le j\le q$. Let $(\xi,\mathcal{N}_q)$ be a structure consisting of a differentiable principal G - bundle $\xi=(E,p,B,G)$ and $\mathcal{N}_q$ a sequence of closed subgroups of G.

Let $E_j=E/H_j,\ j=\overline{0,q},\ H^j_k=H_j/H_k$ and $G^j_k=H_j/N_{jk}$ for $0\le j<k\le q$, where N_{jk} is the largest normal subgroup of K_j included in H_k and H_j/N_{jk} is the factor group of H_j by N_{jk}. Finally, let $p_{jk}:zH_k\in E_k\rightarrow zH_j\in E_j,\ (\forall)z\in E$ for $0\le i<j\le 2$, the canonical map.

The pair $(\xi,\mathcal{N}_q)$ defines the fibre bundles (see [6], MR 53 # 4058):

$$\xi_{jk}=(E_k,p_{jk},E_j,H^j_k,G^j_k)\ ,0\le j<k\le q.$$

The triplet $(\xi;\xi_{0j},\xi_{jq})$, where $\xi_{0j}=(E_j,p_{0j},B,H^0_j,G^0_j)$, $\xi_{jq}=(E,p_{jq},E_j,H_j)$ is called the *refinement of* ξ *defined by* H_j. We have that ξ_{jq} is a principal H_j - bundle for all $0<j<q$.

When $q=2$ and $H_1=H$, the refinement of ξ defined by H is the triple $(\xi;\xi_{01},\xi_{12})$ with $\xi_{01}=(E/H,p_{01},B,G/H,G/N)$ and $\xi_{12}=(E,p_{12},E/H,H)$, where N is the largest normal subgroup of G included in H. This structure can be find in [4].

EXAMPLE 2.1. ([6]) Let $\xi^*=(L_nM,p=\pi_L,M,GL(n;\mathbf{R})$ be the principal bundle of tangent linear frames to n - manifold M. We consider the sequence $\mathcal{N}^*_2=(G=GL(n;\mathbf{R})\supset H=GT(n;\mathbf{R})\supset\{e\})$, where $GT(n;\mathbf{R})=\{\ (a^i_j)\in GL(n;\mathbf{R})\mid a^i_j=0 \text{ for } i>j\ \}$ is the subgroup of upper triangular matrices.

The refinement of ξ^* defined by $GT(n;\mathbf{R})$ is $(\xi^*;\xi^*_{01},\xi^*_{12})$, where $\xi^*_{01} = (\mathcal{D}_nM, p_{01}, M, GL(n;\mathbf{R})/GT(n;\mathbf{R}), GP(n-1;\mathbf{R}))$ is the fibre bundle of tangent flags to M and $\xi^*_{12} = (L_nM, p_{12}, \mathcal{D}_nM, GT(n;\mathbf{R}))$. Here $\mathcal{D}_nM = L_nM/GT(n;\mathbf{R})$ is the manifold of tangent flags to M and $GP(n-1;\mathbf{R})) = GL(n;\mathbf{R})/D(n;\mathbf{R})$ is the real projective group of order $n-1$, since the largest subgroup normal subgroup of $GL(n;\mathbf{R})$ included in $GT(n;\mathbf{R})$ is $D(n;\mathbf{R}) = \{\ (\lambda\delta^i_j) \mid (\forall)\ \lambda \in \mathbf{R}\ \}$. □

REMARK 2.1. (i) If H is a closed subgroup of G such that $N = \{e\}$, then the refinement of $\xi = (E,p,B,G)$ defined by H is $(\xi;\xi_{01},\xi_{12})$, where $\xi_{01} = (E/H, p_{01}, B, G/H, G)$ and $\xi_{12} = (E, p_{12}, E/H, H)$.

(ii) If H is a normal closed subgroup of G, then the refinement of $\xi = (E,p,B,G)$ defined by H is a refinement $(\xi;\xi_{01},\xi_{12})$, where $\xi_{01} = (E/H, p_{01}, B, G/H)$ and $\xi_{12} = (E, p_{12}, E/H, H)$ are principal bundles .□

EXAMPLE 2.2. Let M be a n - manifold and $(T(M), \pi_T, M, \mathbf{R}^n, GL(n;\mathbf{R}))$ be the fibre bundle of tangent vectors on M.

By affine frame at $x \in M$ we mean a triple $u = (x,y,z)$, where $(y,z) \in \pi_T^{-1}(x) \times \pi_L^{-1}(x)$. We denote by $\mathcal{A}_nM$ the set of affine frames on M endowed with the differentiable structure canonically induced from the differentiable structure of M. In a local chart of M, an affine frame at $x \in M$ is given by:

(2.1) $$y = y^i(\frac{\partial}{\partial x^i})_x, \quad z_j = z^i_j(\frac{\partial}{\partial x^i})_x, \quad det(z^i_j) \neq 0$$

and the local coordinates of $u = (x,y,z) \in \mathcal{A}_nM$ are (x^i, y^i, z^i_j), $i,j = \overline{1,n}$.

Let be the affine group $GA(n;\mathbf{R}) = \{\begin{pmatrix} 1 & 0 \\ a^i & g^i_j \end{pmatrix} \in GL(n+1;\mathbf{R}) \mid det(g^i_j) \neq 0\ \}$. The action of the Lie group $G = GA(n;\mathbf{R})$ on the manifold $\mathcal{A}_nM$ is defined by the right translations $\tau_{(a,g)} : G \times \mathcal{A}_nM \rightarrow \mathcal{A}_nM$, $(\forall)\ (a,g) \in G$ with $a = (a^i) \in \mathbf{R}^n$, $g = (g^i_j) \in GL(n;\mathbf{R})$ where:

(2.2) $$((a,g),u) \rightarrow (\tau_{(a,g)}(u) = (x, y + za, zg) \text{ for all } u = (x,y,z) \in \mathcal{A}_nM.$$

Let $\eta = (\mathcal{A}_nM, p_A, M, GA(n;\mathbf{R}))$ be the principal bundle of affine frames to an n - manifold M, where $p_A : \mathcal{A}_NM \rightarrow M$ is the canonical projection given by $p_A(u) = x$, $(\forall)\ u = (x,y,z) \in \mathcal{A}_nM$. We consider the sequence $\mathcal{N}^1_2 = (GA(n;\mathbf{R}) \supset T(n;\mathbf{R}) \supset \{e\})$, where $T(n;\mathbf{R}) = \{\begin{pmatrix} 1 & 0 \\ a^i & \delta^i_j \end{pmatrix}\}$. Since $T(n;\mathbf{R})$ is normal in G, applying Remark 2.1.(ii), the refinement of η defined by $T(n;\mathbf{R})$ is $(\eta;\eta_{01},\eta_{12})$, where $\eta_{01} = (\mathcal{A}_nM/T(n;\mathbf{R}), p_{01}, M, GA(n;\mathbf{R})/T(n;\mathbf{R}))$ and $\eta_{12} = (\mathcal{A}_nM, p^2_{12}, \mathcal{A}_nM/T(n;\mathbf{R}), T(n;\mathbf{R}))$. □

Let Q be a manifold and G a Lie group which acts differentiably on Q. We oonsider $\pi_G : Q \rightarrow Q/G$ the principal bundle with the structure group G. We assume that is given a sequence $\mathcal{N}_2 = (G \supset K \supset \{e\})$ of closed subgroups of G.

If we denote $\eta = (Q, \pi_G, Q/G, G)$, then the pair $(\eta;\mathcal{N}_2)$ determines a refinement $(\eta;\eta_{01},\eta_{12})$ of η defined by K, where
$\eta_{01} = (Q/K, \pi_{GK}, Q/G, G/K, G/N)$ and $\eta_{12} = (Q, \pi_K, Q/K, K)$, and N is the largest

normal subgroup of G included in K.

Let A^G and A^K two connections on Q given by the forms $A^G : TQ \to \mathcal{G}$, $A^K : TQ \to \mathcal{K}$, where $\mathcal{G}$ resp., $\mathcal{K}$ is the Lie algebra of G resp., K.

Let the adjoint bundles $\widetilde{\mathcal{G}} = Ad_G(Q)$, $\widetilde{\mathcal{K}} = Ad_K(Q)$ and the isomorphisms $\alpha_{A^G} : TQ/G \to T(Q/G) \oplus \widetilde{\mathcal{G}}^G$, $\alpha_{A^K} : TQ/K \to T(Q/K) \oplus \widetilde{\mathcal{K}}^K$.

The vector bundles $T(Q/G) \oplus \widetilde{\mathcal{G}}^G \to Q/G$ and $T(Q/K) \oplus \widetilde{\mathcal{K}}^K \to Q/K$ are called the *reduced bundles associated to refinement defined by* $(\eta; \mathcal{N}_2)$.

3. Geometric structures on the fibre bundles of a refinement of a principal $GP(n;\mathbf{R})$ - bundle.

Let us we apply the above considerations in the case the group G is the projective group $GP(n;\mathbf{R})$ and K is the affine group $GA(n;\mathbf{R})$.

Let $G = GL(n+1,\mathbf{R})/D(n+1,\mathbf{R})$ the projective group of order n. The class $[a^i_j]$ determined by the matrix (a^i_j) is denoted by a. The subgroup K of G is determined by all classes $[a^i_j]$ for which the matrix (a^i_j) satisfy the condition $a^{n+1}_h = 0,\ h = \overline{1,n}$ and it may be identified of the affine group of order n. We obtain thus a sequence $\mathcal{N}_2 = (G \supset K \supset \{e\})$, where $G = GP(n;\mathbf{R})$ and $K = GA(n;\mathbf{R})$.

A base for the Lie algebra $\mathcal{G}$ of G is $\{\ \varepsilon^i_j, \varepsilon^i, \varepsilon_j\ \}$ and we have

$$[\varepsilon^i_j, \varepsilon^l_k] = \delta^{ilp}_{kqj}\varepsilon^q_p, \quad [\varepsilon^i_j, \varepsilon_k] = \delta^i_k\varepsilon_j, \quad [\varepsilon^i_j, \varepsilon^k] = -\delta^k_j\varepsilon^i, \quad [\varepsilon^i, \varepsilon^j] = 0. \quad [\varepsilon_i, \varepsilon_j] = 0,$$
$$[\varepsilon^i, \varepsilon_j] = -\gamma^{ip}_{jq}\varepsilon^q_p, \quad \text{where} \quad \delta^{ilp}_{kqj} = \delta^i_k\delta^l_q\delta^p_j - \delta^l_j\delta^i_q\delta^p_k, \quad \gamma^{ip}_{jq} = \delta^i_j\delta^p_q + \delta^i_q\delta^p_j.$$

A base for the Lie algebra $\mathcal{K}$ of K is $\{\ \varepsilon^i_j, \varepsilon_j\ \}$ and we have

$$[\varepsilon^i_j, \varepsilon^l_k] = \delta^{ilp}_{kqj}\varepsilon^q_p, \quad [\varepsilon^i_j, \varepsilon_k] = \delta^i_k\varepsilon_j, \quad [\varepsilon_i, \varepsilon_j] = 0.$$

Let $\pi_G : Q \to Q/G$ the principal bundle having the projective group G as structure group. Let $h^{n+i}_{n+i}(U^{n+i}_{n+i}, \chi^{n+i}_{n+i})$ the local charts of Q with the coordinates (q^i, x^i_j, x^i, x_j). The base of sections of the vector bundle $\widetilde{\mathcal{G}}^G \to Q/G$ is

$$e^i_j = x^h_j\frac{\partial}{\partial x^h_i} + x_j\frac{\partial}{\partial x_i}, \quad e_j = x^h_j\frac{\partial}{\partial x^h} - x_j(x^h_k\frac{\partial}{\partial x^h_k} + x^h\frac{\partial}{\partial x^h} + x_k\frac{\partial}{\partial x_k}), \quad e^i = x^h\frac{\partial}{\partial x^h_i} + \frac{\partial}{\partial x_i}.$$

Let A^G a connection on the principal bundle $\pi_G : Q \to Q/G$ given by the functions $(P^i_{jl}, P^i_l, P_{jl})$ on Q/G. The following relations hold:

$$
(3.1)\qquad \begin{cases}
\widetilde{\nabla}^{A^G}_{\frac{\partial}{\partial q^i}} e^l_k = \delta^{lps}_{qkr} P^r_{pi} e^q_s + \gamma^{lq}_{pk} P^p_i e_q + P_{ki} e^l \\
\widetilde{\nabla}^{A^G}_{\frac{\partial}{\partial q^i}} e^l = P^l_{hi} e^h + \gamma^{ls}_{pk} P^k_i e^p_s, \quad \widetilde{\nabla}^{A^G}_{\frac{\partial}{\partial q^i}} e_k = P^j_{ki} e_j - \gamma^{sq}_{pk} P_{qi} e^p_s \\
\widetilde{B}^{A^G} = \frac{1}{2} B^l_{kij} dq^i \wedge dq^j \otimes e^k_l + \frac{1}{2} B_{lij} dq^i \wedge dq^j \otimes e^l + \frac{1}{2} B^l_{ij} dq^i \wedge dq^j \otimes e_l.
\end{cases}
$$

Let $(q^i, \dot{q}^i, \xi^l_k, \xi_k, \xi^l)$ the local coordinates on $TQ \oplus \widetilde{\mathcal{G}}^G$ and $(q^i, p_i, \mu^l_k, \mu^k, \mu_l)$ the local coordinates on $T^*Q \oplus \widetilde{\mathcal{G}}^{G^*}$. The structure Poisson is given by the following relations:

(3.2.1) $\{q^i, q^j\} = 0,\ \{q^i, p_j\} = \delta^i_j,\ \{q^i, \mu^l_k\} = 0,\ \{q^i, \mu^l\} = 0,\ \{q^i, \mu_l\} = 0;$

(3.2.2) $\{p_i, p_j\} = -B^l_{kij}\mu^k_l - B_{lij}\mu^l - B^l_{ij}\mu_l; \qquad \{p_i, \mu^l_k\} = \mu^l_j P^j_{ik} - \mu^q_p(\delta^p_i P_{qk} + \delta^l_q P_{ki});$

(3.2.3) $\{p_i, \mu^l\} = \mu^k_p(\delta^p_k P^l_i + \delta^l_k P^p_i) + \mu^h P^l_{hi}; \qquad \{\mu^i_j, \mu^l_k\} = \delta^i_k \mu^l_j - \delta^l_j \mu^i_k;$

(3.2.4) $\{\mu^i_j, \mu^l\} = -\delta^l_j \mu^i, \quad \{\mu^i_j, \mu_l\} = \delta^i_l \mu_j, \quad \{\mu^l, \mu_k\} = -(\delta^l_k \delta^p_q + \delta^l_q \delta^p_k)\mu^q_p;$

(3.2.5) $\{\mu^l, \mu^k\} = 0, \quad \{\mu_l, \mu_k\} = 0.$

Let $\overline{\kappa}^G$ the bi-invariant Riemannian metric on G and the Lagrangian $L : TQ \to \mathbf{R}$ given by (1.8). The reduced Lagrangian l^G on $T(Q/G) \oplus \widetilde{\mathcal{G}}^G$ is given by :

$$
(3.3)\qquad \begin{aligned} l^G(q^i, \dot{q}^i, \xi^i_j, \xi_j, \xi^i) = {} & \frac{1}{2}\kappa_{ij}\xi^i\xi^j + \frac{1}{2}\kappa^{jl}_{ik}\xi^i_j\xi^k_l + \frac{1}{2}\kappa^{ij}\xi^i\xi^j + \\ & + \kappa^j_{il}\xi^l\xi^j_i + \kappa^{jl}_i \xi_l \xi^i_j + \frac{1}{2}\kappa_{ih}\xi^i\xi^h + \kappa^l_i \xi_l \xi^i + \frac{1}{2} g_{ij}\dot{q}^i\dot{q}^j. \end{aligned}
$$

and we have

$$
(3.4)\qquad \begin{cases}
p^i_j = \dfrac{\partial l^G}{\partial \xi^i_j} = \kappa^{jl}_{ih}\xi^k_l + \kappa^j_{il}\xi^l + \kappa^{jl}_i \xi_l, \quad p_i = \dfrac{\partial l^G}{\partial \xi_i} = \kappa^l_{ih}\xi^h_l + \kappa_{ih}\xi^h + \kappa^l_i \xi_l \\
p^i = \dfrac{\partial l^G}{\partial \xi_i} = \kappa^{il}_h \xi^h_l + \kappa^i_h \xi^h + \kappa^{ih}\xi_h, \quad \overline{p}_i = \dfrac{\partial l^G}{\partial q^i} = g_{ij}\dot{q}^j.
\end{cases}
$$

The Lagrange-Poincaré equation and Wong's equations for the projective group are given by the relations (3.5.1)-(3.5.4):

$$
(3.5.1)\quad \frac{dp^j_i}{dt} = -[(p^l_h(\delta^j_l P^h_{ik} - \delta^h_i P^j_{lk} + (\delta^h_i P^j_k + \delta^j_i P^h_k)p_h + P_{ik}p^j] \cdot q^k
$$

$$
(3.5.2)\quad \frac{dp_i}{dt} = -[P^h_{ik}p_h - (\delta^l_h P_{ik} + \delta^l_k P_{hi})p^h_l] \cdot q^k
$$

$$
(3.5.3)\quad \frac{dp^i}{dt} = -[P^i_{kh}p_k + (\delta^l_k P^i_h + \delta^i_k P^l_h)p^k_l]\dot{q}^h
$$

$$
(3.5.4)\quad \frac{d\overline{p}_i}{dt} = -(p^l_h B^h_{lij} + p_{ij} B^h_{ij} + p^l B_{lij})\dot{q}^j - \frac{1}{2}\frac{\partial g^{jl}}{\partial q^i}\overline{p}_j\overline{p}_l.
$$

Let $\pi_K : Q \to Q/K$ the principal bundle having the affine group K as structure group and the local coordinates (q^i, η_i) on Q/K. The base of sections of the vector bundle $\widetilde{\mathcal{K}}^K \to Q/K$ is $e^i_j = x^h_j \dfrac{\partial}{\partial x^h_i}$, $\quad e_j = x^h_j \dfrac{\partial}{\partial x^h}$.

Let A^K a connection on the principal bundle $\pi_K : Q \to Q/K$ given by the functions $(A^{hr}_k, A^{hr}, A^h_{kr}, A^h_r)$ on Q/K. The following relations (3.6) hold:

$$
(3.6) \qquad \begin{cases}
\widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial q^r}} e^l_k = (A^p_{kr}\delta^l_q - A^l_{qr}\delta^p_k)e^q_p - A^l_r e_k, \quad \widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial \eta_r}} e^l_k = (A^{pr}_k \delta^l_q - A^{lr}_q \delta^p_k)e^q_p - A^{lr} e_k \\
\widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial q^r}} e_k = A^i_{kr} e_i, \qquad \widetilde{\nabla}^{A^K}_{\frac{\partial}{\partial \eta_r}} e_k = (A^{ir}_k e_i \\
\widetilde{B}^{A^K} = \frac{1}{2}(B^l_{kij} dq^i \wedge dq^j + B^{lh}_{ki} dq^i \wedge d\eta_h + B^{lhi}_k d\eta_h \wedge d\eta_i) \otimes e^k_l + \\
+\frac{1}{2}(B^l_{ij} dq^i \wedge dq^j + B^{lh}_i dq^i \wedge d\eta_h + B^{lhi} d\eta_h \wedge d\eta_i) \otimes e_l.
\end{cases}
$$

Let $(q^i, \eta_i, \dot{q}^i, \dot{\eta}_i, \xi^l_k, \xi^l)$ the local coordinates on $TQ \oplus \widetilde{\mathcal{K}}^K$ and $(q^i, \eta_i, p_i, \lambda^i, \mu^l_k, \mu_l)$ the local coordinates on $T^*Q \oplus \widetilde{\mathcal{K}}^{K^*}$. The structure Poisson is given by the relations:

$$\{q^i, p_j\} = \delta^j_i, \quad \{\eta_i, \lambda^j\} = \delta^j_i, \quad \{p_i, p_j\} = -\frac{1}{2}(B^l_{kij}\mu^k_l + B^l_{ij}\mu_l);$$

$$\{p_i, \lambda^j\} = -\frac{1}{2}(B^{lj}_{ki}\mu^k_l + B^{lj}_i \mu^l), \quad \{\lambda^i, \lambda^j\} = -\frac{1}{2}(B^{lji}_k \mu^k_l + B^{lji}\mu_l);$$

$$\{p_i, \mu^l_k\} = \mu^q_p(A^p_{ki}\delta^l_q - A^l_{qi}\delta^p_k) - \mu_k A^l_i; \quad \{\lambda^r, \mu^l_k\} = \mu^q_p(A^{pr}_k \delta^l_q - A^{lp}_q \delta^r_k) - \mu_k A^{lr};$$

$$\{p_r, \mu_k\} = A^i_{kr}\mu_i, \ \{\lambda^r, \mu_k\} = A^{ir}_k \mu_i; \{\mu^i_j, \mu^l_k\} = -\delta^i_k \mu^l_j + \delta^l_j \mu^i_k, \{\mu^i_k, \mu_j\} = \delta^i_k \mu_j.$$

Let $\overline{\kappa}^K$ the bi-invariant Riemannian metric on K and the Lagrangian $L : TQ \to \mathbf{R}$ given by (1.8). The reduced Lagrangian l^K on $T(Q/K) \oplus \widetilde{\mathcal{K}}^K$ is given by:

$$
(3.7) \qquad l^K(q^i, \eta_i, \dot{q}^i, \dot{\eta}_i, \xi^a_b, \xi^a) = \frac{1}{2}\kappa_{ab}\xi^a\xi^b + \frac{1}{2}\kappa^c_{ba}\xi^a\xi^b_c +
+\frac{1}{2}\kappa^{cd}_{ab}\xi^a_c\xi^b_d + \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j + \frac{1}{2}g^j_i\dot{q}^i\eta_j + \frac{1}{2}g^{ij}\eta_i\eta_j.
$$

and we have :

$$
(3.8) \qquad \begin{cases}
p^b_a = \dfrac{\partial l^K}{\partial \xi^a_b} = \kappa^{bc}_{ad}\xi^d_c + \kappa^b_{ac}\xi^c, \quad p_a = \dfrac{\partial l^K}{\partial \xi^a} = \kappa^c_{ad}\xi^d_c + \kappa_{ab}\xi^b \\
p^i = \dfrac{\partial l^K}{\partial \dot{\eta}_i} = g^i_j \dot{q}^j + g^{ij}\dot{\eta}_j, \qquad \overline{p}_i = \dfrac{\partial l^K}{\partial \dot{q}^i} = g_{ij}\dot{q}^j + g^j_i \dot{\eta}_j.
\end{cases}
$$

The Lagrange-Poincaré equations and Wong's equations for the affine group are given

by the relations (3.9.1)-(3.9.4):

$$\frac{dp_a^b}{dt} = -[p_p^q(A_{qi}^b\delta_q^p - A_{qi}^b\delta_a^p)\dot{q}^i - A_i^b p_a\dot{q}^i + p_q^p(A_a^{pi}\delta_q^p - A_q^{bi}\delta_a^p)\cdot\eta_i - A^{bi}p_a\eta_i] \tag{3.9.1}$$

$$\frac{dp_a}{dt} = -[p_bA_{ai}^b\dot{q}^i + p_bA_a^{bi}\dot{\eta}_i] \tag{3.9.2}$$

$$\frac{dp^i}{dt} = -(p_a^bB_b^{ahi}\dot{\eta}_h + p_a^bB_h^{ai}\dot{q}^h + p_aB^{ahi}\dot{\eta}_h + p_a^bB_{bh}^{ai}\dot{q}^h) - \tag{3.9.3}$$
$$-\frac{1}{2}\frac{\partial\widetilde{g}^{jl}}{\partial\eta_i}p_jp_l - \frac{\partial\widetilde{g}_l^j}{\partial\eta_i}p_jp^l - \frac{1}{2}\frac{\partial\widetilde{g}_{jl}}{\partial\eta_i}p^jp^l$$

$$\frac{d\overline{p}_i}{dt} = -(p_a^bB_{bji}^a\dot{q}^j + p_a^bB_{bi}^{aj}\dot{\eta}_j + p_aB_{ji}^a\dot{q}^i + p_aB_i^{aj}\dot{\eta}_j) - \tag{3.9.4}$$
$$-\frac{1}{2}\frac{\partial\widetilde{g}^{jl}}{\partial q^i}p_jp_l - \frac{\partial\widetilde{g}_l^j}{\partial q^i}p_jp^l - \frac{1}{2}\frac{\partial\widetilde{g}_{jl}}{\partial q^i}p^jp^l,$$

where $\widetilde{g}^{jl}$ are the elements of the inverse of matrix (g_{ij}).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,355-366,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

EXPLICIT p-ADIC q-EXPANSION FOR THE ALTERNATING SUMS OF POWERS

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Abstract. In this paper, we give an explicit p-adic expansion of

$$\sum_{\substack{j=1\\(j,p)=1}}^{np} \frac{(-1)^j}{[j]_q^r}$$

as a power series in n. The coefficients are values of p-adic q-l-function for q-Euler numbers.

§1. Introduction

Let p be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, cf.[1, 4, 6, 14]. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$,

Key words and phrases. p-adic q-integrals, Euler numbers, p-adic l-function.
2000 Mathematics Subject Classification: 11S80, 11B68, 11M99 .

then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Kubota and Leopoldt proved the existence of meromorphic functions, $L_p(s,\chi)$, defined over the p-adic number field, that serve as p-adic equivalents of the Dirichlet L-series, cf.[1, 14,15,16]. These p-adic L-functions interpolate the values

$$L_p(1-n,\chi) = -\frac{1}{n}(1-\chi_n(p)p^{n-1})B_{n,\chi_n}, \text{ for } n \in \mathbb{N} = \{1,2,\cdots,\},$$

where $B_{n,\chi}$ denote the nth generalized Bernoulli numbers associated with the primitive Dirichlet character χ, and $\chi_n = \chi w^{-n}$, with w the *Teichmüller* character, cf.[8,14]. In [14,15], L. C. Washington have proved the following interesting formula:

$$\sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{1}{j^r} = -\sum_{k=1}^{\infty} \binom{-r}{k} (pn)^k L_p(r+k, w^{1-k-r}), \text{ where } \binom{-r}{k} \text{ is binomial coefficient.}$$

To give the q-extension of the above Washington result, the author derived the sums of powers of consecutive q-integers as follows:

$$(*) \qquad \sum_{l=0}^{n-1} q^l [l]_q^{m-1} = \frac{1}{m} \sum_{l=0}^{m-1} \binom{m}{l} q^{ml} \beta_l [n]_q^{m-l} + \frac{1}{m}(q^{mn}-1)\beta_m, \text{ see [7,8] },$$

where β_m are q-Bernoulli numbers. By using (*), we gave an explicit p-adic expansion

$$\sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{q^j}{[j]_q^r} = -\sum_{k=1}^{\infty} \binom{-r}{k} [pn]_q^k L_{p,q}(r+k, w^{1-r-k})$$
$$-(q-1)\sum_{k=1}^{\infty} \binom{-r}{k} [pn]_q^k T_{p,q}(r+k, w^{1-r-k}) - (q-1)\sum_{a=1}^{p-1} B_{p,q}^{(n)}(r, a:F),$$

where $L_{p,q}(s,\chi)$ is p-adic q-L-function (see [7,12]). Indeed, this is a q-extension result due to Washington, corresponding to the case $q=1$, see [14]. Recently, the second author described algorithms to deal with nested symbolic sums over combinations of harmonic sum, binomial coefficients and denominators[13]. In addition, he treated Mellin transforms and the inverse Mellin transformation for functions that are encountered in Feynman diagram calculations. Together with results for the values of the higher harmonic sum at infinity the presented algorithm can be used for the symbolic

evaluation of whole classes of integrals that were thus far intractable [13]. The computation of Feynman diagrams has confronted physicists with classes of integrals that usually hard to be evaluated, both analytically and numerically. In [10], Rim and Kim treated explicit p-integral alternating harmonic sums. Harmonic sum is critical to explain the resonating phenomenon in a nature. It is also important as a solution of simple harmonic oscillating system in classical mechanics and quantum mechanics (see [17]). p-adic harmonic sum can be applied to these physical phenomena. With this application, p-adic harmonic sum also can be used for quantum statistical physics or quantum transportation theory(see [17]). Euler numbers and polynomials in alternating harmonic sum are used for Langevine equation of magnetism which is in the system with viscosity. For a fixed positive integer d with $(p, d) = 1$, set

$$\begin{aligned}
&X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \\
&X_1 = \mathbb{Z}_p, X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\
&a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},
\end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$, (cf.[1,3,4,9,16]). We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x, y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x, y) \to (a, a)$, cf.[3]. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q} \sum_{0\leq j<p^N} q^j f(j) = \sum_{0\leq j<p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p), \text{ cf.[3,4,6,7,8,9,10]},$$

which represents a q-analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit of those sums(as $n \to \infty$) if this limit exists. The q-Volkenborn integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$(1)\ \ I_q(f) = \int_X f(x)d\mu_q(x) = \int_{X_d} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{x=0}^{dp^N-1} f(x)q^x, \text{ cf. [6,7]}\ .$$

It is well known that the familiar Euler polynomials $E_n(z)$ are defined by means of the following generating function:

$$F(z, t) = \frac{2}{e^t + 1}e^{zt} = \sum_{n=0}^{\infty} E_n(z)\frac{t^n}{n!}, \text{ cf.[3,11]}.$$

We note that, by substituting $z=0$, $E_n(0)=E_n$ are the familiar n-th Euler numbers. Over five decades ago, Carlitz defined q-extension of Euler numbers and polynomials, cf.[1, 4, 5]. Recently, author gave another construction of q-Euler numbers and polynomials (see [1, 5, 9]). By using author's q-Euler numbers and polynomials, we gave the alternating sums of powers of consecutive q-integers as follows: For $m \geq 1$, we have

$$2\sum_{l=0}^{n-1}(-1)^l [l]_q^m = (-1)^{n+1}\sum_{l=0}^{m-1}\binom{m}{l} q^{nl} E_{l,q}[n]_q^{m-l} + \left((-1)^{n+1}q^{nm}+1\right)E_{m,q},$$

where $E_{l,q}$ are q-Euler numbers (see [3]). From this result, we can study the p-adic interpolating function for q-Euler numbers and sums of powers due to author [7]. Throughout this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1-q} \text{ cf.[3,11,5,9].}$$

Note that when p is prime $[p]_q$ is an irreducible polynomial in $Q[q]$. Furthermore, this means that $Q[q]/[p]_q$ is a field and consequently rational functions $r(q)/s(q)$ are well defined $\mod [p]_q$ if $(r(q), s(q)) = 1$. In a recent paper [3] the author constructed the new q-extensions of Euler numbers and polynomials. In Section 2, we introduce the q-extension of Euler numbers and polynomials. In Section 3 we construct a new q-extension of Dirichlet's type l-function which interpolates the q-extension of generalized Euler numbers attached to χ at negative integers. The values of this function at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with at negative integers. The purpose of this paper is to construct the new q-extension of generalized Euler numbers attached to χ due to author and prove the existence of a specific p-adic interpolating function which interpolate the q-extension of generalized Bernoulli polynomials at negative integer. Finally, we give an explicit p-adic q-expansion

$$\sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{(-1)^j}{[j]_q^r},$$

as a power series in n. The coefficients are values of p-adic q-l-function for q-Euler numbers.

2. Preliminaries

For any non-negative integer m, the q-Euler numbers, $E_{m,q}$, were represented by

(2)
$$\frac{2}{[2]_q}\int_{\mathbb{Z}_p} q^{-x}[x]_q^m d\mu_{-q}(x) = E_{m,q} = 2\left(\frac{1}{1-q}\right)^m \sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{1}{1+q^i},\ \text{see [3,10]}\ .$$

Note that $\lim_{q\to 1} E_{m,q} = E_m$. From Eq.(2), we can derive the below generating function:

$$F_q(t) = 2e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\frac{1}{1+q^j}(-1)^j(\frac{1}{1-q})^j\frac{t^j}{j!} = \sum_{j=0}^{\infty}E_{n,q}\frac{t^n}{n!}. \tag{3}$$

By using p-adic q-integral, we can also consider the q-Euler polynomials, $E_{n,q}(x)$, as follows:

(4)
$$E_{n,q}(x) = \frac{2}{[2]_q}\int_{\mathbb{Z}_p} q^{-t}[x+t]_q^n d\mu_{-q}(t) = 2(\frac{1}{1-q})^n\sum_{k=0}^{n}\binom{n}{k}\frac{(-q^x)^k}{1+q^k},\ \text{cf.[3,6,7,8,9,10]}.$$

Note that

$$E_{n,q}(x) = \frac{2}{[2]_q}\int_{\mathbb{Z}_p}([x]_q + q^x[t]_q)^n q^{-t} d\mu_{-q}(x) = \sum_{j=0}^{n}\binom{n}{j}q^{jx}E_{j,q}[x]_q^{n-j}. \tag{5}$$

By (4), we easily see that

$$\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!} = F_q(x,t) = 2e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}\frac{(-1)^j}{1+q^j}q^{jx}(\frac{1}{1-q})^j\frac{t^j}{j!}. \tag{6}$$

From (6), we derive

$$F_q(x,t) = 2\sum_{n=0}^{\infty}(-1)^n e^{[n+x]_q t} = \sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}. \tag{7}$$

3. On the q-analogue of Hurwitz's type ζ-function associated with q-Euler numbers

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. It is easy to see that

$$E_{n,q}(x) = [m]_q^n \sum_{a=0}^{m-1} (-1)^a E_{n,q^m}(\frac{a+x}{m}), \text{ see [3,6,10]} ,$$

where m is odd positive integer. From (7), we can easily derive the below formula:

$$E_{k,q}(x) = \frac{d^k}{dt^k} F_q(x,t)|_{t=0} = 2\sum_{n=0}^{\infty} (-1)^n [n+x]_q^k. \tag{8}$$

Thus, we can consider a q-ζ-function which interpolates q-Euler numbers at negative integer as follows:

Definition 1. *For $s \in \mathbb{C}$, define*

$$\zeta_{E,q}(s,x) = 2\sum_{m=1}^{\infty} \frac{(-1)^n}{[n+x]_q^s}.$$

Note that $\zeta_{E,q}(s,x)$ has a meromorphic function in whole complex plane.

By using Definition 1 and Eq.(8), we obtain the following:

Proposition 2. *For any positive integer k, we have*

$$\zeta_{E,q}(-k,x) = E_{k,q}(x).$$

Let χ be the Dirichlet character with conductor $f \in \mathbb{N}$. Then we define the generalized q-Euler numbers attached to χ as

$$F_{q,\chi}(t) = 2\sum_{n=0}^{\infty} e^{[n]_q t} \chi(n)(-1)^n = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}. \tag{9}$$

Note that

$$E_{n,\chi,q} = [f]_q^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_{n,q^f}(\frac{a}{f}), \text{ where } f(= odd) \in \mathbb{N} . \tag{10}$$

By (9), we easily see that

$$\frac{d^k}{dt^k} F_{q,\chi}(t)|_{t=0} = E_{k,\chi,q} = 2\sum_{n=1}^{\infty} \chi(n)(-1)^n [n]_q^k \tag{11}$$

Definition 3. *For $s \in \mathbb{C}$, we define Dirichlet's type l-function as follows:*

$$l_q(s,\chi) = 2\sum_{n=1}^{\infty} \frac{\chi(n)(-1)^n}{[n]_q^s}.$$

From (11) and Definition 3, we can derive the below theorem.

Theorem 4. *For $k \geq 1$, we have*

$$l_q(-k,\chi) = E_{k,\chi,q}.$$

In [3], it was known that

$$2\sum_{l=0}^{n-1}(-1)^l[l]_q^m = \left((-1)^{n+1}q^n E_{m,q}(n) + E_{m,q}\right), \text{ where } m,n \in \mathbb{N}. \tag{12}$$

From (4) and (12), we derive

$$\begin{aligned} &2\sum_{l=0}^{n-1}(-1)^l[l]_q^m \\ &= (-1)^{n+1}\sum_{l=0}^{m-1}\binom{m}{l}q^{nl}E_{l,q}[n]_q^{m-l} + \left((-1)^{n+1}q^{nm}+1\right)\right) E_{m,q}. \end{aligned} \tag{13}$$

Let s be a complex variable, and let a and $F(= odd)$ be the integers with $0 < a < F$. We now consider the partial q-zeta function as follows:

$$H_q(s,a:F) = \sum_{\substack{m\equiv a(F)\\ m>0}} \frac{(-1)^m}{[m]_q^s} = (-1)^a\frac{[F]_q^{-s}}{2}\zeta_{E,q^F}(s,\frac{a}{F}). \tag{14}$$

For $n \in \mathbb{N}$, we note that $H_q(-n,a:F) = (-1)^a\frac{[F]_q^n}{2}E_{n,q^F}(\frac{a}{F})$. Let χ be the Dirichlet's character with conductor $F(= odd)$. Then we have

$$l_q(s,\chi) = 2\sum_{a=1}^{F}\chi(a)H_q(s,a:F). \tag{15}$$

The function $H_q(s,a:F)$ will be called the q-extension of partial zeta function which interpolates q-Euler polynomials at negative integers. The values of $l_q(s,\chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with $l_q(s,\chi)$ at the negative integers in Section 4.

§4. p-ADIC q-l-FUNCTIONS AND SUMS OF POWERS

We define $< x >=< x : q >= \frac{[x]_q}{w(x)}$, where $w(x)$ is the *Teichmüller* character. When $F(= odd)$ is multiple of p and $(a,p)=1$, we define a p-adic analogue of (14) as follows:

$$H_{p,q}(s,a:F) = \frac{(-1)^a}{2} < a >^{-s} \sum_{j=0}^{\infty} \binom{-s}{j} q^{ja} \left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F}, \text{ for } s \in \mathbb{Z}_p. \tag{16}$$

Thus, we note that

(17)

$$\begin{aligned} H_{p,q}(-n,a:F) &= \frac{(-1)^a}{2} < a >^{n} \sum_{j=0}^{n} \binom{n}{j} q^{ja} \left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F} \\ &= \frac{(-1)^a}{2} w^{-n}(a)[F]_q^n E_{n,q^F}(\frac{a}{F}) = w^{-n}(a) H_q(-n,a:F), \text{ for } n \in \mathbb{N}. \end{aligned}$$

We now construct the p-adic analytic function which interpolates q-Euler number at negative integer as follows:

$$l_{p,q}(s,\chi) = 2 \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) H_{p,q}(s,a:F). \tag{18}$$

In [5, 9], it was known that

$$E_{k,\chi,q} = \frac{2}{[2]_{q^f}} \int_X \chi(x)[x]_q^k q^{-x} d\mu_{-q}(x), \text{ for } k \in \mathbb{N}.$$

For $f(= odd) \in \mathbb{N}$, we note that

$$E_{n,\chi,q} = [f]_q^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_{n,q^f}(\frac{a}{f}).$$

Thus, we have

(18-1)

$$\begin{aligned} l_{p,q}(-n,\chi) &= 2 \sum_{\substack{a=1 \\ (p,a)=1}}^{F} \chi(a) H_{p,q}(-n,a:F) = \frac{2}{[2]_{q^f}} \int_{X^*} \chi w^{-n}(x)[x]_q^n q^{-x} d\mu_{-q}(x) \\ &= E_{n,\chi w^{-n},q} - [p]_q^n \chi w^{-n}(p) E_{n,\chi w^{-n},q^p}. \end{aligned}$$

In fact,

$$l_{p,q}(s,\chi)=2\sum_{a=1}^{F}(-1)^a <a>^{-s}\chi(a)\sum_{j=0}^{\infty}\binom{-s}{j}q^{ja}\left(\frac{[F]_q}{[a]_q}\right)^j E_{j,q^F},\ \text{for } s\in\mathbb{Z}_p. \tag{19}$$

This is a p-adic analytic function and has the following properties for $\chi=w^t$:

$$l_{p,q}(-n,w^t)=E_{n,q}-[p]_q^n E_{n,q^p},\ \text{where } n\equiv t\ (\mod p-1), \tag{20}$$

$$l_{p,q}(s,t)\in\mathbb{Z}_p \text{ for all } s\in\mathbb{Z}_p \text{ when } t\equiv 0(\mod p-1). \tag{21}$$

If $t\equiv 0(\mod p-1)$, then $l_{p,q}(s_1,w^t)\equiv l_{p,q}(s_2,w^t)(\mod p)$ for all $s_1,s_2\in\mathbb{Z}_p$, $l_{p,q}(k,w^t)\equiv l_{p,q}(k+p,w^t)(\mod p)$. It is easy to see that

$$\frac{1}{r+k-1}\binom{-r}{k}\binom{1-r-k}{j}=\frac{-1}{j+k}\binom{-r}{k+j-1}\binom{k+j}{j}, \tag{22}$$

for all positive integers r,j,k with $j,k\geq 0$, $j+k>0$, and $r\neq 1-k$. Thus, we note that

$$\frac{1}{r+k-1}\binom{-r}{k}\binom{1-r-k}{j}=\frac{1}{r-1}\binom{-r+1}{k+j}\binom{k+j}{j}. \tag{22-1}$$

From (22) and (22-1), we derive

$$\frac{r}{r+k}\binom{-r-1}{k}\binom{-r-k}{j}=\binom{-r}{k+j}\binom{k+j}{j}. \tag{23}$$

By using (13), we see that

(24)

$$\begin{aligned}\sum_{l=0}^{n-1}\frac{(-1)^{Fl+a}}{[Fl+a]_q^r}&=\sum_{l=0}^{n-1}(-1)^l(-1)^a\left([a]_q+q^a[F]_q[l]_{q^F}\right)^{-r}\\&=-\sum_{s=1}^{\infty}[a]_q^{-r}\left(\frac{[F]_q}{[a]_q}\right)^s q^{as}(-1)^a\binom{-r}{s}\frac{(-1)^n}{2}\sum_{l=0}^{s-1}\binom{s}{l}q^{nFl}E_{l,q^F}[n]_{q^F}^{s-l}\\&-\sum_{s=1}^{\infty}[a]_q^{-r}\left(\frac{[F]_q}{[a]_q}\right)^s q^{as}(-1)^a\binom{-r}{s}\frac{((-q^{Fs})^n-1)}{2}E_{s,q^F}+\frac{(1-(-1)^n)[a]_q^{-r}}{2}(-1)^a.\end{aligned}$$

For $s \in \mathbb{Z}_p$, we define the below T-Euler polynomials:

$$T_{n,q}(s,a:F) = (-1)^a < a >^{-s} \sum_{k=1}^{\infty} \binom{-s}{k} [\frac{a}{F}]_{q^F}^{-k} q^{ak} \left((-1)^n q^{nFk} - 1\right) E_{k,q^F}. \tag{25}$$

Note that $\lim_{q \to 1} T_{n,q}(s,a:F) = 0$, if n is even positive integer. From (23) and (24), we derive

$$\begin{aligned} &\sum_{l=0}^{n-1} \frac{(-1)^{Fl+a}}{[Fl+a]_q^r} \\ &= -\sum_{s=1}^{\infty} \binom{-r}{s} [a]_q^{-r} \left(\frac{[F]_q}{[a]_q}\right)^s \frac{(-q^s)^a}{2} \sum_{l=0}^{s-1} \binom{s}{l} q^{nFl} E_{l,q^F} [n]_{q^F}^{s-l} \\ &- \frac{w^{-r}(a)}{2} T_{n,q}(r,a:F), \text{ where } n \text{ is positive even integer .} \end{aligned} \tag{26}$$

Let n be positive even integer. Then, we evaluate the right side of Eq.(26) as follows:
(27)

$$\begin{aligned} &\sum_{s=1}^{\infty} \binom{-r}{s} [a]_q^{-r} \left(\frac{[F]_q}{[a]_q}\right)^s \frac{(-q^s)^a (-1)^n}{2} \sum_{l=0}^{s-1} \binom{s}{l} q^{nFl} E_{l,q^F} [n]_{q^F}^{s-l} \\ &= \sum_{k=1}^{\infty} \frac{r}{r+k} \binom{-r-1}{k} [a]_q^{-k-r} q^{ak} (-1)^n [Fn]_q^k \frac{(-1)^a}{2} \sum_{l=0}^{\infty} \binom{-r-k}{l} q^{al} \left(\frac{[F]_q}{[a]_q}\right)^l E_{l,q^F}. \end{aligned}$$

It is easy to check that

$$q^{nFl} = \sum_{j=0}^{l} \binom{l}{j} [nF]_q^j (q-1)^j = 1 + \sum_{j=1}^{l} \binom{l}{j} [nF]_q^j (q-1)^j. \tag{28}$$

Let
(29)

$$K_{p,q}(s,a:F) = \frac{(-1)^a}{2} < a >^{-s} \sum_{l=1}^{\infty} \binom{-s}{l} q^{al} \left(\frac{[F]_q}{[a]_q}\right)^l E_{l,q^F} \sum_{j=1}^{l} \binom{l}{j} [nF]_q^j (q-1)^j.$$

Note that $\lim_{q \to 1} K_{p,q}(s,a;F) = 0$. For $F = p$, $r \in \mathbb{N}$, we see that

$$2 \sum_{a=1}^{p-1} \sum_{l=0}^{n-1} \frac{(-1)^{a+pl}}{[a+pl]_q^r} = 2 \sum_{\substack{j=1 \\ (j,p)=1}}^{np} \frac{(-1)^j}{[j]_q^r}. \tag{30}$$

For $s \in \mathbb{Z}_p$, we define p-adic analytically continued function on $\mathbb{Z}_p$ as

$$K_{p,q}(s,\chi) = 2\sum_{a=1}^{p-1}\chi(a)K_{p,q}(s,a:F),$$
$$T_{p,q}(s,\chi) = 2\sum_{a=1}^{p-1}\chi(a)T_{n,q}(s,a:F), \text{ where } k,n \geq 1\ . \tag{31}$$

From (24)-(31), we derive

$$2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{[j]_q^r} = -\sum_{k=1}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k l_{p,q}(r+k,w^{-r-k})$$
$$-\sum_{k=1}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k K_{p,q}(r+k,w^{-r-k}) - T_{p,q}(r,w^{-r}).$$

Therefore we obtain the following theorem:

Theorem 5. *Let p be an odd prime and let $n \geq 1$ be positive even integer. Then we have*

$$2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{[j]_q^r} = -\sum_{k=1}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k l_{p,q}(r+k,w^{-r-k})$$
$$-\sum_{k=1}^{\infty}\frac{r}{r+k}\binom{-r-1}{k}(-1)^n[pn]_q^k K_{p,q}(r+k,w^{-r-k}) - T_{p,q}(r,w^{-r}), \tag{32}$$

where r is positive integer.

Remark. When r is non-positive integer, we can easily derive the value of the left side of Eq.(32) from Eq.(13).

For $q = 1$ in (32), we have

$$2\sum_{\substack{j=1\\(j,p)=1}}^{np}\frac{(-1)^j}{j^r} = -\sum_{k=1}^{\infty}\frac{r}{k+r}\binom{-r-1}{k}(-1)^n(pn)^k l_p(r+k,w^{-r-k}),$$

where n is positive even integer (see [10]).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,367-375,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Fuzzy multi-metric spaces

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Abstract

The purpose of this paper is to introduce the concept of fuzzy multi-metric space by combining Smarandache multi-spaces with fuzzy metric space. Also some characteristics of fuzzy multi-metric space are obtained. Furthermore, we extend the Banach fixed point theorem to (fuzzy) contractive mappings in fuzzy multi-metric spaces.

Keywords. Multi space; multi-metric space; triangular norm; fuzzy metric space.

M.S.C. (2000). 54A40; 54E35; 54E40; 54E45.

1. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [13]. Many authors have introduced the concept of fuzzy metric space in different ways [1-3,5-8]. George and Veeramani [3,4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [8] and defined a Hausdorff topology on this fuzzy metric space. They also showed that every metric induces a fuzzy metric.

The notion of multi-spaces is introduced by Smarandache in [10] under his idea of hybrid mathematics: combining different fields into a unifying field [11]. The definition of multi-metric space is given by Mao [9], combining Smarandache multi-spaces with the classical metric spaces. He also give some characteristics of a multi-metric space.

In this paper, we give the notion of fuzzy multi-metric space by combining Smarandache multi-spaces with the fuzzy metric space in the sense of George and Veeremani [3]. We give some theorems on covergence and continuity in fuzzy multi-metric space. Furthermore, we extend the Banach fixed point theorem to (fuzzy) contractive mappings, in our sense and Grabiec's sense [5], on complete fuzzy multi-metric spaces (in George and Veeramani's sense).

2. Preliminaries

Definition 1 ([12]). *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is continuous t-norm if* $*$ *is satisfying the following conditions:*

(i) $*$ is commutative and associative;
(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2 ([12]). *A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all x, y, $z \in X$, s, $t > 0$,*

(i) $M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(v) $M(x, y, .) : (0, \infty) \to [0,1]$ is continuous.

Remark 1. *In fuzzy metric space X, $M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

Example 1. *Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0,1]$ and let M_d be a fuzzy set on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}$$

for all $k, m, n \in \mathbb{R}^+$. Then $(X, M_d, *)$ is a fuzzy metric space.

Remark 2. *Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and hence M is a fuzzy metric with respect to any continuous t-norm. In the above example by taking $k = m = n = 1$, we get*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

We call this fuzzy metric induced by a metric d the standard fuzzy metric.

Definition 3 ([3]). *Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$$

is called the open ball with center x and radius r with respect to t.

Theorem 1 ([3]). *Every open ball $B(x, r, t)$ is an open set.*

Remark 3. *Let $(X, M, *)$ be a fuzzy metric space. Define $\tau = \{A \subset X :$ for each $x \in X$, there exist $t > 0$, $r \in (0,1)$ such that $B(x, r, t) \subset A\}$. Then τ is a topology on X.*

Remark 4.

(i) Since $\{B(x, \frac{1}{n}, \frac{1}{n}) : n = 1, 2, ...\}$ is a local base at x, the topology τ is first countable.
(ii) Every fuzzy metric space is Hausdorff.

(iii) Let $(X, M, *)$ be an fuzzy metric space and τ be the topology on X induced by the fuzzy metric. Then for a sequence $(x_n)_n$ in X, $x_n \longrightarrow x$ if and only if $M(x_n, x, t) \longrightarrow 1$ as $n \longrightarrow \infty$.
(iv) In a fuzzy metric space every compact set is closed and bounded.

Definition 4 ([3]). *Let $(X, M, N, *, \Diamond)$ be a fuzzy metric space. Then,*

(i) A sequence $(x_n)_n$ in X is said to be Cauchy if for each $\varepsilon > 0$ and each $t > 0$, there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.
(ii) $(X, M, *)$ is called complete if every Cauchy sequence convergent with respect to M.

Definition 5 ([10]). *A multi-metric space is a union $\widetilde{X} = \cup_{i=1}^m X_i$ such that each X_i is a metric space with metric d_i for all i, $1 \leq i \leq m$. When we say a multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$, it means that a multi-metric space with metrics $d_1, d_2, ..., d_m$ such that (X_i, d_i) is a metric space for any integer i, $1 \leq i \leq m$.*

3. Main results

Definition 6. *A fuzzy multi-metric space is a union $\widetilde{X} = \cup_{i=1}^m X_i$ such that each X_i is a fuzzy metric space with fuzzy metric M_i and t-norm $*$, for all i, $1 \leq i \leq m$.*

When we say a fuzzy multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$, it means that a fuzzy multi-metric space with fuzzy metrics $M_1, M_2, ..., M_m$ such that $(X_i, M_i, *)$ is a fuzzy metric space for any integer i, $1 \leq i \leq m$.

Remark 5. *The following two extremal cases are permitted in Definition 6:*

(i) There are integers $i_1, i_2, ..., i_s$ such that $X_{i_1} = X_{i_2} = \cdots = X_{i_s}$, where $i_j \in \{1, 2, ..., m\}$, $1 \leq j \leq s$.
(ii) There are integers $l_1, l_2, ..., l_s$ such that $M_{l_1} = M_{l_2} = \cdots = M_{l_s}$, where $l_j \in \{1, 2, ..., m\}$, $1 \leq j \leq s$.

Definition 7. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space. We define open ball $B(x, r, t)$ with centre $x \in \widetilde{X}$ and radius r, $r \in (0, 1)$, $t > 0$ as $B(x, r, t) = \{y \in \widetilde{X}$: there exists an integer k, $1 \leq k \leq m$ such that $M_k(x, y, t) > 1 - r\}$.*

Remark 6. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space. Define $\tau = \{A \subset \widetilde{X}$: for each $x \in \widetilde{X}$, there exist $t > 0$, $r \in (0, 1)$ such that $B(x, r, t) \subset A\}$. Then τ is a topology on $\widetilde{X}$.*

Remark 7. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a multi-metric space such that each (X_i, d_i) is a metric space for all i, $1 \leq i \leq m$. We define $a * b = ab$ and*

$$M_i(x, y, t) = \frac{t}{t + d_i(x, y)}$$

then $\widetilde{X}$ is a fuzzy multi-metric space. We call those M_i as the standard fuzzy multi-metric induced by d_i. Even if we take $a * b = \min(a, b)$, $\widetilde{X}$ will be a fuzzy multi-metric space. The metric space (X_i, d_i) is complete iff the standard fuzzy metric space $\left(X_i, M_{i_{d_i}}, *\right)$ is complete for all i, $1 \leq i \leq m$. Then the multi-metric space $\widetilde{X}$ is complete if and only if the standard fuzzy multi-metric space is complete.

Corollary 1. *If $M_1, M_2, ..., M_m$ are m fuzzy metrics on a space X, then $M_1 * M_2 * ... * M_m$ is a fuzzy metric on X, where $*$ is min or product t-norm.*

Corollary 2. *If $d_1, d_2, ..., d_m$ are m metrics on a space X, then*

$$\frac{t}{t + d_1(x, y)} * \frac{t}{t + d_2(x, y)} * \cdots * \frac{t}{t + d_m(x, y)}$$

is a fuzzy metric on X where $a * b = ab$.

Definition 8. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space and $(x_n)_n$ be a sequence in $\widetilde{X}$. $(x_n)_n$ is said converge to a point x, $x \in \widetilde{X}$ if for any ε, $\varepsilon \in (0, 1)$ there exist numbers n_0 and i, $1 \leq i \leq m$ such that if $n \geq n_0$ then $M_i(x_n, x, t) > 1 - \varepsilon$, for each $t > 0$.*

For $(x_n)_n$ convergence to a point x, $x \in \widetilde{X}$, we denote it by $\lim_n x_n = x$ or $x_n \longrightarrow x$.

Theorem 2. *$\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space and $(x_n)_n$ be a sequence in $\widetilde{X}$. Then $x_n \longrightarrow x$ iff there exist integers i, $1 \leq i \leq m$ such that $M_i(x_n, x, t) \longrightarrow 1$ as $n \longrightarrow \infty$.*

Theorem 3. *A sequence $(x_n)_n$ in a fuzzy multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$ is convergent if and only if there exist integers n_0 and k, $1 \leq k \leq m$, such that the subsequence $\{x_n : n \geq n_0\}$ is a convergent sequence in $(X_k, M_k, *)$.*

Proof. If $(x_n)_n$ is a convergent sequence in the fuzzy multi-metric space $\widetilde{X}$, by definition for any ε, $\varepsilon \in (0, 1)$, there exist a point x, $x \in \widetilde{X}$ and natural numbers $n_0(\varepsilon)$ and k, $1 \leq k \leq m$, such that if $n \geq n_0(\varepsilon)$, then $M_k(x_n, x, t) > 1 - \varepsilon$ for all $t > 0$. That is, $\{x_n : n \geq n_0(\varepsilon)\} \subset X_k$ and $\{x_n : n \geq n_0(\varepsilon)\}$ is a convergent sequence in $(X_k, M_k, *)$.

If there exist integer n_0 and k, $1 \leq k \leq m$, such that $\{x_n : n \geq n_0\}$ is a convergent subsequence in $(X_k, M_k, *)$, then for any ε, $\varepsilon \in (0, 1)$, by definition there exists a positive integer p_0 and a point x, $x \in \widetilde{X}$ such that $M_k(x_n, x, t) > 1 - \varepsilon$, for all $t > 0$, where $n \geq \max\{n_0, p_0\}$. Hence, $(x_n)_n$ is a convergent sequence in the fuzzy multi-metric space $\widetilde{X}$. □

Theorem 4. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space, $(x_n)_n$, $(y_n)_n$ are sequences in $\widetilde{X}$ and $(t_n)_n \subset (0, \infty)$. If $x_n \longrightarrow x_0$, $y_n \longrightarrow y_0$,*

$t_n \longrightarrow t$ and there is an integer p, $1 \leq p \leq m$ such that $x_0, y_0 \in X_p$, $t > 0$, then $\lim_n M_p(x_n, y_n, t_n) = M_p(x_0, y_0, t)$.

Proof. Since $x_n \longrightarrow x_0$ and $y_n \longrightarrow y_0$ there exist integers n_1 and n_2 such that if $n \geq \max\{n_1, n_2\}$, then $x_n, y_n \in X_p$. Fix $\delta > 0$ such that $\delta < t/2$. Then, there exists an integer n_3 such that $|t_n - t| < \delta$ for all $n \geq n_0 = \max\{n_1, n_2, n_3\}$. Hence,

$$M_p(x_n, y_n, t_n) \geq M_p\left(x_n, x_0, \delta\right) * M_p(x_0, y_0, t - 2\delta) * M_p\left(y_n, y_0, \delta\right)$$

and

$$M_p(x_0, y_0, t + 2\delta) \geq M_p\left(x_n, x_0, \delta\right) * M_p(x_n, y_n, t_n) * M_p\left(y_n, y_0, \delta\right)$$

for all $n \geq n_0$. By taking limits when $n \longrightarrow \infty$, we get

$$\lim_n M_p(x_n, y_n, t_n) \geq 1 * M_p(x_0, y_0, t - 2\delta) * 1$$

and

$$M_p(x_0, y_0, t + 2\delta) \geq 1 * \lim_n M_p(x_n, y_n, t_n) * 1$$

respectively. So, by continuity of the function $t \longrightarrow M_p(x, y, t)$ we obtain

$$\lim_n M_p(x_n, y_n, t_n) = M_p(x_0, y_0, t).$$

□

Theorem 5. *If $(x_n)_n$ is a convergent sequence in a fuzzy multi metric space $\widetilde{X} = \cup_{i=1}^m X_i$, then $(x_n)_n$ has only one limit point.*

Proof. Let $x_n \longrightarrow x_1$, $x_n \longrightarrow x_2$ and $x_1, x_2 \in \widetilde{X}$. Then there exist integer n_0 and i, $1 \leq i \leq m$ such that $x_n \in X_i$ for all $n \geq n_0$. Hence,

$$1 \geq M_i(x_1, x_2, t) \geq M_i\left(x_1, x_n, \frac{t}{2}\right) * M_i\left(x_n, x_2, \frac{t}{2}\right).$$

By taking limits when $n \longrightarrow \infty$, we get

$$1 \geq M_i(x_1, x_2, t) \geq 1 * 1 = 1$$

which implies $x_1 = x_2$. □

Theorem 6. *Any convergent sequence in a fuzzy multi metric space is a bounded points set.*

Proof. It is clear from Theorem 5. □

Definition 9. *A sequence $(x_n)_n$ in a fuzzy multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$ is called Cauchy sequence if for each ε, $\varepsilon \in (0,1)$, $t > 0$, there exist integers $n_0(\varepsilon)$ and s, $1 \leq s \leq m$ such that $M_s(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0(\varepsilon)$.*

Theorem 7. *A Cauchy sequence $(x_n)_n$ in a fuzzy multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$ is convergent if and only if for all k, $1 \leq k \leq m$, $|(x_n)_n \cap X_k|$ is finite or infinite but $(x_n)_n \cap X_k$ is convergent in $(X_k, M_k, *)$.*

Proof. The necessity of conditions is clear from Theorem 3.

Now, we prove the sufficiency. By definition, there exist integers s, $1 \leq s \leq m$ and n_1 such that $x_n \in X_s$ for $n \geq n_1$. If $|(x_n)_n \cap X_k|$ is infinite and $\lim_{n\to\infty}((x_n)_n \cap X_k) = x$, then there must be $k = s$. Denoted by $(x_n)_n \cap X_k = \{x_{k_1}, x_{k_2}, ..., x_{k_n}, ...\}$. For any δ, $\delta \in (0,1)$, there exists an integer n_2, $n_2 \geq n_1$ such that $M_k(x_m, x_n, \frac{t}{2}) > 1-\delta$ and $M_k(x_{k_n}, x, \frac{t}{2}) > 1-\delta$ for all $m, n \geq n_2$ and for all $t > 0$. Since $\delta \in (0,1)$, we can find a ε, $\varepsilon \in (0,1)$, such that $(1-\delta)*(1-\delta) > 1-\varepsilon$. Then, by using Theorem 4, we get that

$$\begin{aligned} M_k(x_n, x, t) &\geq M_k\left(x_n, x_{k_n}, \frac{t}{2}\right) * M_k\left(x_{k_n}, x, \frac{t}{2}\right) \\ &\geq (1-\delta)*(1-\delta) > 1-\varepsilon. \end{aligned}$$

Hence, $\lim_n x_n = x$ which completes the proof. □

Definition 10. *A fuzzy multi-metric space is said to be complete if every Cauchy sequence is convergent.*

Theorem 8. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a complete fuzzy multi-metric space. For a ball sequence $(B(x_n, \varepsilon_n, t))_n$, where $0 < \varepsilon_n < 1$ for $n = 1, 2, ...$, the following conditions hold:*

(i) $B(x_1, \varepsilon_1, t) \supset B(x_2, \varepsilon_2, t) \supset ... \supset B(x_n, \varepsilon_n, t) \supset ...$
(ii) $\lim_n \varepsilon_n = 0$.

Then, $\cap_{n=1}^\infty B(x_n, \varepsilon_n, t)$ only has one point.

Proof. First, we prove that the sequence $(x_n)_n$ is a Cauchy sequence in $\widetilde{X}$. By the condition (i), we know that if $m \geq n$, then $x_m \in B(x_m, \varepsilon_m, t) \subset B(x_n, \varepsilon_n, t)$ for all $t > 0$. Whence, for all i, $1 \leq i \leq m$, $M_i(x_m, x_n, t) > 1-\varepsilon_n$ for $x_m, x_n \in X_i$. For any ε, $\varepsilon \in (0,1)$, since $\lim_n \varepsilon_n = 0$, there exists an integer $n_0(\varepsilon)$ such that if $n > n_0(\varepsilon)$, then $\varepsilon_n < \varepsilon$. Therefore, if $x_n \in X_l$, then $\lim_m x_m = x_n$. Whence, there exists an integer n_0 such that $m \geq n_0$, $x_m \in M_l$ by Theorem 3. Take integers $m, n \geq \max\{n_0, n_0(\varepsilon)\}$. We know that

$$M_l(x_m, x_n, t) > 1-\varepsilon_n > 1-\varepsilon.$$

So, $(x_n)_n$ is a Cauchy sequence.

By the assumption, $\widetilde{X}$ is complete. We know that the sequence $(x_n)_n$ is convergence to a point x_0, $x_0 \in \widetilde{X}$. By conditions (i) and (ii), we have that $M_l(x_0, x_n, t) > 1-\varepsilon_n$ as $m \longrightarrow \infty$. Hence, $x_0 \in \cap_{n=1}^\infty B(x_n, \varepsilon_n, t)$.

Now if there is a point $y \in \cap_{n=1}^\infty B(x_n, \varepsilon_n, t)$, then there must be $y \in X_l$. We get that

$$1 \geq M_l(y, x_0, t) = \lim_n M_l(y, x_n, t) \geq \lim_n (1-\varepsilon_n) = 1$$

for all $t > 0$, by Theorem 4. Therefore, $M_l(y, x_0, t) = 1$ which implies $y = x_0$. □

Definition 11. *Let $\widetilde{X_1}$ and $\widetilde{X_2}$ be two fuzzy multi-metric spaces and f be a mapping from $\widetilde{X_1}$ to $\widetilde{X_2}$, $x_0 \in \widetilde{X_1}$ and $f(x_0) = y_0$. For ε, $\varepsilon \in (0,1)$ if there exists a number δ, $\delta \in (0,1)$ such that for all $x \in B(x_0, \delta, t)$, $f(x) = y \in B(y_0, \varepsilon, t) \subset \widetilde{X_2}$, i.e., $f(B(x_0, \delta, t)) \subset B(y_0, \varepsilon, t)$, for all $t > 0$, then we say f is continuous at point x_0. If f is continuous at every point of $\widetilde{X_1}$, then f is said to a continuous mapping from $\widetilde{X_1}$ to $\widetilde{X_2}$.*

Proposition 1. *Let $\widetilde{X_1}$ and $\widetilde{X_2}$ be two fuzzy multi-metric spaces, f be a continuous mapping from $\widetilde{X_1}$ to $\widetilde{X_2}$ and $(x_n)_n$ be a sequence in $\widetilde{X_1}$. If $x_n \longrightarrow x$, then $f(x_n) \longrightarrow f(x)$.*

4. Fixed points for fuzzy multi-metric spaces

Definition 12. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space and $T : \widetilde{X} \to \widetilde{X}$ be a mapping. $x^* \in \widetilde{X}$ is called a fixed point of T if $Tx^* = x^*$. Denote the number of fixed points of a mapping T in $\widetilde{X}$ by $\Phi^{\ddagger}(T)$.*

Definition 13. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space. We will say the mapping $f : \widetilde{X} \longrightarrow \widetilde{X}$ is fuzzy contractive if there exist $k \in (0,1)$, $1 \leq i, j \leq m$, such that*

$$\frac{1}{M_j(f(x), f(y), t)} - 1 \leq k \left(\frac{1}{M_i(x, y, t)} - 1 \right)$$

for each $x, y \in \widetilde{X}$ and $t > 0$. $k \in (0,1)$, is called contractive constant of f.

Proposition 2. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a multi-metric space where each (X_i, d_i) is a metric space for any integer i, $1 \leq i \leq m$. The mapping $f : \widetilde{X} \longrightarrow \widetilde{X}$ is contractive (a contraction) on the multi-metric space $\widetilde{X}$ with contractive constant k iff f is fuzzy contractive, with contractive constant k, on the standard fuzzy multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$ such that $\left(X_i, M_{i_{d_i}}, *\right)$ standard fuzzy metric space induced by d_i for all $1 \leq i \leq m$.*

Definition 14. *A sequence $(x_n)_n$ in a multi-metric space $\widetilde{X} = \cup_{i=1}^m X_i$ is said to be contractive if there exists $k \in (0,1)$ such that $d_j(x_{n+1}, x_{n+2}) \leq k d_i(x_n, x_{n+1})$, for all $n \in \mathbb{N}$ and $1 \leq i, j \leq m$.*

Definition 15. *Let $\widetilde{X} = \cup_{i=1}^m X_i$ be a fuzzy multi-metric space. We will say that the sequence $(x_n)_n$ in $\widetilde{X}$ is fuzzy contractive if there exists $k \in (0,1)$ such that*

$$\frac{1}{M_j(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{M_i(x_n, x_{n+1}, t)} - 1 \right)$$

for all $t > 0$, $n \in \mathbb{N}$ and $1 \leq i, j \leq m$.

Proposition 3. *Let $\widetilde{X} = \cup_{i=1}^{m} X_i$ be the standard fuzzy multi-metric space induced by the metric d_i on X_i for all $1 \leq i \leq m$. The sequence $(x_n)_n$ in $\widetilde{X}$ is contractive in multi-metric space iff $(x_n)_n$ is fuzzy contractive in standard fuzzy multi-metric space.*

Next, we extend the Banach fixed point theorem to fuzzy contractive mappings of complete fuzzy multi-metric spaces.

Theorem 9. *Let $\widetilde{X} = \cup_{i=1}^{m} X_i$ be a complete fuzzy multi-metric space in which fuzzy contractive sequences are Cauchy. Let $T : \widetilde{X} \longrightarrow \widetilde{X}$ be a fuzzy contractive mapping being k the contractive constant. Then $1 \leq \Phi^{\ddagger}(T) \leq m$.*

Proof. Fix $x \in \widetilde{X}$. Let $x_n = T^n(x)$, $n \in \mathbb{N}$. We have for $t > 0$

$$\frac{1}{M_i(T(x), T^2(x), t)} - 1 \leq k \left(\frac{1}{M_i(x, x_1, t)} - 1 \right)$$

and by induction

$$\frac{1}{M_i(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{M_i(x_n, x_{n+1}, t)} - 1 \right),$$

$n \in \mathbb{N}$.

Then $(x_n)_n$ is a fuzzy contractive sequence, so it is a Cauchy sequence and, hence, $(x_n)_n$ converges to z^*, for some $z^* \in \widetilde{X}$. We will see z^* is a fixed point for T. By Theorem 2, we have

$$\frac{1}{M_i(T(y), T(x_n), t)} - 1 \leq k \left(\frac{1}{M_i(y, x_n, t)} - 1 \right) \longrightarrow 0$$

as $n \longrightarrow \infty$. Then $\lim_n M_i(T(y), T(x_n), t) = 1$ for each $t > 0$, and, therefore, $\lim_n T(x_n) = T(z^*)$, i.e., $\lim_n x_{n+1} = T(z^*)$ and then $T(z^*) = z^*$.

For other chosen points $u_0, v_0 \in X_1$, we can also define recursively $u_{n+1} = Tu_n$, $v_{n+1} = Tv_n$ and get the limit points, there exists an integer i_0 such that, $\lim_n u_n = \lim_n v_n = w^* \in X_{i_0}$, $Tu^* \in X_{i_0}$. Then for $t > 0$ we have

$$\begin{aligned}
\frac{1}{M_{i_0}(z^*, u^*, t)} - 1 &= \frac{1}{M_{i_0}(T(z^*), T(u^*), t)} - 1 \\
&\leq k \left(\frac{1}{M_{i_0}(z^*, u^*, t)} - 1 \right) \\
&= k \left(\frac{1}{M_{i_0}(T(z^*), T(u^*), t)} - 1 \right) \\
&\leq k^2 \left(\frac{1}{M_{i_0}(z^*, u^*, t)} - 1 \right) \\
&\leq \cdots \leq k^n \left(\frac{1}{M_{i_0}(z^*, u^*, t)} - 1 \right) \longrightarrow 0
\end{aligned}$$

as $n \longrightarrow \infty$.

Hence, $M_{i_0}(z^*, u^*, t) = 1$ and then $z^* = u^*$.

Similar consider the points in X_i, $2 \leq i \leq m$, we get $1 \leq \Phi^{\ddagger}(T) \leq m$. $\square$

Now suppose $\left(\widetilde{X}, M_{i_{d_i}}, *\right)$ is a complete standard fuzzy multi-metric space where $(X_i, M_{id_i}, *)$ fuzzy metric space induced by the metric d_i in X_i for all $1 \leq i, j \leq m$. From Remark 7 (X_i, d_i) is complete, then if $(x_n)_n$ is a fuzzy contractive sequence, by Proposition 3 it is contractive in (X_i, d_i), hence convergent. So, from Theorem 9 we have the following corollary, which can be considered the fuzzy version of the classic Banach contraction theorem on complete metric spaces.

Corollary 3. *Let $\widetilde{X}$ be a complete standard fuzzy multi-metric space and let $T : \widetilde{X} \longrightarrow \widetilde{X}$ a fuzzy contractive mapping. Then $1 \leq \Phi^{\ddagger}(T) \leq m$.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.3,377-389,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Statistical Convergence of Double Sequences on Intuitionistic Fuzzy Normed Spaces

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Abstract

The concept of statistical convergence was presented by Steinhaus (1951). This concept was extended to the double sequences by Mursaleen and Edely (2003). In this paper, we define and study statistical analogue of convergence and Cauchy for double sequences on intuitionistic fuzzy normed spaces. Then we give a useful characterization for statistically convergent double sequences. Furthermore, we display an example such that our method of convergence is stronger than the usual convergence for double sequences on intuitionistic fuzzy normed spaces.

KEY WORDS: Natural double density, statistical convergence, continuous $t-$norm, continuous $t-$conorm, intuitionistic fuzzy normed space.

1 Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [29]. Then many authors developed the theory of fuzzy set and applications. The fuzzy logic has been used many fields, like metric and topological spaces [9] , [10] , [16] , [19], theory of functions [4] , [18] , [28] , computer programing [17], econometrics and other fields [1] , [2] , [3] , [8] , [20] , [22] . Also, recently, the concepts of intuitionistic fuzzy metric space has been studied by Park [23], and intuitionistic fuzzy normed space have been studied by Saadati and Park [25] .

In this paper we give statistical analogues of convergence and Cauchy for double sequences which studied in Mursaleen and Osama [21] on intuitionistic fuzzy normed spaces.Also we display an example such that our method of convergence is stronger than the usual convergence for double sequences on intuitionistic fuzzy normed spaces.

Now we recall some notations and definitions which we used in the paper.

Definition 1 [26]*A binary operation* $*: [0,1] \times [0,1] \rightarrow [0,1]$ *is said to be a continuous* $t-$*norm if it satisfies the following conditions:*

(a) $*$ *is associative and commutative,*

(b) $*$ *is continuous,*

(c) $a * 1 = a$ *for all* $a \in [0,1]$,

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous $t-$*norm are* $a * b = ab$ and $a * b = \min(a,b)$ for all $a, b \in [0,1]$.

Definition 2 [26]*A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous* $t-$*conorm if it satisfies the following conditions:*

(a) $\diamond$ *is associative and commutative,*

(b) $\diamond$ *is continuous,*

(c) $a \diamond 0 = a$ *for all* $a \in [0,1]$,

(d) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous $t-$*conorm are* $a \diamond b = \min(a+b, 1)$ and $a \diamond b = \max(a,b)$ for all $a, b \in [0,1]$.

Now we give the concept of intuitionistic fuzzy normed space which has recently introduced by Saadati and Park [25].

Definition 3 [25] *The* $5-$*tuple* $(V, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space* (*IFNS*) *if* V *is a vector space,* $*$ *is a continuous* $t-$*norm,* $\diamond$ *is a continuous* $t-$*conorm, and* μ, ν *fuzzy sets on* $V \times (0, \infty)$ *satisfying the following conditions for every* $x, y \in V$ *and* $s, t > 0$:

(a) $\mu(x,t) + \nu(x,t) \leq 1$,

(b) $\mu(x,t) > 0$,

(c) $\mu(x,t) = 1$ *if and only if* $x = 0$,

(d) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,

(f) $\mu(x, \cdot) : (0, \infty) \to [0,1]$ *is continuous,*

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$,

(h) $\nu(x,t) < 1$,

(i) $\nu(x,t) = 0$ *if and only if* $x = 0$,

(j) $\nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,

(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,

(l) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous,*

(m) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 0$.

In this case (μ,ν) is called an *intuitionistic fuzzy norm.* We can give an example as follow:

Let $(V, \|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. If we define

$$\mu_0(x,t) := \frac{t}{t+\|x\|} \text{ and } \nu_0(x,t) := \frac{\|x\|}{t+\|x\|}.$$

for all $x \in V$ and every $t > 0$, then observe that $(V, \mu, \nu, *, \diamond)$ is an intuitionistic fuzzy normed space.

Before we present the new definitions and the main theorems, we shall recall some concepts which we need.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [24]. A double sequence $x = (x_{jk})_{jk=0}^{\infty}$ is called convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k \geq N$. L is called the Pringsheim limit of x.

A double sequence $x = (x_{jk})$ is said to be Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{pq} - x_{jk}| < \varepsilon$ for all $p \geq j \geq N$, $q \geq k \geq N$.

A double sequence x is bounded if there exists a positive number M such that $|x_{jk}| < M$ for all j and k.

So we can give the (μ,ν) analogue of above two definitions as follow:

Definition 4 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{jk})$ is said to be convergent to $L \in V$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon > 0$ and $t > 0$, there exists $N \in \mathbb{N}$ such that $\mu(x_{jk} - L, t) > 1 - \varepsilon$ and $\nu(x_{jk} - L, t) < \varepsilon$ for all $j, k \geq N$. It is denoted by $(\mu, v)_2 - \lim x = L$ or $x_{jk} \overset{(\mu,\nu)_2}{\to} L$ as $j, k \to \infty$.*

Definition 5 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{jk})$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν) provided that, for every $\varepsilon > 0$ and $t > 0$, there exists $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that $\mu(x_{jk} - x_{pq}, t) > 1 - \varepsilon$ and $\nu(x_{jk} - x_{pq}, t) < \varepsilon$ for all $j, p \geq N$, $k, q \geq M$*

Now we first recall statistical convergence and then in new section, we introduce basic definitions and properties which we mention above .

2 Statistical Convergence of Double Sequence on IFNS

Steinhaus [27] introduced the idea of statistical convergence (see also Fast [11]). If K is a subset of $\mathbb{N}$, the set of natural numbers, then the asymptotic density

of K denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, when $|A|$ denotes the cardinality of the set A. A sequence $x = (x_k)$ of numbers is statistically convergent to L if

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - \lim x = L$.

Statistical convergence has been investigated in a number of paper [6], [7], [12], [13], [14], [15].

Now we recall the concept of statistical convergence of double sequences.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two-dimensional analogue of natural density can be defined as follows.

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \lim_{n,m} \inf \frac{K(n,m)}{nm}.$$

In case the sequence $(K(n,m) \diagup nm)$ has a limit in Pringsheim's sense [24] then we say that K has a double natural density and is defined as

$$\lim_{n,m} \frac{K(n,m)}{nm} = \delta_2(K).$$

If we consider the set of $K = \left\{\left(i^2,\ j^2\right) : i,j \in \mathbb{N}\right\}$, then

$$\delta_2(K) = \lim_{n,m} \frac{K(n,m)}{nm} \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0.$$

Also, if we consider the set of $\{(i,2j) : i,j \in \mathbb{N}\}$ has double natural density $1/2$.

If we set $n = m$, we have a two-dimensional natural density considered by Christopher [5].

Now we recall the concepts of statistically convergent and statistically Cauchy for double sequence as follows:

Definition 6 [21] *A real double sequence $x = (x_{jk})$ is said to be statistically convergent the number ℓ provided that, for each $\varepsilon > 0$, the set*

$$\{(j,k),\ j \leq n \text{ and } k \leq m : |x_{jk} - \ell| \geq \varepsilon\}$$

has double natural density zero. In this case we write $st_2 - \lim_{j,k} x_{jk} = \ell$.

Definition 7 [21]*A real double sequence $x = (x_{jk})$ is said to be statistically Cauchy provided that, for every $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j,p \geq N$, $k,q \geq M$, the set*

$$\{(j,k),\ j \leq n \text{ and } k \leq m : |x_{jk} - x_{pq}| \geq \varepsilon\}$$

has double natural density zero.

Now we give the analogues of these with respect to the intuitionistic fuzzy norm (μ, ν).

Definition 8 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. A real double sequence $x = (x_{jk})$ is statistically convergent to $L \in V$ with respect to the intuitionistic fuzzy norm (μ, ν) provided that, for every $\varepsilon > 0$ and $t > 0$,*

$$K = \{(j,k)\,,\ j \leq n \text{ and } k \leq m : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\} \quad (1)$$

has double natural density zero, i.e., if $K(n,m)$ be the numbers of (j,k) in K

$$\lim_{n,m} \frac{K(n,m)}{nm} = 0. \quad (2)$$

In this case we write $st_{(\mu,\nu)_2} - \lim_{j,k} x_{jk} = L$, where L is said to be $st_{(\mu,\nu)_2}-$limit. Also we denote the set of all statistically convergent double sequences with respect to the intuitionistic fuzzy norm (μ, ν) by $st_{(\mu,\nu)_2}$.

By using (2) and the well-known properties of the double natural density, we easily get the following lemma.

Lemma 9 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. Then, for every $\varepsilon > 0$ and $t > 0$, the following statements are equivalent:*

(*i*) $st_{(\mu,\nu)_2} - \lim_{j,k} x_{jk} = L$

(*ii*) $\delta_2\{(j,k)\,,\ j \leq n \text{ and } k \leq m : \mu(x_{jk} - L, t) \leq 1 - \varepsilon\} = \delta_2\{(j,k)\,,\ j \leq n \text{ and } k \leq m : \nu(x_{jk} - L, t) \geq \varepsilon\} = 0.$

(*iii*) $\delta_2\{(j,k)\,,\ j \leq n \text{ and } k \leq m : \mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon\} = 1.$

(*iv*) $\delta_2\{(j,k)\,,\ j \leq n \text{ and } k \leq m : \mu(x_{jk} - L, t) > 1 - \varepsilon\} = \delta_2\{(j,k)\,,\ j \leq n \text{ and } k \leq m : \nu(x_{jk} - L, t) < \varepsilon\} = 1.$

(*v*) $st_2 - \lim \mu(x_{jk} - L, t) = 1$ and $st_2 - \lim \nu(x_{jk} - L, t) = 0.$

Theorem 10 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If a double sequence $x = (x_{jk})$ is statistically convergent with respect to the intuitionistic fuzzy norms (μ, ν), then the $st_{(\mu,\nu)_2}-$limit is unique.*

Proof. Let $x = (x_{jk})$ be a double sequence. Suppose that $st_{(\mu,\nu)_2} - \lim x = L_1$ and $st_{(\mu,\nu)_2} - \lim x = L_2$. Let $t > 0$ and $\varepsilon > 0$. Choose $r \in (0,1)$ such that $(1-r) * (1-r) \geq 1 - \varepsilon$ and $r \diamond r \leq \varepsilon$. Then, we define the following sets:

$$\begin{aligned}
K_{\mu,1}(r,t) &:= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L_1, t) \leq 1 - r\}\,, \\
K_{\mu,2}(r,t) &:= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L_2, t) \leq 1 - r\}\,, \\
K_{\nu,1}(r,t) &:= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L_1, t) \geq r\}\,, \\
K_{\nu,2}(r,t) &:= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \nu(x_{jk} - L_2, t) \geq r\}\,.
\end{aligned}$$

Since $st_{(\mu,\nu)_2} - \lim x = L_1$, we have

$$\delta_2\{K_{\mu,1}(\varepsilon,t)\} = \delta_2\{K_{\nu,1}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Furthermore, using $st_{(\mu,\nu)_2} - \lim x = L_2$, we get

$$\delta_2\{K_{\mu,2}(\varepsilon,t)\} = \delta_2\{K_{\nu,2}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Now let $K_{\mu,\nu}(\varepsilon,t) := \{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\} \cap \{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. Then observe that $\delta_2\{K_{\mu,\nu}(\varepsilon,t)\} = 0$ which implies $\delta_2\{\mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)\} = 1$. If $(j,k) \in \mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)$, then we have two possible cases. The former is the case of $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}$; and the letter is $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. We first consider that

$$(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}.$$

Then we have

$$\begin{aligned} \mu(L_1 - L_2, t) &\geq \mu(x_{jk} - L_1, \frac{t}{2}) * \mu(x_{jk} - L_2, \frac{t}{2}) \\ &> (1-r) * (1-r) \geq 1 - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\mu(L_1 - L_2, t) = 1$ for all $t > 0$,which yields $L_1 = L_2$. On the other hand, if $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$, then we may write that

$$\begin{aligned} \nu(L_1 - L_2, t) &\leq \nu(x_{jk} - L_1, \frac{t}{2}) \diamond \nu(x_{jk} - L_2, \frac{t}{2}) \\ &< r \diamond r \leq \varepsilon. \end{aligned}$$

Again, since $\varepsilon > 0$ was arbitrary, we have $\nu(L_1 - L_2, t) = 0$ for all $t > 0$, which implies $L_1 = L_2$. Therefore, in all cases, we conclude that the $st_{(\mu,\nu)_2}$−limit is unique. ■

Theorem 11 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. If $(\mu,\nu)_2 - \lim x = L$ for a double sequence $x = (x_{jk})$, then $st_{(\mu,\nu)_2} - \lim x = L$.*

Proof. By hypothesis, for every $\varepsilon > 0$ and $t > 0$, there is a number $N \in \mathbb{N}$ such that

$$\mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon$$

for all $j \geq N$ and $k \geq N$. This guarantees that the set

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\}$$

has at most finitely many terms. Since every finite subset of the natural numbers has double density zero, we immediately see that

$$\delta_2\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\} = 0,$$

whence the result. ■

The following example shows that the converse of Theorem 11 not hold in general.

Example 12 *Let* $(\mathbb{R},|\cdot|)$ *denote the space of real numbers with the usual norm, and let* $a * b = ab$ *and* $a \diamond b = \min\{a+b,1\}$ *for all* $a,b \in [0,1]$. *For all* $x \in \mathbb{R}$ *and every* $t > 0$, *consider*

$$\mu_0(x,t) := \frac{t}{t+|x|} \text{ and } \nu_0(x,t) := \frac{|x|}{t+|x|}.$$

In this case observe that $(\mathbb{R},\mu,\nu,*,\diamond)$ *is an IFNS. Now define a double sequence* $x = (x_{jk})$ *whose terms are given by*

$$x_{jk} := \begin{cases} 1, & \text{if } j \text{ and } k \text{ are squares} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then, for every $0 < \varepsilon < 1$ *and for any* $t > 0$, *let* $K_n(\varepsilon,t) := \{(j,k),\ j \leq n \text{ and } k \leq m : \mu_0(x_{jk},t) \leq 1-\varepsilon \text{ or } \nu_0(x_{jk},t) \geq \varepsilon\}$. *Since*

$$\begin{aligned} K_n(\varepsilon,t) &= \left\{(j,k),\ j \leq n \text{ and } k \leq m : \frac{t}{t+|x_{jk}|} \leq 1-\varepsilon \text{ or } \frac{|x_{jk}|}{t+|x_{jk}|} \geq \varepsilon\right\} \\ &= \left\{(j,k),\ j \leq n \text{ and } k \leq m : |x_{jk}| \geq \frac{\varepsilon t}{1-\varepsilon} > 0\right\} \\ &= \{(j,k),\ j \leq n \text{ and } k \leq m : x_{jk} = 1\} \\ &= \{(j,k),\ j \leq n \text{ and } k \leq m : j \text{ and } k \text{ are squares}\} \end{aligned}$$

we have

$$\delta_2\left(K_n(\varepsilon,t)\right) \leq \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0.$$

Hence, we get $st_{(\mu_0,\nu_0)_2} - \lim x = 0$. *However, since the sequence* $x = (x_{jk})$ *given by (3) is not convergent in the space* $(\mathbb{R},|\cdot|)$, *by Lemma 4.10 of* [25], *we also see that* x *is not convergent with respect to the intuitionistic fuzzy norm* (μ_0,ν_0).

Theorem 13 *Let* $(V,\mu,\nu,*,\diamond)$ *be an IFNS. Then* $st_{(\mu,\nu)_2} - \lim x = L$ *if and only if there exists a subset* $K = \{(j,k)\} \subseteq \mathbb{N}\times\mathbb{N}$, $j,k = 1,2,...$, *such that* $\delta_2(K) = 1$ *and* $(\mu,\nu)_2 - \lim\limits_{\substack{j,k\to\infty \\ (j,k)\in K}} x_{jk} = L$.

Proof. We first assume that $st_{(\mu,\nu)_2} - \lim x = L$. Now, for any $t > 0$ and $j \in \mathbb{N}$, let

$$K_r := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-L,t) \leq 1 - \frac{1}{r} \text{ or } \nu(x_{jk}-L,t) \geq \frac{1}{r}\right\}$$

and

$$\begin{aligned} M_r &= \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-L,t) > 1 - \frac{1}{r} \text{ and } \nu(x_{jk}-L,t) < \frac{1}{r}\right\}, \\ &\quad (r = 1,2,...). \end{aligned}$$

Then $\delta_2(K_r) = 0$ and

(1) $M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ...$

and

(2) $\delta_2(M_r) = 1, \quad r = 1, 2, ...$

Now we have to show that for $(j,k) \in M_r$, (x_{jk}) is convergent to L. Suppose that (x_{jk}) is not convergent to L. Therefore there is $\varepsilon > 0$ such that

$$\{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\}$$

for infinitely many terms.

Let

$$M_\varepsilon = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon\}$$

and $\varepsilon > \frac{1}{r} \quad (r = 1, 2, ...)$.

Then

(3) $\delta_2(M_\varepsilon) = 0$,

and by (1), $M_r \subset M_\varepsilon$. Hence $\delta_2(M_r) = 0$ which contradicts (2). Therefore (x_{jk}) is convergent to L.

Conversely, suppose that there exists a subset $K = \{(j,k)\} \subset \mathbb{N} \times \mathbb{N}$ such that $\delta_2(K) = 1$ and $(\mu,\nu)_2 - \lim_{j,k} x_{jk} = L$, i.e. there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ and $t > 0$

$$\mu(x_{jk} - L, t) > 1 - \varepsilon \text{ and } \nu(x_{jk} - L, t) < \varepsilon, \quad \forall j, k \geq N.$$

Now

$$\begin{aligned} K_\varepsilon &= \{(j,k) \in \mathbb{N} \times \mathbb{N} : \mu(x_{jk} - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - L, t) \geq \varepsilon\} \\ &\subseteq \mathbb{N} \times \mathbb{N} - \{(j_{N+1}, k_{N+1}), \ (j_{N+2}, k_{N+2}), ...\}. \end{aligned}$$

Therefore $\delta_2(K_\varepsilon) \leq 1 - 1 = 0$. Hence x is statistically convergent to L. ■

Definition 14 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS. We say that a double sequence $x = (x_{jk})$ is statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) provided that, for every $\varepsilon > 0$ and $t > 0$, there exist $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that for all $j, p \geq N$, $k, q \geq M$, the set*

$$\{(j,k), \ j \leq n, \ k \leq m : \mu(x_{jk} - x_{pq}, t) \leq 1 - \varepsilon \text{ or } \nu(x_{jk} - x_{pq}, t) \geq \varepsilon\}$$

has double natural density zero.

Now using a similar technique in the proof of Theorem 13 one can get the following result at once.

Theorem 15 *Let $(V, \mu, \nu, *, \diamond)$ be an IFNS, and let $x = (x_{jk})$ be a double sequence whose terms are in the vector space V. Then, the following conditions are equivalent:*

(i) x is a statistically Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν).

(ii) There exists an increasing index sequence $K = \{(j,k)\} \subseteq \mathbb{N} \times \mathbb{N}$, $j,k = 1,2,...$ such that $\delta_2(K) = 1$ and the subsequence $\{x_{jk}\}_{(j,k)\in K}$ is a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν).

Now we show that statistically convergence of double sequences on IFNS has some arithmetical properties similar to properties of the usual convergence on $\mathbb{R}$.

Lemma 16 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. If $st_{(\mu,\nu)_2} - \lim x_{jk} = \xi$ and $st_{(\mu,\nu)_2} - \lim y_{jk} = \eta$ then $st_{(\mu,\nu)_2} - \lim(x_{jk} + y_{jk}) = \xi + \eta$.*

Proof. Let $st_{(\mu,\nu)_2} - \lim x_{jk} = \xi$, $st_{(\mu,\nu)_2} - \lim y_{jk} = \eta$, $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r)*(1-r) \geq 1-\varepsilon$ and $r \diamond r \leq \varepsilon$. Then we define the following sets:

$$\begin{aligned}
K_{\mu,1}(r,t) &: = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(x_{jk}-\xi,t) \leq 1-r\}, \\
K_{\mu,2}(r,t) &: = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \mu(y_{jk}-\eta,t) \leq 1-r\}, \\
K_{\nu,1}(r,t) &: = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu(x_{jk}-\xi,t) \geq r\}, \\
K_{\nu,2}(r,t) &: = \{(j,k) \in \mathbb{N}\times\mathbb{N} : \nu(y_{jk}-\eta,t) \geq r\}.
\end{aligned}$$

Since $st_{(\mu,\nu)_2} - \lim x_{jk} = \xi$, we have

$$\delta_2\{K_{\mu,1}(\varepsilon,t)\} = \delta_2\{K_{\nu,1}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Similarly, since $st_{(\mu,\nu)_2} - \lim y_{jk} = \eta$, we get

$$\delta_2\{K_{\mu,2}(\varepsilon,t)\} = \delta_2\{K_{\nu,2}(\varepsilon,t)\} = 0 \quad \text{for all } t > 0.$$

Now let $K_{\mu,\nu}(\varepsilon,t) := \{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\} \cap \{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. Then observe that $\delta_2\{K_{\mu,\nu}(\varepsilon,t)\} = 0$ which implies $\delta_2\{\mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)\} = 1$. If $(j,k) \in \mathbb{N}\times\mathbb{N}/K_{\mu,\nu}(\varepsilon,t)$, then we have two possible cases. The former is the case of $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}$; and the letter is $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$. We first consider that

$$(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\}.$$

Then we have

$$\begin{aligned}
\mu((x_{jk}-\xi)+(y_{jk}-\eta),t) &\geq \mu(x_{jk}-\xi,\frac{t}{2}) * \mu(y_{jk}-\eta,\frac{t}{2}) \\
&> (1-r)*(1-r) \geq 1-\varepsilon.
\end{aligned}$$

On the other hand, if $(j,k) \in \mathbb{N}\times\mathbb{N}/\{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\}$, then we can write that

$$\begin{aligned}
\nu((x_{jk}-\xi)+(y_{jk}-\eta),t) &\leq \nu(x_{jk}-\xi,\frac{t}{2}) \diamond \nu(x_{jk}-\eta,\frac{t}{2}) \\
&< r \diamond r \leq \varepsilon.
\end{aligned}$$

This show that

$$\delta_2\left(\left\{\begin{array}{r} (j,k)\in\mathbb{N}\times\mathbb{N}:\mu((x_{jk}-\xi)+(y_{jk}-\eta)\,,t)\leq 1-\varepsilon \\ or\quad \nu((x_{jk}-\xi)+(y_{jk}-\eta)\,,t)\geq\varepsilon \end{array}\right\}\right)=0$$

so $st_{(\mu,\nu)_2}-\lim\left(x_{jk}+y_{jk}\right)=\xi+\eta$. ■

Lemma 17 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. If $st_{(\mu,\nu)_2}-\lim x_{jk}=\xi$ and $\alpha\in\mathbb{R}$ then $st_{(\mu,\nu)_2}-\lim\alpha x_{jk}=\alpha\xi$.*

Proof. Let $st_{(\mu,\nu)_2}-\lim x_{jk}=\xi$, $\varepsilon\in(0,1)$ and $t>0$. First of all we consider the case of $\alpha=0$. In this case

$$\mu(0x_{jk}-0\xi,t)=\mu(0,t)=1>1-\varepsilon.$$

Similarly we observe that

$$\nu(0x_{jk}-0\xi,t)=\nu(0,t)=0<\varepsilon$$

for $\alpha=0$. So we obtain $(\mu,\nu)_2-\lim 0x=0$. Then from Theorem 11 we have $st_{(\mu,\nu)_2}-\lim 0x_{jk}=0$.

Now we consider the case of $\alpha\in\mathbb{R}$ $(\alpha\neq 0)$. From definition we can write

$$\delta_2\left(\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(x_{jk}-\xi,t\right)\leq 1-\varepsilon\quad\text{or}\quad\nu\left(x_{jk}-\xi,t\right)\geq\varepsilon\right\}\right)=0.$$

So, if we define the sets:

$$\begin{aligned} K_{\mu,1}(\varepsilon,t)\quad &:\quad=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{jk}-\xi,t)\leq 1-\varepsilon\} \\ K_{\nu,1}(\varepsilon,t)\quad &:\quad=\{(j,k)\in\mathbb{N}\times\mathbb{N}:\nu(x_{jk}-\xi,t)\geq\varepsilon\} \end{aligned}$$

then we can say $\delta_2\left\{K_{\mu,1}\left(\varepsilon,t\right)\right\}=\delta_2\left\{K_{\nu,1}(\varepsilon,t)\right\}=0$ for all $t>0$. Now let $K_{\mu,\nu}(\varepsilon,t)=K_{\mu,1}(\varepsilon,t)\cup K_{\nu,1}(\varepsilon,t)$ then $\delta_2\left\{K_{\mu,\nu}(\varepsilon,t)\right\}=0$ which implies $\delta_2\left\{\mathbb{N}\times\mathbb{N}\diagdown K_{\mu,\nu}(\varepsilon,t)\right\}=1$. If $(j,k)\in\mathbb{N}\times\mathbb{N}\diagdown K_{\mu,\nu}(\varepsilon,t)$ then for $\alpha\in\mathbb{R}$ $(\alpha\neq 0)$

$$\begin{aligned} \mu(\alpha x_{jk}-\alpha\xi,t) &= \mu(x_{jk}-\xi,\frac{t}{|\alpha|}) \\ &\geq \mu(x_{jk}-\xi,t)*\mu(0,\frac{t}{|\alpha|}-t) \\ &= \mu(x_{jk}-\xi,t)*1 \\ &= \mu(x_{jk}-\xi,t)>1-\varepsilon. \end{aligned}$$

Similarly, we observe that for $\alpha\in\mathbb{R}$ $(\alpha\neq 0)$

$$\begin{aligned} \nu(\alpha x_{jk}-\alpha\xi,t) &= \nu(x_{jk}-\xi,\frac{t}{|\alpha|}) \\ &\leq \nu(x_{jk}-\xi,t)\diamond\nu(0,\frac{t}{|\alpha|}-t) \\ &= \nu(x_{jk}-\xi,t)\diamond 0 \\ &= \nu(x_{jk}-\xi,t)<\varepsilon. \end{aligned}$$

This show that

$$\delta_2\left(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(\alpha x_{jk}-\alpha\xi,t)\leq 1-\varepsilon \quad or \quad \nu(\alpha x_{jk}-\alpha\xi,t)\geq\varepsilon\}\right)=0$$

so $st_{(\mu,\nu)_2}-\lim \alpha x_{jk}=\alpha\xi$. ∎

Lemma 18 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS. If $st_{(\mu,\nu)_2}-\lim x_{jk}=\xi$ and $st_{(\mu,\nu)_2}-\lim y_{jk}=\eta$ then $st_{(\mu,\nu)_2}-\lim(x_{jk}-y_{jk})=\xi-\eta$.*

Proof. The proof is clear from Lemma 16 and Lemma 17. ∎

Definition 19 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS . We say that a double sequence $x=(x_{jk})$ is IF-bounded if there exist $t>0$ and $0<r<1$ such that $\mu(x_{jk},t)>1-r$ and $\nu(x_{jk},t)<r$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$.*

Definition 20 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS . For $t>0$, we define open ball $B(x,r,t)$ with center $x\in V$ and radius $0<r<1$, as*

$$B(x,r,t)=\{y\in V:\mu(x-y,t)>1-r\ ,\ \nu(x-y,t)<r\}.$$

It follows from Lemma 16 Lemma 17 and Lemma 18, that the set of all IF-bounded statistically convergent double sequences on IFNS is a linear subspace of the linear normed space $\ell_\infty^{(\mu,\nu)_2}(V)$ of all IF-bounded sequences on IFNS.

Theorem 21 *Let $(V,\mu,\nu,*,\diamond)$ be an IFNS and the set $st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)$ is closed linear subspace of the set $\ell_\infty^{(\mu,\nu)_2}(V)$.*

Proof. It is clear that $st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)\subset\overline{st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)}$. Now we show that $\overline{st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)}\subset st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)$. Let $y\in\overline{st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)}$. Since $B(y,r,t)\cap\left(st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)\right)\neq\varnothing$, there is a $x\in B(y,r,t)\cap\left(st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)\right)$.

Let $t>0$ and $\varepsilon\in(0,1)$. Choose $r\in(0,1)$ such that $(1-r)*(1-r)\geq 1-\varepsilon$ and $r\diamond r\leq\varepsilon$. Since $x\in B(y,r,t)\cap\left(st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)\right)$, there is a set $K\subseteq\mathbb{N}\times\mathbb{N}$ with $\delta(K)=1$ such that

$$\mu\left(y_{jk}-x_{jk},\frac{t}{2}\right)>1-r \quad\text{and}\quad \nu\left(y_{jk}-x_{jk},\frac{t}{2}\right)<r$$

and

$$\mu\left(x_{jk},\frac{t}{2}\right)>1-r \quad\text{and}\quad \nu\left(x_{jk},\frac{t}{2}\right)<r$$

for all $(j,k)\in K$. Then we have

$$\begin{aligned}\mu(y_{jk},t) &= \mu(y_{jk}-x_{jk}+x_{jk},t)\\ &\geq \mu\left(y_{jk}-x_{jk},\frac{t}{2}\right)*\mu\left(x_{jk},\frac{t}{2}\right)\\ &> (1-r)*(1-r)\geq 1-\varepsilon\end{aligned}$$

and

$$\begin{aligned}\nu\left(y_{jk},t\right) &= \nu\left(y_{jk}-x_{jk}+x_{jk},t\right)\\ &\leq \nu\left(y_{jk}-x_{jk},\frac{t}{2}\right)\diamond\nu\left(x_{jk},\frac{t}{2}\right)\\ &< r\diamond r\leq\varepsilon\end{aligned}$$

for all $(j,k)\in K$. Hence

$$\delta_2\left(\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu\left(y_{jk},t\right)>1-\varepsilon \text{ and } \nu\left(y_{jk},t\right)<\varepsilon\}\right)=1$$

and thus $y\in st_{(\mu,\nu)_2}(V)\cap\ell_\infty^{(\mu,\nu)_2}(V)$. ■

Conclusion 22 *In this paper, we obtained results on statistical convergence in intuitionistic fuzzy normed spaces. As every ordinary norm induces a intuitionistic fuzzy norm, the results obtained here are more general than the corresponding of normed spaces.*

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Common Fixed Points and Compatible Maps of Type (P-1) and Type (P-2) in Menger spaces

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April 19, 2007

Abstract

In the present work, we introduce the concepts of compatible maps of type (P-1) and type (P-2), and prove common fixed point theorems for such maps without appeal to continuity in Menger spaces.

Keywords: Menger space; t-norm; Common fixed point; Compatible maps.
AMS Subject Classifications: 54H25, 54E70

1 INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger [11] who was use distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar [21] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [15]. Sessa [16] introduced weakly commuting maps in metric spaces. Jungck [8] enlarged this concept to compatible maps. The notion of compatible maps in Menger spaces has been introduced by Mishra [12]. Cho et al. [2] and Sharma [17,19] gave fuzzy version of compatible maps and proved common fixed point theorems for compatible maps.

In this paper, we introduce the concepts of compatible maps of type (P-1) and type (P-2) in Menger spaces, and show that they are equivalent to compatible maps under certain conditions. In the sequel, we prove common fixed point theorems for compatible maps of type (P-1) (or type (P-2)) and weak compatible maps without continuity in Menger

spaces illustrating with an example which generalize, extend and fuzzify several well known fixed point theorems for contractive type maps on metric spaces, Menger spaces, uniform spaces and fuzzy metric spaces. As an application, we have applied one of our result and probabilistic version of Banach contraction theorem to obtain a fixed point theorem in the product of Menger spaces.

2 Preliminaries

In this section, we recall some definitions and known results in Menger space. For more details we refer the readers to [1,3-7,8-15,18,20-22].

DEFINITION 2.1. *A triangular norm $*$ (shorty t-norm) is a binary operation on the unit interval* $[0,1]$ *such that for all* $a,b,c,d \in [0,1]$ *the following conditions are satisfied:*

(a) $a * 1 = a$;

(b) $a * b = b * a$;

(c) $a * b \le c * d$ whenever $a \le c$ and $b \le d$;

(d) $a * (b * c) = (a * b) * c$.

Some important examples of t-norm are $a*b = \max\{a+b-1, 0\}$ and $a*b = \min\{a,b\}$.

DEFINITION 2.2. *A distribution function is a function $F : [-\infty,\infty] \to [0,1]$ which is left continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0$, $F(\infty) = 1$.*

We will denote by Δ the family of all distribution functions on $[-\infty,\infty]$. H is a special element of Δ defined by

$$H(t) = \begin{cases} 0, \ t \le 0 \\ 1, \ t > 0. \end{cases} .$$

If X is a nonempty set, $F : X \times X \to \Delta$ is called a probabilistic distance on X and $F(x,y)$ is usually detoned by $F_{x,y}$.

DEFINITION 2.3 (21). *The ordered pair (X,F) is called a probabilistic semimetric space (shortly PSM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x,y,z \in X$ and $t,s > 0$,*

(PM-1) $F_{x,y}(t) = H(t) \Longleftrightarrow x = y$;

(PM-2) $F_{x,y} = F_{y,x}$.

If, in addition, the following inequality takes place:

(PM-3) $F_{x,z}(t) = 1,\ F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t+s) = 1$,

then (X,F) is called a probabilistic metric space (shortly PM-space).

The ordered triple $(X,F,*)$ is called Menger probabilistic metric space (shortly Menger space) if (X,F) is a PM-space, $*$ is a t-norm and the following condition is also satisfies: for all $x,y,z \in X$ and $t,s > 0$,

(PM-4) $F_{x,y}(t+s) \ge F_{x,z}(t) * F_{z,y}(s)$.

For every PSM-space (X,F), we can consider the sets of the form

$$U_{\varepsilon,\lambda} = \left\{(x,y) \in X \times X : F_{x,y}(\varepsilon) > 1 - \lambda\right\}.$$

The family $\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)}$ generates a semi-uniformity denoted by U_F and a topology τ_F called the F-topology or the strong topology. Namely,

$$A \in \tau_F \text{ iff } \forall x \in A\ \exists \varepsilon > 0 \text{ and } \lambda \in (0,1) \text{ such that } U_{\varepsilon,\lambda}(x) \subset A.$$

U_F is also generated by the family $\{V_\delta\}_{\delta>0}$ where $V_\delta := U_{\delta,\delta}$.

In [22], it is proved that if $\sup_{t<1}(t*t) = 1$, then U_F is a uniformity, called F-uniformity, which is metrizable.

The F-topology is generated by the F-uniformity and is determined by the F-convergence:

$$x_n \to x \Leftrightarrow F_{x_n,x}(t) \to 1, \forall t > 0.$$

PROPOSITION 2.1 (15). *Let (X, d) be a metric space. Then the metric d induces a distribution function F defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. If t-norm $*$ is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ then $(X, F, *)$ is a Menger space. Further, $(X, F, *)$ is a complete Menger space if (X, d) is complete.*

DEFINITION 2.4 (12). *Let $(X, F, *)$ be a Menger space and $*$ be a continuous t-norm.*

(a) A sequence $\{x_n\}$ in X is said to be *converge* to a point x in X (written $x_n \to x$) if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

(b) A sequence $\{x_n\}$ in X is said to be *Cauchy* if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.

(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

DEFINITION 2.5 (12). *Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.*

LEMMA 2.1 (20). *Let $(X, F, *)$ be a Menger space. If there exists $k \in (0, 1)$ such that*

$$F_{xy}(kt) \geq F_{xy}(t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

3 Compatible Maps of Type (P-1) and Type (P-2)

In this section, we introduce the concept of compatible mappings of type (P-1) and type (P-2) in Menger spaces and show that they are equivalent to compatible mappings under certain conditions.

DEFINITION 3.1. *Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible of type (P) if $F_{ABx_n,BBx_n}(t) \to 1$ and $F_{BAx_n,AAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.*

DEFINITION 3.2. *Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible of type (P-1) if $F_{ABx_n,BBx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.*

DEFINITION 3.3. *Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible of type (P-2) if $F_{BAx_n,AAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.*

REMARK 3.1. *Clearly, if a pair of mappings (A, B) is compatible of type (P-1) then the pair (B, A) is compatible of type (P-2). Further, if A and B compatible mappings of type (P) then the pair (A, B) is compatible of type (P-1) as well as type (P-2).*

The following is an example of pair of self maps in a Menger space which are compatible of type (P-1) and type (P-2) but not compatible.

Example. Let (X, d) be a metric space with the usual metric d where $X = [0, 2]$ and $(X, F, *)$ be the induced Menger space with $F_{x,y}(t) = H(t - d(x, y))$, $\forall x, y \in X$ and $\forall t > 0$. Define self maps A and B as follows:

$$Ax = \begin{cases} 2 - x, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x \leq 2, \end{cases} \quad \text{and} \quad Bx = \begin{cases} x, & \text{if } 0 \leq x < 1, \\ 2, & \text{if } 1 \leq x \leq 2. \end{cases}$$

Take $x_n = 1-1/n$. Then $F_{Ax_n,1}(t) = H(t-(1/n))$ and $\lim_{n\to\infty} F_{Ax_n,1}(t) = H(t) = 1$. Hence $Ax_n \to 1$ as $n \to \infty$. Similarly, $Bx_n \to 1$ as $n \to \infty$. Also $F_{ABx_n,BAx_n}(t) = H(t-(1-1/n))$ and $\lim_{n\to\infty} F_{ABx_n,BAx_n}(t) = H(t-1) \neq 1$, $\forall t > 0$. Hence the pair (A, B) is not compatible. But $F_{ABx_n,BBx_n}(t) = H(t-(2/n))$ and $\lim_{n\to\infty} F_{ABx_n,BBx_n}(t) = H(t) = 1$, $\forall t > 0$. Hence the pair (A, B) is compatible of type (P-1). Similarly, the pair (A, B) is compatible of type (P-2).

Next, we cite the following propositions which gives the condition under which the Definitions 2.5, 3.2 and 3.3 becomes equivalent.

PROPOSITION 3.1. *Let A and B be self maps of a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$.*

(i) If B is continuous then the pair (A, B) is compatible of type (P-1) iff A and B are compatible.

(ii) If A is continuous then the pair (A, B) is compatible of type (P-2) iff A and B are compatible.

Proof. (i) Let $\{x_n\}$ be a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$ and let the pair (A, B) be compatible of type (P-1). Since B is continuous, we have $BAx_n \to Bz$ and $BBx_n \to Bz$. Therefore, by (PM-4), we have

$$F_{ABx_n,BAx_n}(t) \geq F_{ABx_n,BBx_n}(t/2) * F_{BBx_n,BAx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence the mappings A and B are compatible.

Now, let A and B be compatible. Therefore, using the continuity of B, we have

$$F_{ABx_n,BBx_n}(t) \geq F_{ABx_n,BAx_n}(t/2) * F_{BAx_n,BBx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence the mappings A and B are compatible of type (P-1).

(ii) Let $\{x_n\}$ be a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$ and let the pair (A, B) be compatible of type (P-2). Since A is continuous, we have $AAx_n \to Az$ and $ABx_n \to Az$. Therefore, by (PM-4), we have

$$F_{ABx_n,BAx_n}(t) \geq F_{ABx_n,AAx_n}(t/2) * F_{AAx_n,BAx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence the mappings A and B are compatible.

Now, let A and B be compatible. Therefore, using the continuity of A, we have

$$F_{BAx_n,AAx_n}(t) \geq F_{BAx_n,ABx_n}(t/2) * F_{ABx_n,AAx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence the mappings A and B are compatible of type (P-2).

Next, we give some properties of compatible mappings of type (P-1) and type (P-2) which will be used in our results.

PROPOSITION 3.2. *Let A and B be self maps of a Menger space $(X, F, *)$. If the pair (A, B) is compatible of type (P-1) and $Az = Bz$ for some z in X then $ABz = BBz$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = x$ for $n \in \mathbb{N}$ and let $Az = Bz$. Then we have $Ax_n \to Az$ and $Bx_n \to Bz$. Since the pair (A, B) is compatible of type (P-1), we have $F_{ABz,BBz}(t) = F_{ABx_n,BBx_n}(t) \to 1$ as $n \to \infty$. Hence $ABz = BBz$.

PROPOSITION 3.3. *Let A and B be self maps of a Menger space $(X, F, *)$. If the pair (A, B) is compatible of type (P-2) and $Az = Bz$ for some z in X then $BAz = AAz$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = x$ for $n \in \mathbb{N}$ and let $Az = Bz$. Then we have $Ax_n \to Az$ and $Bx_n \to Bz$. Since the pair (A, B) is compatible of type (P-2), we have $F_{BAz,AAz}(t) = F_{BAx_n,AAx_n}(t) \to 1$ as $n \to \infty$. Hence $BAz = AAz$.

PROPOSITION 3.4. *Let A and B be self maps of a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t{*}t \geq t$ for all $t \in [0, 1]$. If the pair (A, B) is compatible of type (P-1) and $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$ then $BBx_n \to Az$ if A is continuous at z.*

Proof. Since A is continuous at z and the pair (A, B) is compatible of type (P-1), we have $ABx_n \to Az$ and $F_{ABx_n,BBx_n}(t) \to 1$ as $n \to \infty$. Therefore

$$F_{Az,BBx_n}(t) \geq F_{Az,ABx_n}(t/2) * F_{ABx_n,BBx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence $BBx_n \to Az$ as $n \to \infty$.

PROPOSITION 3.5. *Let A and B be self maps of a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t{*}t \geq t$ for all $t \in [0, 1]$. If the pair (A, B) is compatible of type (P-2) and $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$ then $AAx_n \to Bz$ if B is continuous at z.*

Proof. Since B is continuous at z and the pair (A, B) is compatible of type (P-2), we have $BAx_n \to Bz$ and $F_{BAx_n,AAx_n}(t) \to 1$ as $n \to \infty$. Therefore

$$F_{Bz,AAx_n}(t) \geq F_{Bz,BAx_n}(t/2) * F_{BAx_n,AAx_n}(t/2) \to 1 * 1 \geq 1$$

as $n \to \infty$. Hence $AAx_n \to Bz$ as $n \to \infty$.

4 Main Results

Let A, B, P, Q, S and T be self maps on a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$, for all $t \in [0, 1]$, satisfying:

(a) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$,

(b) there exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} F_{Px,Qy}(kt) \quad &\geq \quad F_{ABx,STy}(t) * F_{Px,ABx}(t) * F_{Qy,STy}(t) \\ &\quad\; *F_{Px,STy}(\alpha t) * F_{Qy,ABx}((2-\alpha)t) \end{aligned}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$.

For some arbitrary x_0 in X, by (a), we choose x_1 in X such that $Px_0 = STx_1$, and for this x_1 there exists x_2 such that $Qx_1 = ABx_2$. Continuing this process we define the sequence $\{y_n\}$ in X such that

$$Px_{2n} = STx_{2n+1} = y_{2n} \text{ and } Qx_{2n+1} = ABx_{2n+2} = y_{2n+1} \tag{1}$$

for $n = 0, 1, 2, ...$

We need the following lemma proved by Singh and Jain [20] for our result.

LEMMA 4.1. *Let A, B, P, Q, S and T be self maps on a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$, satisfying the conditions (a) and (b). Then the sequence defined by the condition (1) is a Cauchy sequence in X.*

THEOREM 4.1. *Let A, B, P, Q, S and T be self maps on a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$, for all $t \in [0, 1]$, satisfying the conditions (a), (b) and*

(c) one of $P(X), Q(X), AB(X)$ and $ST(X)$ is a complete subspace of X,

(d) (P, AB) is compatible of type (P-1) or type (P-2), and (Q, ST) is weakly compatible.

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof. Let (P, AB) be compatible of type (P-1) and (Q, ST) be weakly compatible. By Lemma 4.1, the sequence $\{y_n\}$ defined by (1) is a Cauchy sequence in X and if $AB(X)$ is complete then $\{y_{2n+1}\}$ has a limit z in $AB(X)$. Let $w \in (AB)^{-1}z$, then $ABw = z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. If we take $x = w$ and $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} F_{Pw,Qx_{2n+1}}(kt) &\geq F_{ABw,STx_{2n+1}}(t) * F_{Pw,ABw}(t) * F_{Qx_{2n+1},STx_{2n+1}}(t) \\ &\quad * F_{Pw,STx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},ABw}((2-\alpha)t). \end{aligned}$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$\begin{aligned} F_{Pw,z}(kt) &\geq F_{ABw,z}(t) * F_{Pw,ABw}(t) * F_{z,z}(t) * F_{Pw,z}(t) * F_{z,ABw}(t) \\ &\geq F_{Pw,z}(t) \end{aligned}$$

which means that $z = Pw$. Hence $z = ABw = Pw$, i.e., w is a coincide point of P and AB. Since the pair (P, AB) is compatible of type (P-1) and $ABw = Pw$, by Proposition 3.2, we have $P(AB)w = AB(AB)w$ or $Pz = ABz$.

Since $P(X) \subseteq ST(X)$, $Pw = z$ implies that $z \in ST(X)$. Let $u \in (ST)^{-1}z$, then $STu = z$. It can easily verified by using similar arguments of the previous part of the proof that $z = Qu$, hence $z = STu = Qu$. Since the pair (Q, ST) is weakly compatible, we have $Q(ST)u = (ST)Qu$ or $Qz = STz$.

If we assume that $ST(X)$ is complete, then the argument analogous to the previous completeness argument establishes w is a coincide point of P and AB, and u is a coincide point of Q and ST. Thus $Pz = ABz$ and $Qz = STz$.

The remaining two cases pertain essentially to the previous cases. Indeed, if $Q(X)$ is complete, then by the condition (a), $z \in Q(X) \subset AB(X)$. Similarly if $P(X)$ is complete, then $z \in P(X) \subset ST(X)$. Thus $Pz = ABz$ and $Qz = STz$.

Now, we prove that $z = Pz$. If we take $x = z$ and $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} F_{Pz,Qx_{2n+1}}(kt) &\geq F_{ABz,STx_{2n+1}}(t) * F_{Pz,ABz}(t) * F_{Qx_{2n+1},STx_{2n+1}}(t) \\ &\quad * F_{Pz,STx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},ABz}((2-\alpha)t). \end{aligned}$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$\begin{aligned} F_{Pz,z}(kt) &\geq F_{Pz,z}(t) * F_{Pz,Pz}(t) * F_{z,z}(t) * F_{Pz,z}(t) * F_{z,Pz}(t) \\ &\geq F_{Pz,z}(t) \end{aligned}$$

which means that $z = Pz$. Hence $z = Pz = ABz$. Similarly, we also have $z = Qz = STz$.

Now, we prove that $z = Bz$. Since $z = Pz = ABz$, $P(Bz) = Bz$ and $AB(Bz) = Bz$. If we take $x = Bz$ and $y = x_{2n+1}$ in (b), we have

$$\begin{aligned} F_{P(Bz),Qx_{2n+1}}(kt) &\geq F_{AB(Bz),STx_{2n+1}}(t) * F_{P(Bz),AB(Bz)}(t) * F_{Qx_{2n+1},STx_{2n+1}}(t) \\ &\quad * F_{P(Bz),STx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},AB(Bz)}((2-\alpha)t). \end{aligned}$$

Taking $n \to \infty$ and $\alpha \to 1$, we have

$$\begin{aligned} F_{Bz,z}(kt) &\geq F_{Bz,z}(t) * F_{Bz,Bz}(t) * F_{z,z}(t) * F_{Bz,z}(t) * F_{z,Bz}(t) \\ &\geq F_{Bz,z}(t) \end{aligned}$$

which means that $z = Bz$. Since $z = ABz$, $z = Bz$ implies that $z = Az$. Hence $z = Pz = Az = Bz$. Similarly we also have $z = Qz = Sz = Tz$. Therefore z is a common fixed point of A, B, P, Q, S and T. It is also easy to prove that if (P, AB) is compatible of type (P-2) and (Q, ST) is weakly compatible then z is a common fixed point of A, B, P, Q, S and T.

For uniqueness of common fixed point, let $v \neq z$ be another common fixed point of A, B, P, Q, S and T. Then, by condition (b) and taking $\alpha \to 1$, we have

$$\begin{aligned} F_{z,v}(kt) &= F_{Pz,Qv}(kt) \geq F_{ABz,STv}(t) * F_{Pz,ABz}(t) * F_{Qv,STv}(t) \\ &\quad * F_{Pz,STv}(t) * F_{Qv,ABz}(t) \\ &\geq F_{z,v}(t) \end{aligned}$$

which means that $z = v$. This completes the proof.

If we take $A = B = S = T = I_X$ in Theorem 4.1, we have the following:

Corollary 4.1. *Let P and Q be self maps on a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$. If there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} F_{Px,Qy}(kt) &\geq F_{x,y}(t) * F_{x,Px}(t) * F_{y,Qy}(t) \\ &\quad * F_{y,Px}(\alpha t) * F_{x,Qy}((2-\alpha)t) \end{aligned}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$, then P and Q have a unique common fixed point.

Remark 4.1. *Theorem 4.1 is a generalization of the results of Singh and Jain [20], and Mishra [12] in the sense of that the condition of compatibility of the first pair of self maps has been restricted to (P-1) type (or (P-2) type) compatibility and no one of the self maps need to be continuous in non-complete Menger space.*

Example. Let (X, d) be a metric space with the usual metric d where $X = [0, 1]$ and $(X, F, *)$ be the induced Menger space with $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X, t > 0$. Clearly $(X, F, *)$ is a Menger space where t-norm $*$ is defined by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let A, B, P, Q, S and T be maps from X into itself defined as

$$Ax = x/5, Bx = x/3, Px = x/6, Qx = 0, Sx = x, Tx = x/2$$

for all $x \in X$. Then

$$P(X) = \left[0, \frac{1}{6}\right] \subset \left[0, \frac{1}{2}\right] = ST(X)$$

and

$$Q(X) = \{0\} \subset \left[0, \frac{1}{15}\right] = AB(X).$$

If we take $k = 1/2$, $t = 1$ and $\alpha = 1$, we see that the condition (b) of the main Theorem is satisfied. Clearly, conditions (c) and (d) of the main Theorem are also satisfied. Moreover, the pair (P, AB) is compatible of type (P-1) and also type (P-2). In fact, if $\lim_{n\to\infty} x_n = 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = 0$ for some $0 \in X$, then

$$\lim_{n\to\infty} F_{P(AB)x_n, AB(AB)x_n}(t) = H(t) = 1.$$

Similarly, (P, AB) is also compatible of type (P-2). The pair (Q, ST) is weakly compatible since they commute at their coincidence point 0. Thus, all conditions of the main Theorem are satisfied and 0 is the unique common fixed point of A, B, P, Q, S and T.

Following Bylka [1], we consider the family G of functions $g : [0, \infty) \to [0, \infty)$ such that g is non-decreasing in $[0, \infty)$ and $\lim_{n\to\infty} g^n(t) = \infty$ for every $t > 0$ where g^n denotes the n-th iteration of g. It is proved in [14] that $g(t) \geq t$ for all $t \geq 0$ and if $F_{xy}(t) \geq F_{xy}(g(t))$ for some $t > 0$ then $x = y$.

THEOREM 4.2. *Let A, B, P, Q, S and T be self maps on a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$, satisfying:*

(a) $P(X) \subseteq ST(X)$, $Q(X) \subseteq AB(X)$,

(b) there exists a function $g \in G$ such that

$$F_{Px,Qy}(t) \geq F_{ABx,STy}(g(t))$$

for all $x, y \in X$ and $t > 0$,

(c) one of $P(X), Q(X), AB(X)$ and $ST(X)$ is a complete subspace of X,

(d) (P, AB) is compatible of type (P-1) or type (P-2), and (Q, ST) is weakly compatible.

Then A, B, P, Q, S and T have a unique common fixed point in X.

Proof. Using the condition (a), we can construct a sequence $\{y_n\}$ in X defined by (1). Then, for all $t > 0$ and $n = 1, 2, ...$, we have

$$\begin{aligned} F_{y_{2n+1},y_{2n+2}}(t) &= F_{Px_{2n+1},Qx_{2n+2}}(t) \\ &\geq F_{ABx_{2n+1},STx_{2n+2}}(g(t)) \\ &= F_{y_{2n},y_{2n+1}}(g(t)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} F_{y_{2n},y_{2n+1}}(t) &= F_{Px_{2n},Qx_{2n+1}}(t) \\ &\geq F_{ABx_{2n},STx_{2n+1}}(g(t)) \\ &= F_{y_{2n-1},y_{2n}}(g(t)). \end{aligned}$$

Therefore

$$F_{y_{2n},y_{2n+1}}(t) \geq F_{y_{2n-1},y_{2n}}(g(t)) \geq ... \geq F_{y_0,y_1}(g^n(t)). \tag{2}$$

Now, we show that the sequence $\{y_n\}$ is a Cauchy sequence in X. Let ε and λ be positive reals. Then, for $m > n, p = m - n$ and by using (1), we have

$$\begin{aligned} F_{y_n,y_m}(\varepsilon) &\geq F_{y_n,y_{n+1}}\left(\frac{\varepsilon}{p}\right) * F_{y_{n+1},y_m}\left(\frac{\varepsilon(p-1)}{p}\right) \\ &\geq F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) * F_{y_{n+1},y_m}\left(\frac{\varepsilon(p-1)}{p}\right) \\ &\geq F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) * F_{y_{n+1},y_{n+2}}\left(\frac{\varepsilon}{p}\right) * F_{y_{n+2},y_m}\left(\frac{\varepsilon(p-2)}{p}\right) \\ &\geq F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) * F_{y_0,y_1}\left(g^{n+1}\left(\frac{\varepsilon}{p}\right)\right) * F_{y_{n+2},y_m}\left(\frac{\varepsilon(p-2)}{p}\right). \end{aligned}$$

Since $\lim_{n\to\infty} g^n(t) = \infty$, we have $g^n\left(\frac{\varepsilon}{p}\right) \le g^{n+1}\left(\frac{\varepsilon}{p}\right)$ and by the hypothesis $t * t \ge t$. Therefore, we have

$$F_{y_n,y_m}(\varepsilon) \ge F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) * F_{y_{n+2},y_m}\left(\frac{\varepsilon(p-2)}{p}\right). \tag{3}$$

Using the induction argument, we obtain from (3) that

$$\begin{aligned} F_{y_n,y_m}(\varepsilon) &\ge F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) * F_{y_{n+k-2},y_{n+k-1}}\left(\frac{\varepsilon}{p}\right) * F_{y_{m-1},y_m}\left(\frac{\varepsilon}{p}\right) \\ &\ge F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) * F_{y_0,y_1}\left(g^{n+k-2}\left(\frac{\varepsilon}{p}\right)\right) * F_{y_0,y_1}\left(g^{m-1}\left(\frac{\varepsilon}{p}\right)\right) \\ &\ge F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right). \end{aligned}$$

Hence, we can choose $n_0 \le n$ such that

$$F_{y_0,y_1}\left(g^n\left(\frac{\varepsilon}{p}\right)\right) > 1-\lambda$$

and then $F_{y_n,y_m}(\varepsilon) > 1-\lambda$ for all $m > n \ge n_0$. This means that $\{y_n\}$ is a Cauchy sequence in X. Let (P, AB) be compatible of type (P-1) and (Q, ST) be weakly compatible. Now suppose that $AB(X)$ is complete then $\{y_{2n+1}\}$ has a limit z in $AB(X)$. Let $w \in (AB)^{-1}z$, then $ABw = z$. We shall use the fact that the subsequence $\{y_{2n}\}$ also converges to z. If we take $x = w$ and $y = x_{2n+1}$ in (b), we have

$$F_{Pw,Qx_{2n+1}}(t) \ge F_{ABw,STx_{2n+1}}(g(t))$$

Taking $n \to \infty$, we have

$$F_{Pw,z}(t) \ge F_{z,z}(g(t)) = 1$$

which implies that $z = Pw$. Hence $z = ABw = Pw$, i.e., w is a coincide point of P and AB. Since the pair (P, AB) is compatible of type (P-1) and $ABw = Pw$, by Proposition 3.2, we have $P(AB)w = AB(AB)w$ or $Pz = ABz$.

Since $P(X) \subseteq ST(X)$, $Pw = z$ implies that $z \in ST(X)$. Let $u \in (ST)^{-1}z$, then $STu = z$. It can easily verified by using similar arguments of the previous part of the proof that $z = Qu$, hence $z = STu = Qu$. Since the pair (Q, ST) is weakly compatible, we have $Q(ST)u = (ST)Qu$ or $Qz = STz$.

If we assume that $ST(X)$ is complete, then the argument analogous to the previous completeness argument establishes w is a coincide point of P and AB, and u is a coincide point of Q and ST. Thus $Pz = ABz$ and $Qz = STz$.

The remaining two cases pertain essentially to the previous cases. Indeed, if $Q(X)$ is complete, then by the condition (a), $z \in Q(X) \subset AB(X)$. Similarly if $P(X)$ is complete, then $z \in P(X) \subset ST(X)$. Thus $Pz = ABz$ and $Qz = STz$.

Now, we prove that $z = Pz$. If we take $x = z$ and $y = x_{2n+1}$ in (b), we have

$$F_{Pz,Qx_{2n+1}}(t) \ge F_{ABz,STx_{2n+1}}(g(t))$$

Taking $n \to \infty$, we have

$$F_{Pz,z}(t) \ge F_{Pz,z}(g(t))$$

which means that $z = Pz$. Hence $z = Pz = ABz$. Similarly we also have $z = Qz = STz$.

Now, we prove that $z = Bz$. Since $z = Pz = ABz$, $P(Bz) = Bz$ and $AB(Bz) = Bz$. If we take $x = Bz$ and $y = x_{2n+1}$ in (b), we have

$$F_{P(Bz),Qx_{2n+1}}(t) \geq F_{AB(Bz),STx_{2n+1}}(g(t))$$

Taking $n \to \infty$, we have

$$F_{Bz,z}(t) \geq F_{Bz,z}(g(t))$$

which means that $z = Bz$. Since $z = ABz$, $z = Bz$ implies that $z = Az$. Hence $z = Pz = Az = Bz$. Similarly we also have $z = Qz = Sz = Tz$. Therefore z is a common fixed point of A, B, P, Q, S and T. It is also easy to prove that if (P, AB) is compatible of type (P-2) and (Q, ST) is weakly compatible then z is a common fixed point of A, B, P, Q, S and T.

It is easy to see that z is unique common fixed point of A, B, P, Q, S and T. This completes the proof.

If we take $g(t) = t/k$ for $k \in (0, 1)$, $P = Q$ and $A = B = S = T = I_X$ in Theorem 4.2, we have the following:

COROLLARY 4.2 (15). *Let P be a self map on a Menger space $(X, F, *)$ with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$. If $P(X)$ is complete and there exists a constant $k \in (0, 1)$ such that*

$$F_{Px,Py}(kt) \geq F_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then P has a unique common fixed point in X.

Note that the proof of Corollary 4.2 also follows from Corollary 4.1 since $P = Q$ and $F_{x,y}(t) = \min\{F_{x,y}(t), F_{x,Px}(t), F_{y,Qy}(t), F_{y,Px}(t), F_{x,Qy}(t)\}$.

REMARK 4.2. *Theorem 4.2 is a generalization of the results of Bylka [1], and Sehgal and Bharucha-Reid [15] in the sense of that the condition of compatibility of the first pair of self maps has been restricted to (P-1) type (or (P-2) type) compatibility and no one of the self maps need to be continuous in non-complete Menger space.*

REMARK 4.3. *In Theorems 4.1 and 4.2, and Corollaries 4.1 and 4.2, the condition "the t-norm $*$ is continuous and $t * t \geq t$ for all $t \in [0, 1]$" can be replaced by the condition "$s * t = \min\{s, t\}$ for all $s, t \in [0, 1]$".*

5 An Application

In this section, we apply Corollaries 4.1 and 4.2 to establish the following result on the product space.

THEOREM 5.1. *Let $(X, F, *)$ be a Menger space with continuous t-norm $*$ and $t * t \geq t$ for all $t \in [0, 1]$ and P and Q be self maps on the product $X \times X$ with values in X. If there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} F_{P(x,y)Q(u,v)}(kt) \geq\ & F_{xu}(t) * F_{yv}(t) * F_{xP(x,y)}(t) * F_{uQ(u,v)}(t) \\ & * F_{uP(x,y)}(\alpha t) * F_{xQ(u,v)}((2-\alpha)t) \end{aligned} \tag{4}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$, then P and Q have a unique common fixed point.

Proof. From (4), we have

$$\begin{aligned} F_{P(x,y)Q(u,y)}(kt) \geq\ & F_{xu}(t) * F_{xP(x,y)}(t) * F_{uQ(u,y)}(t) \\ & * F_{uP(x,y)}(\alpha t) * F_{xQ(u,y)}((2-\alpha)t) \end{aligned}$$

for all x, y, u in X. Therefore, by Corollary 4.1, for each y in X, there exists exactly one point $z(y)$ in X such that

$$P(z(y), y) = z(y) = Q(z(y), y). \tag{5}$$

Now, for any y, y' in X, by (4) with $\alpha = 1$, we have

$$\begin{aligned} F_{P(z(y),y)Q(z(y'),y')}(kt) \geq\; & F_{z(y)z(y')}(t) * F_{yy'}(t) * F_{z(y)P(z(y),y)}(t) \\ & * F_{z(y')Q(z(y'),y')}(t) * F_{z(y')P(z(y),y)}(t) * F_{z(y)Q(z(y'),y')}(t) \end{aligned}$$

that is,

$$\begin{aligned} F_{z(y)z(y')}(kt) &\geq F_{z(y)z(y')}(t) * F_{yy'}(t) * 1 * 1 * F_{z(y')z(y)}(t) * F_{z(y)z(y')}(t) \\ &\geq F_{z(y)z(y')}(t) * F_{yy'}(t) \\ &\geq F_{z(y)z(y')}(t/k^n) * F_{yy'}(t) \\ &\to F_{yy'}(t). \end{aligned}$$

Therefore, Corollary 4.2 yields that the map $z(.)$ of X into itself has exactly one fixed point w in X, i.e. $z(w) = w$. Hence, by (5), $w = z(w) = P(w, w) = Q(w, w)$. This completes the proof.

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TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL.10,NO.3,2008

Volume 10,Number 4 October 2008
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

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EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE
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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,415-430,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Nonlinear Random Multi-valued Variational Inclusion Systems involving (A, η)-accretive Mappings in Banach Spaces[1]

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Abstract In this paper, by using the new random resolvent operator technique associated with (A, η)-accretive mappings, we analyze and establish existence theorem for a new nonlinear random multi-valued variational inclusion systems involving (A, η)-accretive mappings in Banach spaces. Our results generalize some results of other recent works on strongly monotone quasi-variational inclusions, nonlinear implicit quasi-variational inclusions and nonlinear mixed quasi-variational inclusion systems.

Key words and phrases: Relaxed random cocoercive mapping, nonlinear random multi-valued variational inclusion system, (A, η)-accretive mapping, random resolvent operator technique, random iterative algorithm and convergence.

AMS Subject classification: 49J40, 47H05, 47H19

1. Introduction

Very recently, by using the iterative technique and Nadlers theorem, Wu et al. [36] construct a new iterative algorithm for solving the following system of nonlinear inclusions in Banach spaces and prove some new existence results of solutions for the system of nonlinear inclusions and discuss the convergence of the sequences generated by the algorithm. As an application, authors also show the existence of solution for a system of functional equations arising in dynamic programming of multistage decision processes.

In this paper, for any given elements $f : \Lambda \to E$ and $g : \Omega \to E$, and any real-valued random variables $\lambda_1(s)$, $\lambda_2(t) > 0$, we shall consider the following nonlinear random multi-valued variational inclusion systems:

Find $x : \Lambda \to E, y : \Omega \to E$ such that $Range(p) \bigcap domM_1(t, \cdot) \neq \emptyset$ and

$$\begin{cases} y(t) - x(s) - \lambda_1(s)(N_1(s, F(t, y(t)), v(t)) - f(s)) \in \lambda_1(s)M_1(s, p(s, x(s))), \\ \quad \forall v(t) \in T(t, y(t)), \\ x(s) - y(t) - \lambda_2(t)(N_2(t, G(s, x(s)), u(s)) - g(t)) \in \lambda_2(t)M_2(t, y(t)), \\ \quad \forall u(s) \in S(s, x(s)), \end{cases} \tag{1.1}$$

[1] This work was supported by the Scientific Research Fund of Sichuan Provincial Education Department (2006A106).

where E and E are two separable real Banach spaces, $(\Lambda, \mathcal{A}, \mu)$ and $(\Omega, \mathcal{C}, \nu)$ are two complete σ-finite measure spaces, $S : \Lambda \times E \to 2^E$ and $T : \Omega \times E \to 2^E$ are multi-valued mappings, $M_1 : \Lambda \times E \to 2^E$ and $M_2 : \Omega \times E \to 2^E$ are any nonlinear mappings such that for all $(s,t) \in \Lambda \times \Omega$, $M_1(s,\cdot) : E \to 2^E$ is an (A_1, η_1)-accretive mapping and $M_2(t,\cdot) : E \to 2^E$ is an (A_2, η_2)-accretive mapping, $N_1 : \Lambda \times E \times E \to E$, $N_2 : \Omega \times E \times E \to E$, $F : \Omega \times E \to E$, G, $p : \Lambda \times E \to E$, $A_1 : \Lambda \times E \to E$, $A_2 : \Omega \times E \to E$, $\eta_1 : \Lambda \times E \times E \to E$ and $\eta_2 : \Omega \times E \times E \to E$ are single-valued mappings, 2^{E_i} denotes the family of all the nonempty subsets of E_i for $i = 1, 2$.

The study of such types of problems is motivated by an increasing interest in the random equations involving the random operators in view of their need in dealing with probabilistic models in applied sciences is very important. In recent years, Ahmad and Bazán [2], Chang [4], Chang and Huang [5], Cho et al. [8], Ganguly and Wadhwa [12], Huang [14], Huang and Cho [15], Huang et al. [16], Khan et al. [19], Lan [21], Noor and Elsanousi [32] introduced and studied the research works in these fascinating areas, the random variational inequality problems, random quasi-variational inequality problems, random variational inclusion problems and random quasi-complementarity problems, respectively.

On the other hand, it is well known that variational inequality type methods have been applied widely to problems arising from model equilibria problems in economics, optimization and control theory, operations research, transportation network modelling, and mathematical programming. Very recently, in order to study extensively variational inequalities and variational inclusions, which are providing mathematical models to some problems arising in economics, mechanics, and engineering science, Lan et al. [11] first introduced the concept of (A, η)-accretive mappings, which generalizes the existing η-subdifferential operators, maximal η-monotone operators, generalized monotone operators (named H-monotone operators), A-monotone operators, (H, η)-monotone operators, (A, η)-monotone operators in Hilbert spaces, H-accretive operators, generalized m-accretive mappings and (H, η)-accretive operators in Banach spaces. The authors also studied some properties of (A, η)-accretive mappings and defined resolvent operators associated with (A, η)-accretive mappings, which improved and generalized the corresponding results of recent works in [1, 6, 9, 10, 11, 20, 22, 24, 26, 27, 28, 31, 33, 38].

Furthermore, the determinate form of the problem (1.1) was studied by Lan et al. [26] when $M_1(s,\cdot)$ and $M_2(t,\cdot)$ are generalized m-accretive mappings for a given determinate element $(s,t) \in \Lambda \times \Omega$, and for appropriate and suitable choices of N_i, M_i, A_i, η_i, λ_i $(i = 1, 2)$, S, T, F, G, f, g, p and E, it is easy to see that the problem (1.1) includes a number (systems) of (random) quasi-variational inclusions, generalized (random) quasi-variational inclusions, (random) quasi-variational inqualities, (random) implicit quasi-variational inequalities studied by many authors as special cases, see, for example:

Example 1.1. Let $S : \Lambda \times E \to E$ and $T : \Omega \times E \to E$ be single-valued mappings. Then for each fixed elements $(f(s), g(t)) \in E \times E$ and any real-valued random variables $\lambda_1(t)$, $\lambda_2(t) > 0$, the problem (1.1) reduces to finding $x : \Lambda \to E, y : \Omega \to E$ such that $Range(p) \bigcap domM_1(t,\cdot) \neq \emptyset$ and

$$\begin{cases} y(t) - x(s) - \lambda_1(s)(N_1(s, F(t, y(t)), T(t, y(t))) - f(s)) \in \lambda_1(s) M_1(s, p(s, x(s))), \\ x(s) - y(t) - \lambda_2(t)(N_2(t, G(s, x(s)), S(s, x(s))) - g(t)) \in \lambda_2(t) M_2(t, y(t)). \end{cases} \tag{1.2}$$

Example 1.2. Suppose that $p \equiv I$, the identity mapping, $f(s) = 0$ for all $s \in \Lambda$, $g(t) = 0$ for all $t \in \Omega$, $N_1(\cdot, x, y) = x + y$ for all $x, y \in E$ and $N_2(\cdot, z, w) = z + w$ for

all $z, w \in E$. Then the problem (1.2) is equivalent to the following system of general nonlinear mixed random quasi-variational inclusions in Banach spaces: find $(x(s), y(t)) \in E \times E$ such that

$$\begin{cases} 0 \in x(s) - y(t) + \lambda_1(s)(F(t, y(t)) + T(t, y(t))) + \lambda_1(s)M_1(s, x(s)), \\ 0 \in y(t) - x(s) - \lambda_2(t)(G(s, x(s)) + S(s, x(s))) + \lambda_2(t)M_2(t, y(t)). \end{cases} \tag{1.3}$$

The parametric form of the problem (1.3) was studied by Jeong [17] when M, N are m-accretive mappings and $\lambda_1(s)$ and $\lambda_2(t)$ are constants. Further, the determinate form of the problem (1.3) was introduced and studied by Agarwal et al. [1] when $E = \mathcal{H}$ is a Hilbert spaces, M_1, M_2 are two maximal monotone mappings.

Now, for each fixed $(s, t) \in \Lambda \times \Omega$, the solution set $Q(s, t)$ of the problem (1.1) is denoted as

$$\begin{aligned} &Q(s,t) \\ &= \{(x(s), y(t)) \in E \times E : \exists u(s) \in S(s, x(s)) \,\text{and}\, v(t) \in T(t, y(t)), \text{ such that} \\ &\qquad y(t) - x(s) - \lambda_1(s)(N_1(s, F(t, y(t)), v(t)) - f(s)) \in \lambda_1(s)M_1(s, p(s, x(s))) \\ &\qquad \text{and}\, x(s) - y(t) - \lambda_2(t)(N_2(t, G(s, x(s)), u(s)) - g(t)) \in \lambda_2(t)M_2(t, y(t))\}. \end{aligned}$$

In this paper, by using the new random resolvent operator technique associated with (A, η)-accretive mappings, our main aim is to study the behavior of the solution set $Q(\omega, \lambda)$, and the conditions on these mappings $N_i, M_i, S, T, F, G, f, g, p, A_i, \eta_i, \lambda_i$ and E_i for $i = 1, 2$ under which the function $Q(s, t)$ is nonempty and the generalized random iterative procedures with errors for the element of this solution set $Q(s, t)$ in q-uniformly smooth Banach spaces is convergence. Our results generalize some results of other recent works on strongly monotone quasi-variational inclusions, nonlinear implicit quasi-variational inclusions and nonlinear mixed quasi-variational inclusion systems.

2. Preliminaries

Throughout this paper, we suppose that $(\Omega, \mathcal{A}, \mu)$ is a complete σ-finite measure space and E is a separable real Banach space endowed with dual space E^*, the norm $\|\cdot\|$ and the dual pair $\langle \cdot, \cdot \rangle$ between E and E^*. We denote by $\mathcal{B}(E)$ the class of Borel σ-fields in E. Let 2^E and $CB(E)$ denote the family of all the nonempty subsets of E, the family of all the nonempty bounded closed sets of E, respectively. The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is well known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and J_q is single-valued if E^* is strictly convex (see, for example, [37]). If $E = \mathcal{H}$ is a Hilbert space, then J_2 becomes the identity mapping of $\mathcal{H}$. In what follows we shall denote the single-valued generalized duality mapping by j_q.

The modules of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}\|x + y\| + \|x - y\| - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space E is called *uniformly smooth* if $\lim\limits_{t\to 0}\frac{\rho_E(t)}{t}=0$ and E is called *q-uniformly smooth* if there exists a constant $c>0$ such that $\rho_E\le ct^q$, where $q>1$ is a real number.

It is well known that Hilbert spaces, L_p(or l_p) spaces, $1<p<\infty$, and the Sobolev spaces $W^{m,p}$, $1<p<\infty$, are all q-uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [37] proved the following result.

Lemma 2.1. Let $q>1$ be a given real number and E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q>0$ such that for all x, $y\in E$, $j_q(x)\in J_q(x)$, there holds the following inequality

$$\|x+y\|^q\le\|x\|^q+q\langle y,j_q(x)\rangle+c_q\|y\|^q.$$

In this paper, we will use the following definitions and lemmas.

Definition 2.1. A operator $x:\,\Omega\to E$ is said to be *measurable* if for any $E\in\mathcal{B}(E)$, $\{t\in\Omega:\,x(t)\in E\}\in\mathcal{A}$.

Definition 2.2. A operator $F:\,\Omega\times E\to E$ is called a *random operator* if for any $x\in E$, $F(t,x)=y(t)$ is measurable. A random operator F is said to be *continuous* (resp. *linear, bounded*) if for any $t\in\Omega$, the operator $F(t,\cdot):\,E\to E$ is continuous (resp. linear, bounded).

Similarly, we can define a random operator $b:\,\Omega\times E\times E\to E$. We shall write $F_t(x)=F(t,x(t))$ and $b_t(x,y)=b(t,x(t),y(t))$ for all $t\in\Omega$ and $x(t)$, $y(t)\in E$.

It is well known that a measurable operator is necessarily a random operator.

Definition 2.3. A multi-valued operator $G:\,\Omega\to 2^E$ is said to be *measurable* if for any $E\in\mathcal{B}(E)$, $G^{-1}(E)=\{t\in\Omega:\,G(t)\cap E\neq\emptyset\}\in\mathcal{A}$.

Definition 2.4. A operator $u:\,\Omega\to E$ is called a *measurable selection* of a multi-valued measurable operator $\Gamma:\,\Omega\to 2^E$ if u is measurable and for any $t\in\Omega$, $u(t)\in\Gamma(t)$.

Definition 2.5. A multi-valued operator $F:\,\Omega\times E\to 2^E$ is called a *random multi-valued operator* if, for any $x\in E$, $F(\cdot,x)$ is measurable. A random multi-valued operator $F:\,\Omega\times E\to CB(E)$ is said to be $\hat{\mathbf{H}}$*-continuous*, if for any $t\in\Omega$, $F(t,\cdot)$ is continuous in $\hat{\mathbf{H}}(\cdot,\cdot)$, where $\hat{\mathbf{H}}(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined as follows: for any given D, $K\in CB(E)$,

$$\hat{\mathbf{H}}(D,K)=\max\Big\{\sup_{x\in D}\inf_{y\in K}d(x,y),\sup_{y\in K}\inf_{x\in D}d(x,y)\Big\}.$$

Definition 2.6. Let E be a q-uniformly smooth Banach space. Then a random operator $g:\Omega\times E\to E$ is said to be

(i) *m-relaxed accretive* in the second argument, if

$$\langle g_t(x)-g_t(y),j_q(x(t)-y(t))\rangle\ge -m(t)\|x(t)-y(t)\|^q,\quad\forall x(t),\,y(t)\in E,\,t\in\Omega,$$

where $m(t)$ is a real-valued random variable;

(ii) *s-cocoercive* in the second argument, if there exists a real-valued random variable $s(t)>0$ such that

$$\langle g_t(x)-g_t(y),j_q(x(t)-y(t))\rangle\ge s(t)\|g_t(x)-g_t(y)\|^q,\quad\forall x(t),\,y(t)\in E,\,t\in\Omega;$$

(iii) γ-*relaxed cocoercive* in the second argument, if there exists a positive real-valued random variable $\gamma(t)$ such that

$$\langle g_t(x) - g_t(y), j_q(x(t) - y(t))\rangle \geq -\gamma(t)\|g_t(x) - g_t(y)\|^q, \quad \forall x(t),\, y(t) \in E,\, t \in \Omega;$$

(iv) (α, ϵ)-*relaxed cocoercive* in the second argument, if there exist positive real-valued random variables $\alpha(t)$ and $\epsilon(t)$ such that

$$\langle g_t(x) - g_t(y), j_q(x(t) - y(t))\rangle \geq -\alpha(t)\|g_t(x) - g_t(y)\|^q + \epsilon(t)\|x(t) - y(t)\|^q,$$

for all $x(t),\, y(t) \in E,\, t \in \Omega$;

(v) μ-*Lipschitz continuous* if there exists a real-valued random variable $\mu(t) > 0$ such that

$$\|g_t(x) - g_t(y)\| \leq \mu(t)\|x(t) - y(t)\|,\, \forall x(t),\, y(t) \in E,\, t \in \Omega.$$

Remark 2.1. Clearly, every m-cocoercive mapping is m-relaxed cocoercive, while each r-strongly monotone mapping is $(r + r^2, 1)$-relaxed cocoercive with respect to I. Further, we can find some operators which are cocoercive and relaxed cocoercive. See, for example, [23, 33, 35].

Definition 2.7. Let E be a q-uniformly smooth Banach space, $\eta : \Omega \times E \times E \to E$ and $A, H : \Omega \times E \to E$ be random single-valued operators. Then a multi-valued measurable operator $M : \Omega \times E \to 2^E$ is said to be

(1) σ-$\hat{\mathbf{H}}$-*Lipschitz continuous*, if there exists a measurable function $\sigma : \Omega \to (0, +\infty)$ such that for any $t \in \Omega$,

$$\hat{\mathbf{H}}(M_t(x), M_t(y)) \leq \sigma(t)\|x(t) - y(t)\|, \quad \forall x(t),\, y(t) \in E;$$

(2) η-*accretive* if for all $x(t),\, y(t) \in E,\, u(t) \in M_t(x),\, v(t) \in M_t(y),\, t \in \Omega$,

$$\langle u(t) - v(t), j_q(\eta_t(x, y))\rangle \geq 0;$$

(3) *strictly* η-*accretive* if for all $x(t),\, y(t) \in E,\, u(t) \in M_t(x),\, v(t) \in M_t(y),\, t \in \Omega$,

$$\langle u(t) - v(t), j_q(\eta_t(x, y))\rangle \geq 0,$$

and equality holds if and only if $u(t) = v(t)$ for all $t \in \Omega$;

(4) r-*strongly* η-*accretive*, if there exists a real-valued random variable $r(t) > 0$ such that

$$\langle u(t) - v(t), j_q(\eta_t(x, y))\rangle \geq r(t)\|x(t) - y(t)\|^2,$$

for all $x(t),\, y(t) \in E,\, u(t) \in M_t(x),\, v(t) \in M_t(y),\, t \in \Omega$;

(5) α-*relaxed* η-*accretive*, if there exists a real-valued random variable $\alpha(t) > 0$ such that for all $x(t),\, y(t) \in E,\, u(t) \in M_t(x),\, v(t) \in M_t(y),\, t \in \Omega$,

$$\langle u(t) - v(t), j_q(\eta_t(x, y))\rangle \geq -\alpha(t)\|x(t) - y(t)\|^q;$$

(6) m-*accretive*, if M is accretive and $(I + \rho(t)M_t)(E) = E$ for all $t \in \Omega$ and real-valued random variable $\rho(t) > 0$, where I denotes the identity operator on E and $M_t(\cdot) = M(t, \cdot)$ for all $t \in \Omega$;

(7) *generalized m-accretive*, if M is η-accretive and $(I+\rho(t)M_t))(E) = E$ for all $t \in \Omega$ and (equivalently, for some) $\rho(t) > 0$;

(8) *H-accretive* if M is accretive and $(H_t+\rho(t)M_t)(E) = E$ for all $t \in \Omega$ and $\rho(t) > 0$, where $H_t(\cdot) = H(t, \cdot)$ for all $t \in \Omega$;

(9) *(H, η)-accretive*, if M is η-accretive and $(H_t + \rho(t)M_t)(E) = E$ for all $t \in \Omega$ and $\rho(t) > 0$;

(10) *(A, η)-accretive* with real-valued random variable $m(t)$ if

(i) M is m-relaxed η-accretive, (ii) $(A_t + \rho(t)M_t)(E) = E$ for every $t \in \Omega$ and $\rho(t) > 0$, where $A_t(\cdot) = A(t, \cdot)$ for all $t \in \Omega$.

In a similar way, we can define strictly η-accretivity and strongly η-accretivity of the single-valued mapping A.

Remark 2.1. For appropriate and suitable choices of m, A, η and E, it is easy to see that Definition 2.7 includes a number of definitions of monotone operators and accretive mappings (see [25]).

Definition 2.8. The operator $\eta : \Omega\times E\times E \to E$ is said to be *τ-Lipschitz continuous* if there exists a real-valued random variable $\tau(t) > 0$ such that

$$\|\eta_t(x, y)\| \le \tau(t)\|x(t) - y(t)\|, \quad \forall x(t),\, y(t) \in E,\, t \in \Omega.$$

Definition 2.9. Let $A : \Omega \times E \to E$ be a strictly η-accretive mapping and $M : \Omega \times E \to 2^E$ be an (A, η)-accretive mapping. For any given measurable function $\rho : \Omega \to (0, \infty)$, the *resolvent operator* $J^{\rho(t),A_t}_{\eta_t,M_t} : E \to E$ is defined by:

$$J^{\rho(t),A_t}_{\eta_t,M_t}(u) = (A_t + \rho(t)M_t)^{-1}(u), \quad \forall u \in E,\, t \in \Omega.$$

Lemma 2.2. ([25]) Let E be a q-uniformly smooth Banach space and $\eta : \Omega\times E\times E \to E$ be τ-Lipschitz continuous, $A : \Omega \times E \to E$ be a r-strongly η-accretive mapping and $M : \Omega \times E \to 2^E$ be an (A, η)-accretive mapping. Then the resolvent operator $J^{\rho(t),A_t}_{\eta_t,M_t} : E \to E$ is $\frac{\tau^{q-1}(t)}{r(t)-\rho(t)m(t)}$-Lipschitz continuous, i.e.,

$$\|J^{\rho(t),A_t}_{\eta_t,M_t}(x) - J^{\rho(t),A_t}_{\eta_t,M_t}(y)\| \le \frac{\tau^{q-1}(t)}{r(t) - \rho(t)m(t)}\|x - y\|, \quad \forall x, y \in E,\, t \in \Omega,$$

where $\rho(t) \in (0, \frac{r(t)}{m(t)})$ is a measurable function for all $t \in \Omega$.

3. Random Iterative Algorithms

In this section, we suggest and analyze a new class of iterative methods and construct some new random iterative algorithms with errors for solving problem (1.1).

Lemma 3.1. ([3]) Let $M : \Omega \times E \to CB(E)$ be a H-continuous random multi-valued mapping. Then for any measurable mapping $x : \Omega \to E$, the multi-valued mapping $M(\cdot, x(\cdot)) : \Omega \to CB(E)$ is measurable.

Lemma 3.2. ([3]) Let $M, V : \Omega \times E \to CB(E)$ be two measurable multi-valued mappings, $\epsilon > 0$ be a constant and $x : \Omega \to E$ be a measurable selection of M. Then there exists a measurable selection $y : \Omega \to E$ of V such that for any $t \in \Omega$,

$$\|x(t) - y(t)\| \le (1 + \epsilon)\hat{\mathbf{H}}(M(t), V(t)).$$

Lemma 3.3. For each fixed $(s,t) \in \Lambda \times \Omega$, an element $(x(s), y(t)) \in Q(s,t)$ is a solution to the problem (1.1) if and only if there are $(x(s), y(t)) \in E \times E$ such that

$$\begin{cases} p_s(x) \in J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}}[A_{1s}(p_s(x)) + y(t) - x(s) \\ \qquad\qquad -\lambda_1(s)(N_{1s}(F_t(y), T_t(y)) - f(s))], \\ y(t) \in J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}}[A_{2t}(y) + x(s) - y(t) \\ \qquad\qquad -\lambda_2(t)(N_{2t}(G_s(x), S_s(x)) - g(t))], \end{cases}$$

where $J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}} = (A_{1s} + \lambda_1(s)M_{1s})^{-1}$ and $J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}} = (A_{2t} + \lambda_2(t)M_{2t})^{-1}$ are the corresponding resolvent operator of an (A_1, η_1)-accretive mapping $M_1(s, \cdot)$, (A_2, η_2)-accretive mapping $M_2(t, \cdot)$, respectively, A_i is an r_i-strongly monotone mapping for $i = 1, 2$.

Proof. The proof directly follows from the Definition 2.9 and some simple arguments.

Based on Lemma 3.3, we can develop a new iterative algorithm for solving general nonlinear random equation (1.1) as follows:

Let $S : \Lambda \times E \to CB(E)$ and $T : \Omega \times E \to CB(E)$ be a multi-valued mappings and $\alpha_n : \Lambda \to (0,1]$ and $\beta_n : \Omega \to (0,1]$ be two measurable step size functions for all $n \in N$.

For any given $(z_0(\cdot), w_0(\cdot)) \in E \times E$, we choose $(x_0(\cdot), w_0(\cdot)) \in E \times E$ such that

$$p_s(x_0) = J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}}(z_0), \quad y_1(t) = J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}}(w_0),$$

then it is easy to know that $x_0 : \Lambda \to E$ is measurable. Further, by Lemma 3.1 and Himmelberg [13], we know that for the chosen $x_0(\cdot)$ and $w_0(\cdot)$, the multi-valued mapping $S(\cdot, x_0(\cdot))$ $T(\cdot, w_0(\cdot))$ are measurable. Let

$$\begin{aligned} z_1(s) \quad & \in (1-\alpha_0)z_0(s) + \alpha_0[A_{1s}(p_s(x_0)) + y_0(t) - x_0(s) \\ & \qquad -\lambda_1(s)(N_{1s}(F_t(y_0), T_t(y_0)) - f(s))] + \alpha_0 d_0(s) + e_0(s), \\ w_1(t) \quad & \in (1-\alpha_0(t))w_0(t) \\ & \qquad +\alpha_0[A_{2t}(y_0) + x_0(s) - y_0(t) - \lambda_2(t)(N_{2t}(G_s(x_0), S_s(x_0)) - g(t))] + h_0(t), \end{aligned}$$

where $\lambda_1(s)$, $\lambda_2(t)$ and A_{1s}, A_{2t}, p_s, N_{1s}, N_{2t}, G_s, F_t are the same as in (1.1). Then it is easy to know that $z_1 : \Lambda \to E$ and $w_1 : \Omega \to E$ are measurable.

For $z_1(\cdot) \in E$ and $w_1(\cdot) \in E$, we take $x_1(\cdot) \in E$ and $y_1(\cdot) \in E$ such that

$$p_s(x_1) = J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}}(z_1), \quad y_1(t) = J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}}(w_1),$$

then it is easy to know that $x_1 : \Lambda \to E$ and $y_1 : \Omega \to E$ are measurable. Let

$$\begin{aligned} z_2(s) \quad & \in (1-\alpha_1)z_1(s) + \alpha_1[A_{1s}(p_s(x_1)) + y_1(t) - x_1(s) \\ & \qquad -\lambda_1(s)(N_{1s}(F_t(y_1), T_t(y_1)) - f(s))] + \alpha_1 d_1(s) + e_1(s), \\ w_2(t) \quad & \in (1-\alpha_1)w_1(t) \\ & \qquad +\alpha_1[A_{2t}(y_1) + x_1(s) - y_1(t) - \lambda_2(t)(N_{2t}(G_s(x_1), S_s(x_1)) - g(t))] + h_1(t). \end{aligned}$$

By induction, we can get an iterative algorithm for solving the nonlinear operator equation problem (1.1) as follows:

Algorithm 3.1.

STEP 1. For any given $(z_0(\cdot), w_0(\cdot)) \in E \times E$, choose $(x_0(\cdot), y_0(\cdot)) \in E \times E$.

STEP 2. Let

$$\begin{cases} p_s(x_n) = J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}}(z_n), \\ y_n(t) = J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}}(w_n), \\ z_{n+1}(s) \in (1-\alpha_n)z_n(s) + \alpha_n[A_{1s}(p_s(x_n)) + y_1(t) - x_n(s) \\ \qquad -\lambda_1(s)(N_{1s}(F_t(y_n), T_t(y_n)) - f(s))] + \alpha_n d_n(s) + e_n(s), \\ w_{n+1}(t) \in (1-\alpha_n)w_n(t) \\ \qquad +\beta_n[A_{2t}(y_n) + x_n(s) - y_n(t) - \lambda_2(t)(N_{2t}(G_s(x_n), S_s(x_n)) - g(t))] + h_n(t), \\ n = 0, 1, 2, \cdots. \end{cases} \tag{3.1}$$

STEP 3. Choose sequence $\{\alpha_n\}$ and $d_n(s)$, $e_n(s)$, $h(t)$ such that for $n \geq 0$, $\{\alpha_n\}$ is a sequence in $(0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $d_n(s)$, $e_n(s)$, $h(t) \in E$ $(n \geq 0)$ are real-valued random errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

(i) $d_n(s) = d'_n(s) + d''_n(s)$;

(ii) $\lim_{n\to\infty} \|d'_n(s)\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|e''_n(s)\| < \infty$, $\sum_{n=0}^{\infty} \|e_n(s)\| < \infty$, $\sum_{n=0}^{\infty} \|h_n(t)\| < \infty$.

STEP 4. If z_{n+1}, w_{n+1}, x_n, y_n, d_n, e_n, $h(t)$ and α_n satisfy (3.1) to sufficient accuracy, stop; otherwise, set $n := n+1$ and return to **STEP 2.**.

Remark 3.1. From Algorithm 3.1, we can get corresponding algorithms for solving Examples 1.1, 1.2 and other special cases. See, for example, [7, 10, 11, 18, 26, 28, 34, 36, 38] and the references therein.

4. Main Results

In this section, we will prove the existence of solution of the problem (1.1) and the convergence of the iterative sequences generated by the algorithm introduced in Section 3.

Lemma 4.1. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that

$$a_{n+1} \leq (1-t_n)a_n + b_n t_n + c_n, \quad \forall n \geq n_0,$$

where $t_n \in [0,1]$ with $\sum\limits_{n=0}^{\infty} t_n = \infty$, $\lim\limits_{n\to\infty} b_n = 0$, and $\sum\limits_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0 (n \to \infty)$.

Proof. The proof directly follows from the proof of Lemma 2 in Liu [29].

Theorem 4.1. Let E be a q-uniformly smooth Banach space, $A_i : \mathcal{B}_i \to \mathcal{B}_i$ be r_i-strongly monotonefor all $i = 1, 2$, A_1 be π-Lipschitz continuous, $S : \Lambda \times E \to CB(E)$ be κ_1-$\hat{\mathbf{H}}$-Lipschitz continuous in the second variable, $T : \Omega \times E \to CB(E)$ be κ_2-$\hat{\mathbf{H}}$-Lipschitz continuous in the second variable, $M_1 : \Lambda \times E \times E \to 2^E$ be (A_1, η_1)-accretive with real-valued random variable $m_1(s)$ in the second variable and $M_2 : E \times E \times \Lambda \to 2^E$ be (A_2, η_2)-accretive with real-valued random variable $m_2(t)$ in the second variable. Let $\eta_1 : \Lambda \times E \times E \to E$ be τ_1-Lipschitz continuous, $\eta_2 : \Omega \times E \times E \to E$ be τ_2-Lipschitz continuous, $N_1 : \Lambda \times E \times E \to E$ be (γ_1, α_1)-relaxed cocoercive and μ_1-Lipschitz continuous in the second variable, $N_2 : \Omega \times E \times E \to E$ be (γ_2, α_2)-relaxed cocoercive and μ_2-Lipschitz continuous in the second variable, and let N_1 be β_1-Lipschitz continuous in the third variable, and N_2 be β_2-Lipschitz continuous in the third variable. Let $G : \Lambda \times E \to E$ be ξ_1-Lipschitz continuous in the second variable, $F : \Omega \times E \to E$ be

ξ_2-Lipschitz continuous in the second variable, $p : \Lambda \times E \to E$ be δ-strongly monotone and σ-Lipschitz continuous in the second variable and $p_1 : \Lambda \times E \to E$ defined by $p_1(s,x) = A_1(s,p(s,x)) = A_{1s}(p_s(x))$ for all $(s,x) \in \Lambda \times E$ be ϖ-strongly monotone and ς-Lipschitz continuous in the second variable. If there exist real-valued random variables $\lambda_1(s) \in (0, \frac{r_1(s)}{m_1(s)})$, $\lambda_2(t) \in (0, \frac{r_2(t)}{m_2(t)})$ such that

$$\begin{cases} \frac{\tau_1^{q-1}(s)}{r_1(s)-\lambda_1(s)m_1(s)} = \frac{\tau_2^{q-1}(t)}{r_2(t)-\lambda_2(t)m_2(t)}, \quad k = \sqrt[q]{1 - q\delta(s) + c_q\sigma^q(s)} < 1, \\ \sqrt[q]{1 - q\lambda_2(t)\alpha_2(s) + c_q\lambda_2^q(t)\mu_2^q(t)\xi_1^q(s) + q\lambda_2(t)\gamma_2^q(t)\xi_1^q(s)} \\ < \tau_2^{1-q}(t)(r_2(t) - \lambda_2(t)m_2(t))[1 - k - \frac{\tau_1^{q-1}(s)\sqrt[q]{1-q\varpi(s)+c_q\varsigma^q(s)}}{r_1(s)-\lambda_1(s)m_1(s)}] - \lambda_2(t)\kappa_1(s)\beta_2(t), \\ \sqrt[q]{1 - q\lambda_1(s)\alpha_1(t) + c_q\lambda_1^q(s)\mu_1^q(s)\xi_2^q(t) + q\lambda_1(s)\gamma_1^q(s)\xi_2^q(t)} \\ < \tau_1^{1-q}(s)(r_1(s) - \lambda_1(s)m_1(s))[1 - \frac{\tau_2^{q-1}(t)\sqrt[q]{1-qr_2(t)+c_q\pi^q(t)}}{r_2(t)-\lambda_2(t)m_2(t)}] - \lambda_1(s)\beta_1(s)\kappa_2(t), \end{cases} \tag{4.1}$$

where c_{q_1}, c_{q_2} are the constants as in Lemma 2.1, then for each $(s,t) \in \Lambda \times \Omega$, the following results follows.

(1) There exist $x^*(s), y^*(t) \in E$ such that $(x^*(s), y^*(t))$ is a solution of the problem (1.1), i.e., the solution set $Q(s,t)$ of the problem (1.1) is nonempty.

(2) $x_n(s) \to x^*(s)$ and $y_n(t) \to y^*(t)$ as $n \to \infty$, where $\{x_n(s)\}$ and $\{y_n(t)\}$ are the iterative sequences generated by Algorithm 3.1.

Proof. In the sequel, from Lemma 3.3, we first define mappings $\Phi : \Lambda \times \Omega \times E \times E \to E$ and $\Psi : \Lambda \times \Omega \times E \times E \to E$ as follows

$$\begin{aligned} \Phi(s,t,x,v) &= x(s) - p_s(x) \\ &\quad + J_{\eta_{1s},M_{1s}}^{\lambda_1(s),A_{1s}}[A_{1s}(p_s(x)) + y(t) - x(s) - \lambda_1(s)(N_{1s}(F_t(y),v) - f(s))], \\ \Psi(s,t,u,y) &= J_{\eta_{2t},M_{2t}}^{\lambda_2(t),A_{2t}}[A_{2t}(y) + x(s) - y(t) - \lambda_2(t)(N_{2t}(G_s(x),u) - g(t))] \end{aligned} \tag{4.2}$$

for all $(s,t,x,y) \in \Lambda \times \Omega \times E \times E$.

Now define $\|\cdot\|_*$ on $E \times E$ by

$$\|(x,y)\|_* = \|x\| + \|y\|, \quad \forall (x,y) \in E \times E.$$

It is easy to see that $(E \times E, \|\cdot\|_*)$ is a Banach space (see [10]). By (4.2), for any given $\lambda_1(s) > 0$ and $\lambda_2(t) > 0$, define $P : \Lambda \times \Omega \times E \times E \to 2^E \times 2^E$ by

$$\begin{aligned} P(s,t,x,y) = \{(\Phi(s,t,x,v), \Psi(s,t,u,y)) : \quad & \forall u \in S_s(x), v \in T_t(y) \text{ and} \\ & (s,t,x,y) \in \Lambda \times \Omega \times E \times E\}. \end{aligned}$$

For any $(s,t,x,y) \in \Lambda \times \Omega \times E \times E$, since $S_s(x) \in CB(E)$, $T_t(y) \in CB(E)$, p, A_1, A_2, η_1, η_2, N_1, N_2, $J_{\eta_{1s},M_{1s}}^{\lambda_1(s),A_{1s}}$, $J_{\eta_{2t},M_{2t}}^{\lambda_2(t),A_{2t}}$ are continuous, we have $P(s,t,x,y) \in CB(E \times E)$. Now for each fixed $(s,t,x,y) \in \Lambda \times \Omega \times E \times E$, we prove that $P(s,t,x,y)$ is a multi-valued contractive mapping.

In fact, for any (s,t,x,y), $(s,t,\hat{x},\hat{y}) \in \Lambda \times \Omega \times E \times E$ and any $(a_1,a_2) \in P(s,t,x,y)$, there exist $u \in S_s(x)$, $v \in T_t(y)$ such that

$$\begin{aligned} a_1 &= x(s) - p_s(x) \\ &\quad + J_{\eta_{1s},M_{1s}}^{\lambda_1(s),A_{1s}}[A_{1s}(p_s(x)) + y(t) - x(s) - \lambda_1(s)(N_{1s}(F_t(y),v) - f(s))], \\ a_2 &= J_{\eta_{2t},M_{2t}}^{\lambda_2(t),A_{2t}}[A_{2t}(y) + x(s) - y(t) - \lambda_2(t)(N_{2t}(G_s(x),u) - g(t))]. \end{aligned}$$

Note that $S_s(\hat{x}) \in CB(E)$, $T_t(\hat{y}) \in CB(E)$, it follows from Nadler's result [30] that there exist $\hat{u} \in S_s(\hat{x})$ and $\hat{v} \in T_t(\hat{y})$ such that

$$\|u-\hat{u}\| \leq \hat{\mathbf{H}}(S_s(x), S_s(\hat{x})), \quad \|v-\hat{v}\| \leq \hat{\mathbf{H}}(T_t(y), T_t(\hat{y})). \tag{4.3}$$

Setting

$$\begin{aligned} b_1 \quad &= \hat{x}(s) - p_s(\hat{x}) \\ &\quad + J^{\lambda_1(s), A_{1s}}_{\eta_{1s}, M_{1s}}[A_{1s}(p_s(\hat{x})) + \hat{y}(t) - \hat{x}(s) - \lambda_1(s)(N_{1s}(F_t(\hat{y}), \hat{v}) - f(s))], \\ b_2 \quad &= J^{\lambda_2(t), A_{2t}}_{\eta_{2t}, M_{2t}}[A_{2t}(\hat{y}) + \hat{x}(s) - \hat{y}(t) - \lambda_2(t)(N_{2t}(G_s(\hat{x}), \hat{u}) - g(t))], \end{aligned}$$

we have $(b_1, b_2) \in P(s, t, \hat{x}, \hat{y})$. It follows from Lemma 2.2 that

$$\begin{aligned} &\|a_1 - b_1\| \\ &\leq \|x(s) - \hat{x}(s) - (p_s(x) - p_s(\hat{x}))\| \\ &\quad + \| J^{\lambda_1(s), A_{1s}}_{\eta_{1s}, M_{1s}}[A_{1s}(p_s(x)) + y(t) - x(s) - \lambda_1(s)(N_{1s}(F_t(y), v) - f(s))] \\ &\qquad - J^{\lambda_1(s), A_{1s}}_{\eta_{1s}, M_{1s}}[A_{1s}(p_s(\hat{x})) + \hat{y}(t) - \hat{x}(s) - \lambda_1(s)(N_{1s}(F_t(\hat{y}), \hat{v}) - f(s))]\}\| \\ &\leq \|x(s) - \hat{x}(s) - (p_s(x) - p_s(\hat{x}))\| \\ &\quad + \frac{\tau_1^{q-1}(s)}{r_1(s) - \lambda_1(s) m_1(s)} \{\|x(s) - \hat{x}(s) - [A_{1s}(p_s(x)) - A_{1s}(p_s(\hat{x}))]\| \\ &\qquad + \|y(t) - \hat{y}(t) - \lambda_1(s)[N_{1s}(F_t(y), v) - N_{1s}(F_t(\hat{y}), v)]\| \\ &\qquad + \lambda_1(s)\|N_{1s}(F_t(\hat{y}), v) - N_{1s}(F_t(\hat{y}), \hat{v})\|\}. \end{aligned} \tag{4.4}$$

By the assumptions on p, N_1, A_1, F, T and (4.3), we have

$$\|x(s) - \hat{x}(s) - (p_s(x) - p_s(\hat{x}))\|^q \leq (1 - q\delta(s) + c_q \sigma^q(s))\|x(s) - \hat{x}(s)\|^q, \tag{4.5}$$

$$\begin{aligned} &\|x(s) - \hat{x}(s) - [A_{1s}(p_s(x)) - A_{1s}(p_s(\hat{x}))]\|^q \\ &\leq (1 - q\varpi(s) + c_q \varsigma^q(s))\|x(s) - \hat{x}(s)\|^q, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\|y(t) - \hat{y}(t) - \lambda_1(s)[N_{1s}(F_t(y), v) - N_{1s}(F_t(\hat{y}), v)]\|^q \\ &\leq \|y(t) - \hat{y}(t)\|^q + c_q \lambda_1^q(s)\|N_{1s}(F_t(y), v) - N_{1s}(F_t(\hat{y}), v)\|^q \\ &\quad - q\lambda_1(s)\langle N_{1s}(F_t(y), v) - N_{1s}(F_t(\hat{y}), v), j_q(y(t) - \hat{y}(t))\rangle \\ &\leq (1 - q\lambda_1(s)\alpha_1(t) + c_q \lambda_1^q(s)\mu_1^q(s)\xi_2^q(t) + q\lambda_1(s)\gamma_1^q(s)\xi_2^q(t))\|y(t) - \hat{y}(t)\|^q, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\|N_{1s}(F_t(\hat{y}), v) - N_{1s}(F_t(\hat{y}), \hat{v})\| \\ &\leq \beta_2(s)\|v - \hat{v}\| \leq \beta_2(s)\hat{\mathbf{H}}(T_t(y), T_t(\hat{y})) \leq \beta_1(s)\kappa_2(t)\|y(t) - \hat{y}(t)\|, \end{aligned} \tag{4.8}$$

where c_q is the constants as in Lemma 2.1. Combining (4.5)-(4.8) with (4.4), we infer

$$\|a_1 - b_1\| \leq \theta_1\|x(s) - \hat{x}(s)\| + \vartheta_1\|y(t) - \hat{y}(t)\|, \tag{4.9}$$

where

$$\theta_1 = \sqrt[q]{1 - q\delta(s) + c_q \sigma^q(s)} + \frac{\tau_1^{q-1}(s)\sqrt[q]{1 - q\varpi(s) + c_q \varsigma^q(s)}}{r_1(s) - \lambda_1(s) m_1(s)},$$

$$\begin{aligned} \vartheta_1 \quad &= \frac{\tau_1^{q-1}(s)}{r_1(s) - \lambda_1(s) m_1(s)}[\lambda_1(s)\beta_1(s)\kappa_2(t) \\ &\quad + \sqrt[q]{1 - q\lambda_1(s)\alpha_1(t) + c_q \lambda_1^q(s)\mu_1^q(s)\xi_2^q(t) + q\lambda_1(s)\gamma_1^q(s)\xi_2^q(t)}]. \end{aligned}$$

On the other hand, by the assumptions of S, A_2, N_2, G and (3.6), we can obtain

$$\begin{aligned}
&\|y(t)-\hat{y}(t)-[A_{2t}(y)-A_{2t}(\hat{y})]\|^q \le (1-qr_2(t)+c_q\pi^q(t))\|y(t)-\hat{y}(t)\|^q,\\
&\|x(s)-\hat{x}(s)-\lambda_2(t)(N_{2t}(G_s(x),u)-N_{2t}(G_s(\hat{x}),u))\|^q\\
&\le (1-q\lambda_2(t)\alpha_2(s)+c_q\lambda_2^q(t)\mu_2^q(t)\xi_1^q(s)+q\lambda_2(t)\gamma_2^q(t)\xi_1^q(s))\|x(t)-\hat{x}(t)\|^q,\\
&\|N_{2t}(G_s(\hat{x}),u)-N_{2t}(G_s(\hat{x}),\hat{u})\| \le \beta_2(t)\kappa_1(s)\|x(t)-\hat{x}(t)\|,
\end{aligned}$$

and

$$\begin{aligned}
\|a_2-b_2\| \quad &\le \frac{\tau_2^{q-1}(t)}{r_2(t)-\lambda_2(t)m_2(t)}\{\|y(t)-\hat{y}(t)-[A_{2t}(y)-A_{2t}(\hat{y})]\|\\
&\quad +\|x(s)-\hat{x}(s)-\lambda_2(t)(N_{2t}(G_s(x),u)-N_{2t}(G_s(\hat{x}),u))\|\\
&\quad +\lambda_2(t)\|N_{2t}(G_s(\hat{x}),u)-N_{2t}(G_s(\hat{x}),\hat{u})\|\}\\
&\le \theta_2\|x(s)-\hat{x}(s)\|+\vartheta_2\|y(t)-\hat{y}(t)\|, \qquad (4.10)
\end{aligned}$$

where

$$\begin{aligned}
\theta_2 \quad &= \frac{\tau_2^{q-1}(t)}{r_2(t)-\lambda_2(t)m_2(t)}[\lambda_2(t)\kappa_1(s)\beta_2(t)\\
&\quad +\sqrt[q]{1-q\lambda_2(t)\alpha_2(s)+c_q\lambda_2^q(t)\mu_2^q(t)\xi_1^q(s)+q\lambda_2(t)\gamma_2^q(t)\xi_1^q(s)}\,],
\end{aligned}$$

$$\vartheta_2=\frac{\tau_2^{q-1}(t)\sqrt[q]{1-qr_2(t)+c_q\pi^q(t)}}{r_2(t)-\lambda_2(t)m_2(t)}.$$

It follows from (4.9) and (4.10) that

$$\|a_1-b_1\|+\|a_2-b_2\| \le \upsilon(\|x(s)-\hat{x}(s)\|+\|y(t)-\hat{y}(t)\|), \qquad (4.11)$$

where

$$\upsilon=\max\{\theta_1+\theta_2,\vartheta_1+\vartheta_2\}.$$

It follows from condition (4.1) that $\upsilon<1$. Hence, from (4.11), we get

$$\begin{aligned}
d((a_1,a_2),P(s,t,\hat{x},\hat{y})) \quad &= \inf_{(b_1,b_2)\in P(s,t,\hat{x},\hat{y})}(\|a_1-b_1\|+\|a_2-b_2\|)\\
&\le \upsilon\|(x(s),y(t))-(\hat{x}(s)-\hat{y}(t))\|_*.
\end{aligned}$$

Since $(a_1,a_2)\in P(s,t,x,y)$ is arbitrary, we obtain

$$\sup_{(a_1,a_2)\in P(s,t,x,y)} d((a_1,a_2),P(s,t,\hat{x},\hat{y})) \le \upsilon\|(x(s),y(t))-(\hat{x}(s)-\hat{y}(t))\|_*.$$

By using the same argument, we can prove

$$\sup_{(b_1,b_2)\in P(s,t,\hat{x},\hat{y})} d(P(s,t,x,y),(b_1,b_2)) \le \upsilon\|(x(s),y(t))-(\hat{x}(s)-\hat{y}(t))\|_*.$$

It follows from the definition of the Hausdorff metric $\hat{\mathbf{H}}$ on $CB(E\times E)$ that

$$\hat{\mathbf{H}}(P(s,t,x,y),P(s,t,\hat{x},\hat{y})) \le \upsilon\|(x(s),y(t))-(\hat{x}(s)-\hat{y}(t))\|_*$$

for all $(s,x,\hat{x}) \in \Lambda\times E\times E$, $(t,y,\hat{y}) \in \Omega\times E\times E$, i.e., $P(s,t,x,y)$ is a multi-valued contractive mapping, which is uniform with respect to $(s,t) \in \Lambda\times\Omega$. By a fixed point theorem of Nadler [30], for each $(s,t) \in \Lambda\times\Omega$, $P(s,t,x,y)$ has a fixed point $(x^*(s), y^*(t)) \in E\times E$, i.e., $(x^*(s), y^*(t)) \in P(s,t,x^*(s),y^*(t))$. By the definition of P, we know that there exist $u^*(s) \in S_s(x^*)$ and $v^*(t) \in T_t(y^*)$ such that (4.2) holds. Thus, it follows from Lemma 3.3 that $(x^*(s), y^*(t)) \in Q(s,t)$ is a solution of the problem (1.1) and so $Q(s,t) \neq \emptyset$ for all $(s,t) \in \Lambda\times\Omega$.

Next, we prove the conclusion (2). Let $(x^*(s), y^*(t)) \in Q(s,t)$ is a solution of the problem (1.1). It follows from Lemma 3.3 that

$$\begin{aligned} &p_s(x^*) \in J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}}[A_{1s}(p_s(x^*)) + y^*(t) - x^*(s) - \lambda_1(s)(N_{1s}(F_t(y^*),T_t(y^*)) - f(s))],\\ &y^*(t) \in J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}}[A_{2t}(y^*) + x^*(s) - y^*(t) - \lambda_2(t)(N_{2t}(G_s(x^*),S_s(x^*)) - g(t))],\end{aligned}$$

i.e.,

$$\begin{cases} p_s(x^*) = J^{\lambda_1(s),A_{1s}}_{\eta_{1s},M_{1s}}(z^*), \quad y^*(t) = J^{\lambda_2(t),A_{2t}}_{\eta_{2t},M_{2t}}(w^*),\\ z^*(s) = A_{1s}(p_s(x^*)) + y^*(t) - x^*(s) - \lambda_1(s)(N_{1s}(F_t(y^*),v^*) - f(s)),\\ \quad \forall v^* \in T_t(y^*),\\ w^*(t) = A_{2t}(y^*) + x^*(s) - y^*(t) - \lambda_2(t)(N_{2t}(G_s(x^*),u^*) - g(t)),\\ \quad \forall u^* \in S_s(x^*). \end{cases} \tag{4.12}$$

Since $S_s(x^*)$, $S_s(x_n)$, $T_t(y^*)$, $T_n(y_n) \in CB(E)$ for all $n \geq 0$, for any given $n \geq 0$ and $\varepsilon > 0$, it follows from Nadler [30] that there exist $u_n \in S_s(x_n)$, $v_n \in T_t(y_n)$ such that

$$\|u_n - u^*\| \leq (1+\varepsilon)\hat{\mathbf{H}}(S_s(x_n), S_s(x^*)), \quad \|v_n - v^*\| \leq (1+\varepsilon)\hat{\mathbf{H}}(T_t(y_n), T_t(y^*)).$$

Thus, from (3.1), (4.12) and the proofs of (4.9) and (4.10), for all $v_n \in T_t(y_n)$ and $v^* \in T_t(y^*)$, we have

$$\begin{aligned} &\|z_{n+1}(s) - z^*(s)\|\\ &\leq (1-\alpha_n)\|z_n(s) - z^*(s)\| + \alpha_n(\|d'_n(s)\| + \|d''_n(s)\|) + \|e_n(s)\|\\ &\quad +\alpha_n\|x_n(s) - x^*(s) - (A_{1s}(p_s(x_n)) - A_{1s}(p_s(x^*)))\|\\ &\quad +\alpha_n\|y_n(t) - y^*(t) - \lambda_1(s)(N_{1s}(F_t(y_n),v_n) - N_{1s}(F_t(y^*),v_n))\|\\ &\quad +\alpha_n\lambda_1(s)\|N_{1s}(F_t(y^*),v_n) - N_{1s}(F_t(y^*),v^*)\|\\ &\leq (1-\alpha_n)\|z_n(s) - z^*(s)\| + \alpha_n\|d'_n(s)\| + (\|d''_n(s)\| + \|e_n(s)\|)\\ &\quad +\alpha_n\sqrt[q]{1 - q\varpi(s) + c_q\varsigma^q(s)}\|x_n(s) - x^*(s)\|\\ &\quad +\alpha_n[\sqrt[q]{1 - q\lambda_1(s)\alpha_1(t) + c_q\lambda_1^q(s)\mu_1^q(s)\xi_2^q(t) + q\lambda_1(s)\mu_1^q(s)\xi_2^q(t)}\\ &\qquad +\lambda_1(s)\beta_1(s)\kappa_2(t)(1+\varepsilon)]\|y_n(t) - y^*(t)\| \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &\|w_{n+1}(t) - w^*(t)\|\\ &\leq (1-\alpha_n)\|w_n(t) - w^*(t)\|\\ &\quad +\alpha_n\|y_n(t) - y^*(t) - (A_{2t}(y_n) - A_{2t}(y^*))\|\\ &\quad +\alpha_n\|x_n(s) - x^*(s) - \lambda_2(t)(N_{2t}(G_s(x_n),u_n) - N_{2t}(G_s(x^*),u_n))\| \end{aligned}$$

$$
\begin{aligned}
&+\alpha_n\lambda_2(t)\|N_{2t}(G_s(x^*),u_n)-N_{2t}(G_s(x^*),u^*)\|+\|h(t)\| \\
&\le (1-\alpha_n)\|w_n(t)-w^*(t)\| \\
&\quad+\alpha_n\sqrt[q]{1-qr_2(t)+c_q\pi^q(t)}\|y_n(t)-y^*(t)\| \\
&\quad+\alpha_n[\sqrt[q]{1-q\lambda_2(t)\alpha_2(s)+c_q\lambda_2^q(t)\mu_2^q(t)\xi_1^q(s)+q\lambda_2(t)\mu_2^q(t)\xi_1^q(s)} \\
&\qquad+\lambda_2(t)\beta_2(t)\kappa_1(s)(1+\varepsilon)]\|x_n(s)-x^*(s)\|+\|h(t)\|. \qquad (4.14)
\end{aligned}
$$

On the other hand, by Lemma 2.2 and (4.5) we know that

$$
\begin{aligned}
\|y_n(t)-y^*(t)\| &\le \|J_{\eta_{2t},M_{2t}}^{\lambda_2(t),A_{2t}}(w_n)-J_{\eta_{2t},M_{2t}}^{\lambda_2(t),A_{2t}}(w^*)\| \\
&\le \frac{\tau_2^{q-1}(t)}{r_2(t)-\lambda_2(t)m_2(t)}\|w_n(t)-w^*(t)\| \qquad (4.15)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|x_n(s)-x^*(s)\| \\
&\le \|x_n(s)-x^*(s)-(p_s(x_n)-p_s(x^*))\|+\|J_{\eta_{1s},M_{1s}}^{\lambda_1(s),A_{1s}}(z_n)-J_{\eta_{1s},M_{1s}}^{\lambda_1(s),A_{1s}}(z^*)\| \\
&\le \sqrt[q]{1-q\delta(s)+c_q\sigma^q(s)}\|x_n(s)-x^*(s)\|+\frac{\tau_1^{q-1}(s)}{r_1(s)-\lambda_1(s)m_1(s)}\|z_n(s)-z^*(s)\|,
\end{aligned}
$$

which implies that

$$
\|x_n(s)-x^*(s)\| \le \frac{\frac{\tau_1^{q-1}(s)}{r_1(s)-\lambda_1(s)m_1(s)}}{1-\sqrt[q]{1-q\delta(s)+c_q\sigma^q(s)}}\|z_n(s)-z^*(s)\|. \qquad (4.16)
$$

Combining (4.13), (4.14) with (4.15) and (4.16), we get

$$
\begin{aligned}
&\|z_{n+1}(s)-z^*(s)\|+\|w_{n+1}(t)-w^*(t)\| \\
&\le [1-\alpha_n+\alpha_n\iota(\varepsilon)](\|z_n(s)-z^*(s)\|+\|w_n(t)-w^*(t)\|) \\
&\quad+\alpha_n\|d_n'(s)\|+(\|d_n''(s)\|+\|e_n(s)\|+\|h(t)\|), \qquad (4.17)
\end{aligned}
$$

where $\iota(\varepsilon)=\max\{\theta(\varepsilon),\vartheta(\varepsilon)\}$,

$$
\begin{aligned}
\theta(\varepsilon) &= \frac{\frac{\tau_1^{q-1}(s)}{r_1(s)-\lambda_1(s)m_1(s)}}{1-\sqrt[q]{1-q\delta(s)+c_q\sigma^q(s)}}\{\sqrt[q]{1-q\varpi(s)+c_q\varsigma^q(s)}+\lambda_2(t)\beta_2(t)\kappa_1(s)(1+\varepsilon) \\
&\quad+\sqrt[q]{1-q\lambda_2(t)\alpha_2(s)+c_q\lambda_2^q(t)\mu_2^q(t)\xi_1^q(s)+q\lambda_2(t)\gamma_2^q(t)\xi_1^q(s)}\},
\end{aligned}
$$

$$
\begin{aligned}
\vartheta(\varepsilon) &= \frac{\tau_2^{q-1}(t)}{r_2(t)-\lambda_2(t)m_2(t)}\{\sqrt[q]{1-qr_2(t)+c_q\pi^q(t)}+\lambda_1(s)\beta_1(s)\kappa_2(t)(1+\varepsilon) \\
&\quad+\sqrt[q]{1-q\lambda_1(s)\alpha_1(t)+c_q\lambda_1^q(s)\mu_1^q(s)\xi_2^q(t)+q\lambda_1(s)\gamma_1^q(s)\xi_2^q(t)}\}
\end{aligned}
$$

Let $\varepsilon \to 0$. Then we have $\theta(\varepsilon) \to \theta$, $\vartheta(\varepsilon) \to \vartheta$ and $\iota(\varepsilon) \to \iota$, where $\iota = \max\{\theta, \vartheta\}$ and

$$\begin{aligned}\theta &= \frac{\frac{\tau_1^{q-1}(s)}{r_1(s)-\lambda_1(s)m_1(s)}}{1-\sqrt[q]{1-q\delta(s)+c_q\sigma^q(s)}}\{\sqrt[q]{1-q\varpi(s)+c_q\varsigma^q(s)}+\lambda_2(t)\beta_2(t)\kappa_1(s)\\ &\quad +\sqrt[q]{1-q\lambda_2(t)\alpha_2(s)+c_q\lambda_2^q(t)\mu_2^q(t)\xi_1^q(s)+q\lambda_2(t)\gamma_2^q(t)\xi_1^q(s)}\},\end{aligned}$$

$$\begin{aligned}\vartheta &= \frac{\tau_2^{q-1}(t)}{r_2(t)-\lambda_2(t)m_2(t)}\{\sqrt[q]{1-qr_2(t)+c_q\pi^q(t)}+\lambda_1(s)\beta_1(s)\kappa_2(t)\\ &\quad +\sqrt[q]{1-q\lambda_1(s)\alpha_1(t)+c_q\lambda_1^q(s)\mu_1^q(s)\xi_2^q(t)+q\lambda_1(s)\gamma_1^q(s)\xi_2^q(t)}\}\end{aligned}$$

Since $0 < \iota < 1$, $1-\iota > 0$, it follows from (4.17) that

$$\begin{aligned}&\|z_{n+1}(s)-z^*(s)\|+\|w_{n+1}(t)-w^*(t)\|\\ &\le [1-\alpha_n(1-\iota)](\|z_n(s)-z^*(s)\|+\|w_n(t)-w^*(t)\|)\\ &\quad +\alpha_n(1-\iota)\cdot\frac{1}{(1-\iota)}\|d'_n(s)\|+(\|d''_n(s)\|+\|e_n(s)\|+\|h(t)\|). \qquad (4.18)\end{aligned}$$

Since $\sum\limits_{n=0}^{\infty}\alpha_n = \infty$, it follows from Lemma 4.1 and (4.18) that

$$\|(z_n(s), w_n(t)) - (z^*(s), w^*(t))\|_* \to 0 \quad \text{as} \quad n \to \infty,$$

i.e., $z_n(s) \to z^*(s)$ for all $s \in \Lambda$ and $w_n(t) \to w^*(t)$ for all $w \in \Omega$. Hence, by (4.15) and (4.16), we know that the sequence $\{x_n(s)\}$ converges to $x^*(s)$ for all $s \in \Lambda$ and the sequence $\{y_n(t)\}$ converges to $y^*(t)$ for all $t \in \Omega$. This completes the proof.

Remark 4.1. We note that Hilbert space and L_p (or l_p) $(2 \le p < \infty)$ spaces are 2-uniformly smooth Banach spaces. Further, in Theorem 4.1, if N_i is strongly accretive in the second variable, i.e., when $\gamma_i = 0$ $(i = 1, 2)$ in Theorem 4.1, then we can obtain the corresponding results. Our results improve and generale the known results in [7, 10, 11, 18, 26, 28, 34, 36, 38]

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,431-440,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Topological Dynamic Classification of Antitriangular Maps[1]

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[**Abstract**]:We classify antitriangular maps on I^n by using topological dynamics. More precisely, we prove that the following properties are equivalent: (1) zero topological entropy; (2) $UR(F) = R(F)$; (3) type less than or equal to 2^∞; and (4) $AP(F) = \{(x_1, x_2, \cdots, x_n) \in I^n : \lim F^{2^n}(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_n)\}$. These result allow us to decide when the behavior of these n-dimensional dynamical systems is complicated.

Keywords: zero topological entropy; uniformly recurrent points; recurrent points; almost periodic points

AMS(2000) Subject Classification: 58F10, 54H20

§1 Introduction

we consider n-dimensional maps $F : I^n \to I^n$ with

$$F(x_1, x_2, \cdots, x_n) = (f_n(x_n), f_{n-1}(x_{n-1}), \cdots, f_1(x_1))$$

where $(x_1, x_2, \cdots, x_n) \in I^n = [0, 1]\times[0, 1]\times\cdots\times[0, 1]$ and $f_i : I \to I, i = 1, 2, \cdots, n$ are continuous maps. When $n = 2$, these maps have been proposed to give a mathematical description of a competitive production process called Cournot duopoly. And then, we say that F is a Cournot map on I^2, f_1 and f_2 are called reaction maps. The Cournot duopoly has been studied in literatures [7], [8], [10], [13], [15] and [16]. In these papers, when the behavior of these two-dimensional dynamical systems is complicated is studied. In our paper, we studied when the behavior of these n-dimensional dynamical systems is complicated.

It is well known that positive topological entropy implies a complicated dynamical behavior for continuous maps defined on I. More precisely, when $f : I \to I$ is continuous, the topological entropy of f is positive if and only if f^n has a horseshoe for some $n \in N$. Roughly speaking, this implies the existence of a closed invariant subset $Y \subset I$ holding that $f^n \mid Y$ is conjugated to a shift of k symbols for some $k \in N$(see[17]or[2]). Additionally, $h(f) = 0$ implies that$f$ has a restricted type of periodic orbits and simple topological dynamics. Let $UR(f), R(f)$and $AP(f)$ denote the sets of all uniformly recurrent points, all recurrent points and all almost periodic points, resp.

[1]This work supported by the National Science Foundation of China (10661001), the Postdoctor Scientific Foundation of Central South University and the Foundation of GuangXi University (X061022);
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In [17], the authors proved the following Theorem 1.1:

Theorem 1.1 Let $f : I \to I$ be continuous , the following properties are equivalent:

(a) $h(f) = 0$,

(b) the period of any periodic point is a power of two ,

(c) $UR(f) = R(f)$,

(d) $AP(f) = \{x \in I : \lim_{n\to\infty} f^{2^n}(x) = x\}$.

Usually, Theorem 1.1 is used as a criterion to determine whether an interval map f has a complicated dynamical behavior. Theorem 1.1 could be expected to be extended to more general setting, for instance, to continuous maps defined on I^n. However, Theorem 1.1 fails in general in case of a type of n-dimensional maps, called triangular maps with the form $T(x_1, x_2, \cdots, x_n) = (f_1(x), f_2(x_1, x_2), \cdots, f_n(x_1, x_2, \cdots, x_n))$. For example, two-dimensional triangular maps have the same periodic structure as one-dimensional maps [12]. But there is a T of type 2^∞ such that $h(T) > 0$ (see[13] or [4]). Examples of triangular maps holding h(T)=0, type 2^∞ but $UR(f) \neq R(f)$ can be found in [15].

Peiodic structure of antitriangular maps and interval maps are quite similar but not exactly the same [5]. However, the dynamics of n-dimensional antitriangular maps are more closely related to one-dimensional dynamics than triangular maps. In this paper, we proved the following theorem 4.1, which generalizes Theorem 1.1 to n-dimensional antitriangular maps.

Theorem 4.1 Let

$$F(x_1, x_2, \cdots, x_n) = (f_n(x_n), f_{n-1}(x_{n-1}), \cdots f_1(x_1))$$

an antiargular ,Then the following properties are equivalent :

(a)h(F)=0,

(b)The period of any periodic point of F is a power of 2 ,

(c)UR(F)=R(F),

(d)$AP(F) = \{(x_1, x_2, \cdots, x_n) \in I^n : \lim F^{2^s}(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_n)\}$.

§2 Preliminaries

Let $\varphi : X \to X$ be a continuous map on a compact metric space X. φ^0 means the identity map , $\varphi^1 = \varphi$ and $\varphi^{n+1} = \varphi \circ \varphi^n, n \geq 0$, where $\circ$ denotes the composition of maps, $Orb_\varphi(z) = \{\varphi^n(z)\}_{n=0}^\infty$ will denotes the orbit of $z \in X$. $z \in X$ is said to be periodic if $\varphi^n(z) = z$ for some $n \in N$. The smallest positive integer satisfying this condition is called the order or period of z. Let $Per(\varphi)$ denote the set of periods of φ.

In the case of an interval map φ, we say that it has type 2^∞ if $Per(\varphi) = \{2^n : n \in N\}$. A antitriangular map $F(x_1, x_2, \cdots, x_n) = (f_n(x_n), f_{n-1}(x_{n-1}), \cdots, f_1(x_1)$ has type 2^∞ if $f_1 \circ f_n, f_2 \circ f_{n-1}, \cdots, f_n \circ f_1$ have type 2^∞.

The set of periodic points is denoted by $P(\varphi)$. $AP(\varphi)$ denotes the set of almost periodic points, that is, those point $x \in X$ such that for any neighborhood $V = V(x)$ of x, there exists an $N = N(V) \in N$ such that $\varphi^{kN}(x) \in V$, for every $k \geq 0$. $UR(\varphi)$ denotes the set of uniformly recurrent points, that is, those points $x \in X$ such that for any neighborhood $V = V(x)$ of x, there exists $N = N(V) \in N$ such that $\{\varphi^r(x), \varphi^{r+1}(x), \cdots, \varphi^{r+N-1}(x)\} \cap V \neq \emptyset$ for all $r > 0$. Finally, the ω-limit set of φ at the point x is denoted by $w(x, \varphi)$, that is, those points $y \in X$ such that there exists a sequence $(n_i)_{i=1}^{\infty}$ such that $\varphi^{n_i}(x) \to y$ as $n_i \to \infty$. $R(\varphi) = \{x \in X : x \in w(x, \varphi)\}$ is the set of recurrent points. It is well known that (see [17])

$$P(\varphi) \subseteq AP(\varphi) \subseteq UR(\varphi) \subseteq R(\varphi)$$

The definition of topological entropy of $\varphi, h(\varphi)$ can be seen in [1,2] or [19]. For more information on these topics see for instance [17].

Let $f : I \to I$ be a continuous map of zero topological entropy. In order to generalize Theorem 1.1 to n-dimensional antitriangular maps, we need some additional information concerning the structure of infinite ω-limit sets of continuous interval maps with zero topological entropy. In [18], we can see that for any infinite $\omega(x, f)$ there is a sequence of compact intervals

$$J_0 \supset J_1 \supset J_2 \supset \cdots J_i \supset J_{i+1} \supset \cdots$$

such that for each $k \in N$ the following statements hold :

(a) $\{f^i(J_k)\}_{i=1}^{2^k}$ are pairwise disjoint and $f^{2^k}(J_k) = J_k$.

(b) $J_{k+1} \bigcup f^{2^k}(J_{k+1}) \subset J_k$

(c) $w(x, f) \subset \bigcup_{i=0}^{2^k-1} f^i(J_k)$. In particular , $w(x, f) \subset \bigcap_{k=0}^{\infty} \bigcup_{i=0}^{2^k-1} f^i(J_k)$

(d) For each $i \in N, w(x, f) \bigcap f^i(J_k) \neq \emptyset$.

In order to write easily the sets $f^j(J_i)$ for $i \in N$ and $0 \leq j < 2^j$, we consider the sets $X_i = \{0, 1\}^i$ whose elements are finite sequences of i elements composed of 0's and 1's . On each X_i, the adding machine transformation $A : X_i \to X_i$ is defined by $A(\theta_1, \theta_2, \cdots, \theta_i) := (\theta_1, \theta_2, \cdots, \theta_i) + (1, 0, \cdots, 0)$ for all $(\theta_1, \theta_2, \cdots, \theta_i) \in X_i$, where the addition is modulo 2 from the left to the right. For example, $A(1, 1, 0, 0, 1, \cdots, 1) = (0, 0, 1, 0, 1, \cdots, 1)$. The map A can extend to infinite sequences as follows . Let $X_\infty = \{0, 1\}^\infty$, for any $\alpha = (\alpha_i)_{i=1}^{\infty} \in X_\infty$,define $A(\alpha) = \alpha + (1, \mathbf{0})$,where $\mathbf{0}= (0, 0, \cdots)$. Notice that if $\mathbf{1}= (1, 1, \cdots, 1)$, then $A(\mathbf{1}) = \mathbf{0}$. For $\alpha \in X_\infty$ and $i \in N$, define $\alpha \mid_i= (\alpha_1, \alpha_2, \cdots, \alpha_i) \in X_i$ by taking the first i elements of the sequence α. For $\theta \in X_i$ and $\upsilon \in X_j$, let

$$\theta * \upsilon = (\theta_1, \theta_2, \cdots, \theta_i, \upsilon_1, \upsilon_2, \cdots, \upsilon_j) \in X_{i+j}.$$

If $\alpha \in X_\infty$ and $\theta \in X_i$, we similarly define $\theta * \alpha \in X_\infty$. Let $\bar{0} = 1$ and $\bar{1} = 0$ and in a similar way $\bar{\alpha} \in X_\infty$ and $\bar{\theta} \in X_i$ are defined . For a chosen $i \in N$, let $K_{\mathbf{0}|_\mathbf{i}} = J_i$, and for $0 \leq j \leq 2^i$.

$K_{A^j(\mathbf{0}|_\mathbf{i})} = f^j(K_{\mathbf{0}|_\mathbf{i}}) = f^j(J_i)$. Notice that for all $\alpha \in X_\infty, \alpha|_i = A^j(\mathbf{0}|_\mathbf{i})$ for each $i \in N$ and some $0 \leq j \leq 2^i$. Also ,$A^{2^i}(\mathbf{0} \mid_\mathbf{i}) = \mathbf{0} \mid_\mathbf{i}$. Hence the sequence of compact intervals $(K_{\alpha|_i})_{i=1}^{\infty}$

decreases to a compact (possibly degenerate) interval K_α with all the the previous notation, conditions (a)-(d)can be rewritten as follows ([8]).

Theorem 2.1 Let $f : I \to I$ be continuous with $h(f) = 0$ and $\omega(x, f)$ be an infinite $\omega-$limit set for some $x \in I$, then there exists a sequence of compact intervals

$$J_0 \supset K_{(0)} \supset K_{(0,0)} \supset \cdots \supset K_{\mathbf{0}|_\mathbf{i}} \supset \cdots$$

such that for each $k \in N$:

(a) $\{f^j(K_{\mathbf{0}|_\mathbf{k}})\}_{j=1}^{2^k}$ are pairwise disjoint and $f^{2^k}(K_{\mathbf{0}|_\mathbf{k}}) = K_{\mathbf{0}|_\mathbf{k}}$.

(b)For each $\theta \in X_k, K_{\theta*0} \bigcup K_{\theta*1} \subset K_\theta$.

(c)$w(x, f) \subset \bigcup_{\theta \in X_k} K_\theta$. In particular, $w(x, f) \subset \bigcap_{k=0}^{\infty} \bigcup_{\theta \in X_k} = \bigcup_{\alpha \in X_\infty} K_\alpha$.

(d)For each $\theta \in X_k, w(x, f) \bigcap K_\theta \neq \emptyset$.

(e)If K_α is non-degenerate ($| K_\alpha |> 0$) for some $\alpha \in X_\infty$, then K_α is a wandering interval (that is, $f^j(K_\alpha) \bigcap f^i(K_\alpha) = \emptyset$, for all $0 \leq i < j$) .

For any pair of disjoint interval $J, K \subseteq I$, Write $dist(J, K) = \inf\{d(x, y) : x \in J, y \in K\}$. Int(J) denotes the interior of J.

§3 Auxiliary results

Let $f : I \to I$ be continuous interval maps with $h(f) = 0$. Let

$$\omega(x, f) \subset \bigcap_{k=0}^{\infty} \bigcup_{\theta \in X_\infty} K_\theta = \bigcup_{\alpha \in X_\infty} K_\alpha$$

be an infinite w-limit set of f. For all $\alpha \in X_\infty$ and $k \in N$, let $K^+_{\alpha|_k}$ and $K^-_{\alpha|_k}$ be the right and left-side components of $K_{\alpha|_k} \setminus K_\alpha$. Then

Proposition 3.1[8] Under the above assumption , let $x \in R(f)$ be such that

$$x \in w(x, f) \subset \bigcap_{k=0}^{\infty} \bigcup_{\theta \in X_k} = \bigcup_{\alpha \in X_\infty} K_\alpha,$$

then there exists an increasing sequence of positive integers $(n_i)_{i=1}^{\infty}$, with

$$\lim_{i \to \infty} n_i = \infty,$$

such that for all $k \in N$ and $i \geq 1$ one of the following possibilities holds:

(a)If $x \in K_\alpha$ with $| K_\alpha |= 0$, then $f^{k2^{n_i}}(x) \in K_{\alpha|n_i}$,

(b)If $x \in K_\alpha$ with $| K_\alpha |> 0$, then either $f^{2^{n_i}(1+2k)}(x) \in K^+_{\alpha|n_i}$ or $f^{2^{n_i}(1+2k)}(x) \in K^-_{\alpha|n_i}$.

Now, we consider an arbitrary n-dimensional antitriangular map

$$F(x_1, x_2, \cdots, x_n) = (f_n(x_n), f_{n-1}(x_{n-1)}, \cdots, f_1(x_1)),$$

where $f_i : I \to I, i = 1, 2, \cdots, n$ are continuous interval maps. Noticing that for all $k \in N$, it holds that

$$F^{2k}(x_1, x_2, \cdots, x_n) = ((f_n \circ f_1)^k(x_1), (f_{n-1} \circ f_2)^k(x_2), \cdots, (f_1 \circ f_n)^k(x_n)), (3.1)$$

and

$$F^{2k+1}(x_1, x_2, \cdots, x_n) = (f_n \circ (f_1 \circ f_n)^k(x_n), f_{n-1} \circ (f_2 \circ f_{n-1})^k(x_{n-1}), \cdots, f_1 \circ (f_n \circ f_1)^k(x_1)), (3.2)$$

So, the dynamical behavior of F must be closely related to the dynamical behavior of $f_n \circ f_1, f_{n-1} \circ f_2, \cdots, f_1 \circ f_n$.

From the above, we have the following Proposition:

Proposition 3.2 Under the above conditions we have that

(a)$P(F) = P(f_n \circ f_1) \times P(f_{n-1} \circ f_2) \times \cdots \times P(f_1 \circ f_n) = P(F^2)$

(b)$AP(F) = AP(f_n \circ f_1) \times AP(f_{n-1} \circ f_2) \times \cdots \times AP(f_1 \circ f_n) = AP(F^2)$

(c)$UR(F) = UR(F^2) \subset UR(f_n \circ f_1) \times UR(f_{n-1} \circ f_2) \times \cdots \times UR(f_1 \circ f_n)$

(d)$R(F) = R(F^2) \subset R(f_n \circ f_1) \times R(f_{n-1} \circ f_2) \times \cdots \times R(f_1 \circ f_n)$

(c)$h(F) = h(f_n \circ f_1) = h(f_{n-1} \circ f_2) = \cdots = h(f_1 \circ f_n)$

Proof. The proof is similar to [14,10,3] and is omitted.

In order to prove our main result, we must investigate the structure of the sets R(F) and UR(F) for antitriangular maps of zero topological entropy, that is, when $f_n \circ f_1, f_{n-1} \circ f_2, \cdots, f_1 \circ f_n$ have zero topological entropy . we study in this section the relationship among $R(f_n \circ f_1), R(f_{n-1} \circ f_2), \cdots, R(f_1 \circ f_n)$ and $R(F)$.

We start with the following results in which we maintain the notation of Theorem 2.1 and write $K_{\alpha_1}(f_n \circ f_1), K_{\alpha_2}(f_{n-1} \circ f_2), \cdots, K_{\alpha_n}(f_1 \circ f_n)$ to indicate that the system of intervals depends on the compositions $f_n \circ f_1, f_{n-1} \circ f_2, \cdots, f_1 \circ f_n$, respectively.

Proposition 3.3 Let $x_i \in R(f_{n-i+1} \circ f_i), i = 1, 2, \cdots, n$. Assume that $w(x_i, f_{n-i+1} \circ f_i), i = 1, 2, \cdots, n$ are infinite. If $\{x_i\} = K_{\alpha_i}(f_{n-i+1} \circ f_i)$ for some $\alpha_i \in X_\infty, i = 1, 2, \cdots, n$, then $(x_1, x_2, \cdots, x_n) \in R(F) \bigcap UR(F)$.

Proof Let $\varepsilon > 0$ be arbitary , by Theorem 2.1, there must exist an $m \in N$ such that

$$\max_{1 \le i \le n} \{|K_{\alpha_i|m}(f_{n-i+1} \circ f_i)|\} < \varepsilon,$$

then , for any open neighborhood U of $(x_1, x_2, \cdots, x_n)$ there is an $m \in N$ such that

$$K_{\alpha_1|m}(f_n \circ f_1) \times K_{\alpha_2|m}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n|m}(f_1 \circ f_n) \subseteq U$$

Applying (3.1) and Theorem 2.1, we obtain

$$F^{k2^{m+1}}(x_1, x_2, \cdots, x_n) \in K_{\alpha_1|m}(f_n \circ f_1) \times K_{\alpha_2|m}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n|m}(f_1 \circ f_n) \subseteq U$$

for all $k \in N$. It follows that $(x_1, x_2, \cdots, x_n) \in AP(F)$. By (2.1) and Proposition 3.2, $(x_1, x_2, \cdots, x_n) \in AP(F) \subset UR(F) \subset R(F)$, which ends the proof.

Let $\alpha \in X_\infty$ and $K_\alpha = K_\alpha(\varphi) = [z, w]$ be as in Theorem 2.1, where $z, w \in I$ is a continuous interval map of zero topological entropy. Define

$$z(\varphi) = \{n \in N : \varphi^{2^n}(K_\alpha) \subset K_{\alpha|_n * \overline{\alpha_{n+1}}} \subset K^-_{\alpha|_n}\},$$

and

$$w(\varphi) = \{n \in N : \varphi^{2^n}(K_\alpha) \subset K_{\alpha|_n * \overline{\alpha_{n+1}}} \subset K^+_{\alpha|_n}\}.$$

Proposition 3.4 Let $x_i \in K_{\alpha_i}(f_{n-i+1} \circ f_i)$ be such that $| K_{\alpha_i}(f_{n-i+1} \circ f_i) |> 0$, for every $i \in \{1, 2, \cdots, n\}$. Assume that $x_i \in R(f_{n-i+1} \circ f_i)$, for every $i \in \{1, 2, \cdots, n\}$. If $\bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$ is infinite,then $(x_1, x_2, \cdots, x_n) \in R(F) \bigcap UR(F)$.

Proof. Without loss of generality, we suppose that $K_{\alpha_i}(f_{n-i+1} \circ f_i) = [x_i, x_{i_0}]$ for some $x_{i_0} \in I, i \in \{1, 2, \cdots, n\}$.

Since that $\bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$ is infinite, then there exists a sequence of positive integers $(n_j)_{j=1}^\infty$, with $\lim_{i\to\infty} n_j = \infty$ such that $n_j \in \bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$ for all $j \in N$. Hence $(f_{n-i+1} \circ f_i)^{2^{n_j}}(x_i) \in K^-_{\alpha_i|_{n_j}}(f_{n-i+1} \circ f_i)$, for $i = 1, 2, \cdots, n$. On the other hand , applying Theorem 2.1 , for any open neighborhood U of $(x_1, x_2, \cdots, x_n)$ there exists an $n \in N$ such that $\prod_{i=1}^n K^-_{\alpha_i|_m}(f_{n-i+1} \circ f_i) \subseteq U$ for $m \geq n$, In particular, we obtain an $j \in N$ such that $\prod_{i=1}^n K^-_{\alpha_i|_{n_j}}(f_{n-i+1} \circ f_i) \subseteq U$. By using (3.7), we have

$$\begin{aligned} F^{2^{n_j+1}}(x_1, x_2, \cdots, x_n) &= ((f_n \circ f_1)^{2^{n_j}}(x_1), (f_{n-1} \circ f_2)^{2^{n_j}}(x_2), \cdots, (f_1 \circ f_n)^{2^{n_j}}(x_n)) \\ &\in \textstyle\prod_{i=1}^n K^-_{\alpha_i|_{n_j}}(f_{n-i+1} \circ f_i) \subset U; \end{aligned}$$

By Proposition 3.1(b) and (3.1), we have

$$\begin{aligned} &F^{2^{n_j+1}(1+2k)}(x_1, x_2, \cdots, x_n) \in ((f_n \circ f_1)^{2^{n_j}(1+2k)}(K_{\alpha_1}(f_n \circ f_1)), (f_{n-1} \circ f_2)^{2^{n_j}(1+2k)}(K_{\alpha_2}(f_{n-1} \circ f_2)), \cdots, (f_1 \circ f_n)^{2^{n_j}(1+2k)}(K_{\alpha_n}(f_1 \circ f_n))) \\ &\subset ((f_n \circ f_1)^{2^{n_j+1}k}(K_{\alpha_1|_{n_j} * \overline{\alpha_{1,n_j+1}}}(f_n \circ f_1)), (f_{n-1} \circ f_2)^{2^{n_j+1}k}(K_{\alpha_2|_{n_j} * \overline{\alpha_{2,n_j+1}}}(f_{n-1} \circ f_2)), \cdots, (f_1 \circ f_n)^{2^{n_j+1}k}(K_{\alpha_n|_{n_j} * \overline{\alpha_{n,n_j+1}}}(f_1 \circ f_n))) \\ &= K_{\alpha_1|_{n_j} * \overline{\alpha_{1,n_j+1}}}(f_n \circ f_1) \times K_{\alpha_2|_{n_j} * \overline{\alpha_{2,n_j+1}}}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n|_{n_j} * \overline{\alpha_{n,n_j+1}}}(f_1 \circ f_n) \\ &\subset K^-_{\alpha_1|_{n_j}}(f_n \circ f_1) \times K^-_{\alpha_2|_{n_j}}(f_{n-1} \circ f_2) \times \cdots \times K^-_{\alpha_n|_{n_j}}(f_1 \circ f_n)) \subset U \text{ for all } k \in N. \end{aligned}$$

Then, according to the definition of uniformly recurrent point, we obtain $(x_1, x_2, \cdots, x_n) \in UR(F) \subset R(F)$.

Proposition 3.5 Let $x_i \in K_{\alpha_i}(f_{n-i+1} \circ f_i)$ be such that $| K_{\alpha_i}(f_{n-i+1} \circ f_i) |> 0$, for every $i \in \{1, 2, \cdots, n\}$. Assume that $(x_1, x_2, \cdots, x_n) \in R(F)$, then $\bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$ is infinite.

Proof. Suppose $K_{\alpha_i}(f_{n-i+1} \circ f_i) = [x_i, x_{i_0}]$ for some $x_{i_0} \in I, i \in \{1, 2, \cdots, n\}$(the other cases are similar to this .) Now let $(x_1, x_2, \cdots, x_n) \in R(F)$ and $\bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$ finite. Let $m = \max \bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$. By Proposition 3.1(b), we have that $K_{\alpha_i|_k}(f_{n-i+1} \circ f_i)) \neq \emptyset$ for each $i \in \{1, 2, \cdots, n\}$ and any$k \in N$. Moreover , according to Theorem 2.1 (e), we known that $Int(K_{\alpha_i}(f_{n-i+1} \circ f_i))) \bigcap Orb_{f_{n-i+1} \circ f_i)}(x_i) = \emptyset, i = 1, 2, \cdots, n$.

On the other hand, for each $i \in \{1, 2, \cdots, n\}$, we have either

$$((f_{n-i+1} \circ f_i)^{2^j}(K_{\alpha_j}(f_{n-i+1} \circ f_i)) \subset K_{\alpha_j|_j * \overline{(\alpha_j, j+1)}}(f_{n-i+1} \circ f_i)) \subset K^+_{\alpha_j|_j}(f_{n-i+1} \circ f_i)$$

or

$$((f_{n-i+1} \circ f_i)^{2^j}(K_{\alpha_j}(f_{n-i+1} \circ f_i)) \subset K_{\alpha_j|_j * \overline{(\alpha_j, j+1)}}(f_{n-i+1} \circ f_i)) \circ f_i)) \subset K^-_{\alpha_j|_j}(f_{n-i+1} \circ f_i).$$

Let $U = Int(K^-_{\alpha_1|_l}(f_n \circ f_1) \times K^-_{\alpha_2|_l}(f_{n-1} \circ f_2) \times \cdots \times K^-_{\alpha_n|_l}(f_1 \circ f_n))$ for some $l > m$. Being $(x_1, x_2, \cdots, x_n) \in R(F) = R(F^2)$, let $s \in N$ satisfy that

$$F^{2s}(x_1, x_2, \cdots, x_n) \in K^-_{\alpha_1|_k}(f_n \circ f_1)) \times K^-_{\alpha_2|_k}(f_{n-1} \circ f_2) \times \cdots \times K^-_{\alpha_n|_k}(f_1 \circ f_n))).$$

By Theorem 2.1 and (3.7),$s = 2^l q$ for some $q \in N$(notice that we can assume that $2^l \leq s$). If q is odd ,$q = 1 + 2k$, then by Proposition 3.1(b) and (3.1) we would have that
$F^{2^l+1}(K_{\alpha_1}(f_n \circ f_1) \times K_{\alpha_2}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n}(f_1 \circ f_n))$
$\subset K^-_{\alpha_1|_k}(f_n \circ f_1) \times K_{\alpha_2|_k}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n|_k}(f_1 \circ f_n)$
and so $l \in \bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$, a contradiction . If q is even, $q = l.2^l, l$ odd , $r > 0$, by (3.1), we would have

$$F^{2^{l+r+1}}(K_{\alpha_1}(f_n \circ f_1) \times K_{\alpha_2}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n}(f_1 \circ f_n)) \subset K^-_{\alpha_1|_l}(f_n \circ f_1) \times K^-_{\alpha_2|_l}(f_{n-1} \circ f_2) \times \cdots \times K^-_{\alpha_n|_l}(f_1 \circ f_n))$$

By(3.4) and since

$$K^-_{\alpha_1|_{l+r}}(f_n \circ f_1) \subset K^-_{\alpha_1|_l}(f_n \circ f_1), K^-_{\alpha_2|_{l+r}}(f_{n-1} \circ f_2) \subset K^-_{\alpha_2|_l}(f_{n-1} \circ f_2), \cdots, K^-_{\alpha_n|_{l+r}}(f_1 \circ f_n) \subset K^-_{\alpha_n|_l}(f_1 \circ f_n)$$

we have that

$$F^{2^{l+r+1}}(K_{\alpha_1}(f_n \circ f_1) \times K_{\alpha_2}(f_{n-1} \circ f_2) \times \cdots \times K_{\alpha_n}(f_1 \circ f_n))$$
$$\subset K^-_{\alpha_1|_{l+r}}(f_n \circ f_1) \times K^-_{\alpha_2|_{l+r}}(f_{n-1} \circ f_2) \times \cdots \times K^-_{\alpha_n|_{l+r}}(f_1 \circ f_n))$$

and then we can similarly conclude that $l + r \in \bigcap_{i=1}^n x_i(f_{n-i+1} \circ f_i)$ which is a contradiction.
Proposition 3.6 Let $x_i \in R(f_{n-i+1} \circ f_i), i = 1, 2, \cdots, n$. Assume that $w(x_i, f_{n-i+1} \circ f_i), (i = 1, 2, \cdots, n)$ are infinite. Let $I_0 \subset \{1, 2, \cdots, n\}$.If $x_i \in K_{\alpha_i}(f_{n-i+1} \circ f_i)$with $| K_{\alpha_i}(f_{n-i+1} \circ f_i) | > 0$ for some $\alpha_i \in X_\infty, i \in I_o; x_i = K_{\alpha_i}(f_{n-i+1} \circ f_i)$ for some $\alpha_i \in X_\infty, i \in \{1, 2, \cdots, n\}/I_0$ then $(x_1, x_2, \cdots, x_n) \in R(F) \bigcap UR(F)$.

Proof Without lose generality,we suppose that $I_0 = \{1, 2, \cdots, p\}$. Let $\varepsilon > 0$, Assume , for instance , that $K_{\alpha_i}(f_{n-i+1} \circ f_i) = [x_i, x_{i0}]$ for some $x_{i0} \in I, i \in \{1, 2, \cdots, p\}$ (the other cases are analogous). By the proof of Theorem 3.3 and Theorem 3.5, we would have that

$$\bigcap_{i \in I_o} x_i(f_{n-i+1} \circ f_i)$$

is infinite.

By Theorem 2.1, we have that

$$\lim_{m \to \infty} \max\{| K^-_{\alpha_1|_m}(f_n \circ f_1) |, \cdots, | K^-_{\alpha_p|_m}(f_{n-p+1} \circ f_p) |, | K_{\alpha_{p+1}|_m}(f_{n-p} \circ f_{p+1}) |, \cdots, | K_{\alpha_n|_m}(f_1 \circ f_n) |\} = 0,$$

then , for any open neighborhood U of $(x_1, x_2, \cdots, x_n)$, there is a $k \in N$ such that

$$K^-_{\alpha_1|m}(f_n \circ f_1) \times \cdots \times \ K^-_{\alpha_p|m}(f_{n-p+1} \circ f_p) \times \ K_{\alpha_{p+1}|m}(f_{n-p} \circ f_{p+1}) \times \cdots \times \ K_{\alpha_n|m}(f_1 \circ f_n)\} \subseteq U$$

for all $m_i \geq k, i = 1, 2, \cdots, n$. On the other hand ,by Proposition 3.1 (b) there is an increasing sequence of positive integer $(n_i)_{i=1}^{\infty}$ such that

$$(f_n \circ f_1)^{2n_i(1+2l)}(x_1) \in K^-_{\alpha_1|n_i}(f_n \circ f_1), \cdots, (f_{n-p+1} \circ f_p)^{2n_i(1+2l)}(x_p) \in K^-_{\alpha_p|n_i}(f_{n-p+1} \circ f_p)$$

for all $l \in N$, By Proposition 3.1 (a) $(f_j \circ f_{n-j+1})^{2^n l}(x_j) \in K_{\alpha_j|n}(f_{n-j+1} \circ f_j)$ for all $n, l \in N, j = p+1, p+2, \cdots, n$. Then there is an $n_i \in N$ such that, applying (3.1) $F^{2^{n_i+1}(1+2l)}(x_1, x_2, \ldots, x_n)$

$$\in K^-_{\alpha_1|n_i}(f_n \circ f_1) \times \cdots \ K^-_{\alpha_p|n_i}(f_{n-p+1} \circ f_p) \times \ K_{\alpha_{p+1}|n_i}(f_{n-p} \circ f_{p+1}) \times \cdots \times K_{\alpha_n|n_i}(f_1 \circ f_n)\} \subseteq U$$

for all $l \in N$. Therefore, $(x_1, x_2, \cdots, x_n) \in UR(F)$ and by (2.1) $(x_1, x_2, \cdots, x_n) \in R(F)$.

The following result concludes our study of the set $R(F)$and $UR(F)$ for antiargular maps of zero entropy .

Theorem 3.7 Let $F(x_1, x_2, \cdots, x_n) = ((f_n \circ f_1)(x_1), (f_{n-1} \circ f_2)(x_2), \cdots, (f_1 \circ f_n)(x_n))$ an antitriangular map having zero topological entropy. Then $(x_1, x_2, \cdots, x_n) \in UR(F)$ if and only if $(x_1, x_2, \cdots, x_n) \in R(F)$.

Proof According to (2.1) and Proposition 3.2 , we must prove that $R(F^2) \subseteq UR(F^2)$. So let $(x_1, x_2, \cdots, x_n) \in R(F^2)$. Noting that, by Proposition 3.2 (d) and Theorem 1.1, it holds that $x_i \in R(f_{n-i+1} \circ f_i) = UR(f_{n-i+1} \circ f_i), i = 1, 2, \cdots, n$. We distinguish two cases: (a) $w(x_i, f_{n-i+1} \circ f_i)$ is finite if $i = 1, 2, \cdots, n$; (b) $w(x_i, f_{n-i+1} \circ f_i)$ is finite if $i = 1, 2, \cdots, m$, and $w(x_i, f_{n-i+1} \circ f_i)$ is infinite if $i = m+1, m+2, \cdots, n$.

(a) If x_i is a periodic point of $f_{n-i+1} \circ f_i, i = 1, 2, \cdots, n$ then according to (2.1) and Proposition 3.2 $(x_1, x_2, \cdots, x_n) \in P(F) \subset UR(F)$.

(b) Now let x_i be a periodic of $f_{n-i+1} \circ f_i$ of period $2^{s_i}, i = 1, 2, \cdots, m$. and $w(x_i, f_{n-i+1} \circ f_i)$, infinite,$i = m+1, m+2, \cdots, n$.. Let $s = max\{s_i | i = 1, 2, \ldots, m\}, U$ be an open neighaborhood of $(x_1, x_2, \ldots, x_n)$.Following the proof of Propositions 3.3-3.6,for any neighborhood $V_i$of $x_i, i = m+1, m+2, \cdots, n$. there exists a positive integer $t \geq s$ such that $(f_{n-i+1} \circ f_i)^{2^t(1+2k)}(x_i) \in V_i$ for all $k \in N, (i = m+1, m+2, \cdots, n.)$ And then

$F^{2^{t+1}(1+2k)}(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_m, (f_{n-m} \circ f_{m+1})^{2^t(1+2k)}(x_{m+1}), \cdots,$ $(f_1 \circ f_n)^{2^t(1+2k)}(x_n) \in V_1 \times V_2 \times \cdots \times V_n \subset U$ and therefore $(x_1, x_2, \cdots, x_n) \in UR(F^2)$.

§4 Main theorem

In this section, we will prove our main result

Theorem 4.1 Let

$$F(x_1, x_2, \cdots, x_n) = (f_n(x_n), f_{n-1}(x_{n-1}), \cdots f_1(x_1))$$

an antiargular ,Then the following properties are equivalent :

(a)h(F)=0,

(b)The period of any periodic point of F is a power of 2 ,

(c)UR(F)=R(F),

(d)$AP(F) = \{(x_1, x_2, \cdots, x_n) \in I^n : \lim F^{2^s}(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_n)\}$.

Proof $(a) \Rightarrow (b)$ By Proposition 3.2 (e),$h(F) = h(f_{n-i+1} \circ f_i), i = 1, 2, \cdots, n$. By Theorem 1.1 $f_{n-i+1} \circ f_i (i = 1, 2, \cdots, n)$ have only periodic points with period a power of two . By (3.1) and Proposition 3.2(a) F must have only periodic pionts with period a power of two .

$(b) \Rightarrow (a)$ By (3.1) and Proposition 3.2 (a),$f_{n-i+1} \circ f_i, i = 1, 2, \cdots, n$ have only periodic points with period a power of two . By Theorem 1.1 and Proposition 3.2 (e),$h(F) = h(f_{n-i+1} \circ f_i) = 0, (i = 1, 2, \cdots, n)$.

$(a) \Rightarrow (c)$ By Proposition 3.2 ,$h(F) = h(f_{n-i+1} \circ f_i) = 0, (i = 1, 2, \cdots, n)$. By Theorem 3.7 it holds that $UR(F) = R(F)$.

$(c) \Rightarrow (a)$ If $UR(F) = R(F)$, then by proposition 3.2 ,$UR(F^2) = R(F^2)$. Applying again Proposition 3.2 , $UR(f_{n-i+1} \circ f_i) = R(f_{n-i+1} \circ f_i)$. Then , by Theorem 1.1 and 3.2(e), we conclude that $h(F) = h(f_{n-i+1} \circ f_i) = 0, i = 1, 2, \cdots, n$.

$(a) \iff (d)$ If $h(F) = 0$, then $h(f_{n-i+1} \circ f_i) = 0, i = 1, 2, \cdots, n$. and by Theorem 1.1, we obtain $AP(f_{n-i+1} \circ f_i) = \{x_i \in I : \lim (f_{n-i+1} \circ f_i)^{2^s}(x_i) = x_i\}$.Notice that $AP(F) = AP(F^2) = AP(f_n \circ f_1) \times AP(f_{n-1} \circ f_{n-1}) \times \cdots \times AP(f_1 \circ f_n)$ (Proposition 3.2 (b)). Then ,By (3.1) $AP(F) = \{(x_1, x_2, \cdots, x_n) \in I^n : \lim F^{2^s}(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_n)\}$.

The converse implication is analogous and so the proof concludes.

§5 Conclusion

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,441-451,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Some Results in Intuitionistic Fuzzy Metric Spaces

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June 1, 2007

Abstract

In this paper we prove some results of metric spaces including Uniform continuity theorem and Ascoli-Arzela theorem for intuitionistic fuzzy metric spaces. We also prove that every intuitionistic fuzzy metric space has a countably locally finite basis and use this result to conclude that every intuitionistic fuzzy metric space is metrizable.

Key Words: Intuitionistic fuzzy metric space, Cauchy sequence, completeness, uniform continuity, equicontinuity.

2000 AMS Subject Classifications: 54A40, 54E70

1 Introduction

One of the most important problems in fuzzy topology, which may have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by Elnaschie [9, 10], is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors [3, 11, 13] from different points of views. In particular, George and Veeramani [7] have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek [13]. Using the idea of intuitionistic fuzzy sets [1, 2], Park [17] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George

and Veeramani [7]. In this paper, we introduce the notion of uniform continuity and equicontinuity in an intuitionistic fuzzy metric space and prove Uniform continuity theorem for intuitionistic fuzzy metric space. We prove that if an equicontinuous sequence of functions from an intuitionistic fuzzy metric space X to a complete intuitionistic fuzzy metric space Y converges for each point of a dense subset of X, then it converges at each point of X and the limit function is continuous. Using this result we prove Ascoli-Arzela theorem for intuitionistic fuzzy metric space. Finally, we prove that in an intuitionistic fuzzy metric space X every open cover admits a countably locally finite refinement which covers X. We use this result to prove that every intuitionistic fuzzy metric space has a countably locally finite base.

2 Intuitionistic fuzzy metric spaces

Definition 2.1 [18] A binary operation $* : [0.1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:
(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c,\ b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 2.2 [18] A binary operation $\diamond : [0.1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\diamond$ is satisfying the following conditions:
(a) $\diamond$ is commutative and associative;
(b) $\diamond$ is continuous;
(c) $a \diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c,\ b \leq d$ and $a, b, c, d \in [0,1]$.

Note 2.3 The notions of t-norms and t-conorms are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [15] in his study of statistical metric spaces. Several examples for these notions were proposed by many authors (see [4, 5, 6, 8, 12, 18, 19]).

Because there are three redundant conditions in definition of intuitionistic fuzzy metric space in [17], we modify the definition as follows:

Definition 2.4 A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X,\ s, t > 0$,
(a) $M(x, y, t) + N(x, y, t) \leq 1$;
(b) $M(x, y, t) > 0$;
(c) $M(x, y, t) = 1$ if and only if $x = y$;
(d) $M(x, y, t) = M(y, x, t)$;
(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(f) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
(g) $N(x, y, t) = N(y, x, t)$;
(h) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;
(i) $N(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.5 Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated [14], i.e. $x \diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in X$.

Example 2.6 [17] Let (X, d) be a metric space. Denote $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + m\ d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{kt^n + m\ d(x, y)},$$

for all $h, k, m, n \in \mathbf{R}^+$. Then $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space. We call this (M_d, N_d) as the intuitionistic fuzzy metric induced by d.

Example 2.7 [17] Let $X = \mathbf{N}$. Define $a * b = \max\{0, a + b - 1\}$ and $a \diamond b = a + b - ab$ for all $a, b \in [0, 1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ as follows:

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y, \\ \frac{y}{x} & \text{if } y < x, \end{cases} \qquad N(x, y, t) = \begin{cases} \frac{y-x}{y} & \text{if } x \leq y, \\ \frac{x-y}{x} & \text{if } y < x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space.

Remark 2.8 Note that, in the above example, t-norm $*$ and t-conorm $\diamond$ are not associated. And there exists no metric d on X satisfying

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)},$$

where $M(x, y, t)$ and $N(x, y, t)$ are as defined in above example. Also note that the above functions (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$.

Definition 2.9 [17] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space, and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ is called the open ball with center x and radius r with respect to t. Define $\tau_{(M,N)} = \{A \subset X : \text{for each } x \in A, \text{ there exist } t > 0 \text{ and } r \in (0, 1) \text{ such that } B(x, r, t) \subset A\}$. Then $\tau_{(M,N)}$ is a topology on X. Clearly, this topology is Hausdorff and first countable.

Theorem 2.10 [17] *Let* $(X, M, N, *, \diamond)$ *be a intuitionistic fuzzy metric space and* $\tau_{(M,N)}$ *be the topology on* X *induced by the intuitionistic fuzzy metric. Then for a sequence* $\{x_n\}$ *in* X, $x_n \to x$ *if and only if* $M(x_n, x, t) \to 1$ *and* $N(x_n, x, t) \to 0$ *as* $n \to \infty$.

Definition 2.11 [17] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.

(b) $(X, M, N, *, \diamond)$ is called complete if every Cauchy sequence is convergent with respect to $\tau_{(M,N)}$.

Theorem 2.12 *Every intuitionistic fuzzy metric space is normal.*

Proof Let $(X, M, N, *, \diamond)$ be given intuitionistic fuzzy metric space and F, G be two disjoint closed subsets of X. Let $x \in F$. Then $x \in G^c$. Since G^c is open, there exist $t_x > 0$ and $r_x \in (0, 1)$ such that $B(x, r_x, t_x) \cap G = \emptyset$. Similarly, there exist $t_y > 0$ and $r_y \in (0, 1)$ such that $B(y, r_y, t_y) \cap F = \emptyset$ for all $y \in G$. Let $s = \min\{r_x, t_x, r_y, t_y\}$. Then we choose a $s_0 \in (0, s)$ such that $(1 - s_0) * (1 - s_0) > 1 - s$. Put $U = \bigcup_{x \in F} B(x, s_0, \frac{s}{2})$ and $V = \bigcup_{y \in G} B(y, s_0, \frac{s}{2})$. Then U and V are open sets such that $F \subset U$ and $G \subset V$. Now we claim that $U \cap V = \emptyset$. Let $z \in U \cap V$. Then there exist $x \in F$ and $y \in G$ such that $z \in B(x, s_0, \frac{s}{2})$ and $z \in B(y, s_0, \frac{s}{2})$. Now, we have

$$\begin{aligned} M(x, y, s) &\geq M(x, z, \frac{s}{2}) * M(y, z, \frac{s}{2}) \\ &\geq (1 - s_0) * (1 - s_0) \\ &> 1 - s. \end{aligned}$$

Hence $y \in B(x, s, s)$. But since $s < r_x, t_x$, $B(x, s, s) \subset B(x, r_x, t_x)$ and thus $B(x, r_x, t_x) \cap G \neq \emptyset$, which is a contradiction. Hence X is normal.

Remark 2.13 From above theorem we can easily deduce that every metrizable space is normal. Since every intuitionistic fuzzy metric space is normal, Urysohn lemma and Tietze extension theorem are true in the case of intuitionistic fuzzy metric space.

3 Some theorems in intuitionistic fuzzy metric spaces

Definition 3.1 A function f from an intuitionistic fuzzy metric space X to an intuitionistic fuzzy metric space Y is said to be uniformly continuous if for given $r \in (0, 1)$ and $t > 0$, there exist $r_0 \in (0, 1)$ and $t_0 > 0$ such that $M(x, y, t_0) > 1 - r_0$ implies $M(f(x), f(y), t) > 1 - r$ for all $x, y \in X$.

Theorem 3.2 (**Uniform continuity theorem**) *If* f *is continuous function from a compact intuitionistic fuzzy metric space* X *to an intuitionistic fuzzy metric space* Y, *then* f *is uniformly continuous.*

Proof Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $(1-r) * (1-r) > 1-s$. Since $f : X \to Y$ is continuous, for each $x \in X$, we can find $r_x \in (0,1)$ and $t_x > 0$ such that $M(x, y, t_x) > 1 - r_x$ implies $M\left(f(x), f(y), \frac{t}{2}\right) > 1 - r$. But $r_x \in (0,1)$ and then we can find $s_x \in (0, r_x)$ such that $(1-s_x) * (1-s_x) > 1 - r_x$. Since X is compact and $\{B(x, s_x, \frac{t_x}{2}) : x \in X\}$ is an open covering of X, there exist $x_1, x_2, \ldots, x_k$ in X such that $X = \bigcup_{i=1}^{k} B(x_i, s_{x_i}, \frac{t_{x_i}}{2})$. Put $s_0 = \min s_{x_i}$ and $t_0 = \min \frac{t_{x_i}}{2}$, $i = 1, 2, \ldots, k$. For any $x, y \in X$, if $M(x, y, t_0) > 1 - s_0$, then $M(x, y, \frac{t_{x_i}}{2}) > 1 - s_{x_i}$. Since $x \in X$, there exists a x_i such that $M(x, x_i, \frac{t_{x_i}}{2}) > 1 - s_{x_i}$. Hence we have $M(f(x), f(x_i), \frac{t}{2}) > 1 - r$. Now

$$\begin{aligned} M(y, x_i, t_{x_i}) &\geq M\left(x, y, \frac{t_{x_i}}{2}\right) * M\left(x, x_i, \frac{t_{x_i}}{2}\right) \\ &\geq (1 - s_{x_i}) * (1 - s_{x_i}) > 1 - r_{x_i}. \end{aligned}$$

Therefore, $M\left(f(y), f(x_i), \frac{t}{2}\right) > 1 - r$. Now we have

$$\begin{aligned} M(f(x), f(y), t) &\geq M\left(f(x), f(x_i), \frac{t}{2}\right) * M\left(f(y), f(x_i), \frac{t}{2}\right) \\ &\geq (1-r) * (1-r) > 1-s. \end{aligned}$$

Hence f is uniformly continuous.

Remark 3.3 Let f be an uniformly continuous function from the intuitionistic fuzzy metric space X into the intuitionistic fuzzy metric space Y. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.

Theorem 3.4 *Every compact intuitionistic fuzzy metric space is separable.*

Proof Let $(X, M, N, *, \diamond)$ be the given compact intuitionistic fuzzy metric space. Let $r \in (0,1)$ and $t > 0$. Since X is compact, there exist $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} B(x_i, r, t)$. In particular, for each $n \in \mathbf{N}$, we can choose a finite subset A_n such that $X = \bigcup_{a \in A_n} B(a, \frac{1}{n}, \frac{1}{n})$. Let $A = \bigcup_{n \in \mathbf{N}} A_n$. Then A is countable. We claim that $X \subset \overline{A}$. Let $x \in X$. Then for each $n \in \mathbf{N}$, there exists $a_n \in A_n$ such that $x \in B(a_n, \frac{1}{n}, \frac{1}{n})$. Thus a_n converges to x. But since $a_n \in A$ for all n, $x \in \overline{A}$. Hence A is dense in X and thus X is separable.

Definition 3.5 Let X be any nonempty set and $(Y, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be converge uniformly to a function f from X to Y if for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(f_n(x), f(x), t) > 1 - r$ for all $n \geq n_0$ and for all $x \in X$.

Definition 3.6 A family $\mathcal{F}$ of functions from an intuitionistic fuzzy metric space X to a complete intuitionistic fuzzy metric space Y is said to be equicontinuous if for given $r \in (0,1)$ and $t > 0$, there exist $r_0 \in (0,1)$ and $t_0 > 0$ such that $M(x, y, t_0) > 1 - r_0$ implies $M(f(x), f(y), t) > 1 - r$ for all $f \in \mathcal{F}$.

Lemma 3.7 *Let $\{f_n\}$ be an equicontinuous sequence of functions from an intuitionistic fuzzy metric space X to a complete intuitionistic fuzzy metric space Y. If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges for each point of X and the limit function is continuous.*

Proof Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $(1-r) * (1-r) * (1-r) > 1-s$. Since $\mathcal{F} = \{f_n\}$ is an equicontinuous family, for given $r \in (0,1)$ and $t > 0$, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that for each $x, y \in X$, $M(x,y,t_1) > 1-r_1 \Rightarrow M(f_n(x), f_n(y), \frac{t}{3}) > 1-r$ for all $f_n \in \mathcal{F}$. Since D is dense in X, there exists $y \in B(x, r_1, t_1) \cap D$ and $\{f_n(y)\}$ converges for that y. Since $\{f_n(y)\}$ is a Cauchy sequence, for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(f_n(y), f_m(y), \frac{t}{3}) > 1-r$ for all $m, n \geq n_0$. Now for any $x \in X$, we have

$$
\begin{aligned}
&M(f_n(x), f_m(x), t) \\
&\quad \geq M(f_n(x), f_n(y), \frac{t}{3}) * M(f_n(y), f_m(y), \frac{t}{3}) * M(f_m(x), f_m(y), \frac{t}{3}) \\
&\quad \geq (1-r) * (1-r) * (1-r) \\
&\quad > 1-s.
\end{aligned}
$$

for all $m, n \geq n_0$. Hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{f_n(x)\}$ converges. Let $f(x) = \lim f_n(x)$. We claim that f is continuous. Let $s_o \in (0,1)$ and $t_0 > 0$ be given. Then we can find $r_0 \in (0,1)$ such that $(1-r_0)*(1-r_0)*(1-r_0) > 1-s_0$. Since $\mathcal{F}$ is equicontinuous, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that $M(x,y,t_2) > 1-r_2 \Rightarrow M(f_n(x), f_n(y), \frac{t_0}{3}) > 1-r_0$ for all $f_n \in \mathcal{F}$. Since $f_n(x)$ converges to $f(x)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_1 \in \mathbf{N}$ such that $M(f_n(x), f(x), \frac{t_0}{3}) > 1-r_0$ for all $n \geq n_1$. Also since $f_n(y)$ converges to $f(y)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_2 \in \mathbf{N}$ such that $M(f_n(y), f(y), \frac{t_0}{3}) > 1-r_0$ for all $n \geq n_2$. Now for all $n \geq \max\{n_1, n_2\}$, we have

$$
\begin{aligned}
&M(f(x), f(y), t_0) \\
&\quad \geq M\left(f(x), f_n(x), \frac{t_0}{3}\right) * M\left(f_n(x), f_n(y), \frac{t_0}{3}\right) * M\left(f_n(y), f(y), \frac{t_0}{3}\right) \\
&\quad \geq (1-r_0) * (1-r_0) * (1-r_0) \\
&\quad > 1-s_0.
\end{aligned}
$$

Hence f is continuous.

Theorem 3.8 (Ascoli-Arzela theorem) *Let X be a compact intuitionistic fuzzy metric space and Y be a complete intuitionistic fuzzy metric space. Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y. If $\{f_n\}_{n \in \mathbf{N}}$ is a sequence in $\mathcal{F}$ such that $\overline{\{f_n(x) : n \in \mathbf{N}\}}$ is a compact subset of Y for each $x \in X$, then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.*

Proof Since X is a compact intuitionistic fuzzy metric space, by Theorem 3.4, X is separable. Let $D = \overline{\{x_i : i = 1, 2, \ldots\}}$ be a countable dense subset of X. By hypothesis, for each i, $\overline{\{f_n(x_i) : n \in \mathbf{N}\}}$ is compact subset of Y. Since every intuitionistic fuzzy metric space is first countable space [17], every compact subset of Y is sequentially compact. Thus by standard argument, we have a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each $i = 1, 2, \ldots$. By Lemma 3.7, there exists a continuous function f from X to Y such that $\{g_n(x)\}$ converges to $f(x)$ for all $x \in X$. Now we claim that g_n converges uniformly to f on X. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that $(1-r) * (1-r) * (1-r) > 1-s$. Since $\mathcal{F}$ is equicontinuous, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that $M(x, y, t_1) > 1-r_1 \Rightarrow M(g_n(x), g_n(y), \frac{t}{3}) > 1-r$ for all n. Since X is compact, by Theorem 3.2, f is uniformly continuous. Hence for given $r \in (0,1)$ and $t > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that $M(x, y, t_2) > 1 - r_2 \Rightarrow M(f(x), f(y), \frac{t}{3}) > 1 - r$ for all $x, y \in X$. Let $r_0 = \min\{r_1, r_2\}$ and $t_0 = \min\{t_1, t_2\}$. Since X is compact and D is dense in X, $X = \bigcup_{i=1}^{k} B(x_i, r_0, t_0)$ for some k. Thus for each $x \in X$, there exists i, $i \le k$, such that $M(x, x_i, t_0) > 1-r_0$. But since $r_0 = \min\{r_1, r_2\}$ and $t_0 = \min\{t_1, t_2\}$, we have, by the equicontinuity of $\mathcal{F}$, $M(g_n(x), g_n(x_i), \frac{t}{3}) > 1 - r$ and we also have, by the uniform continuity of f, $M(f(x), f(x_i), \frac{t}{3}) > 1-r$. Since $\{g_n(x_j)\}$ converges to $f(x_j)$, for $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(g_n(x_j), f(x_j), \frac{t}{3}) > 1 - r$ for all $n \ge n_0$, and for all $j = 1, 2, \cdots, n$. Now for each $x \in X$, we have

$$\begin{aligned}
&M(g_n(x), f(x), t) \\
&\quad \ge M(g_n(x), g_n(x_i), \frac{t}{3}) * M(g_n(x_i), f(x_i), \frac{t}{3}) * M(f(x_i), f(x), \frac{t}{3}) \\
&\quad \ge (1-r) * (1-r) * (1-r) \\
&\quad > 1-s.
\end{aligned}$$

Hence g_n converges uniformly to f on X.

Lemma 3.9 *Let X be any nonempty set and (Y, d) be a metric space. Let $(Y, M, N, *, \diamond)$ be the induced intuitionistic fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y converges uniformly to a function f from X to Y with respect to d if and only if $\{f_n\}$ converges uniformly to f with respect to the intuitionistic fuzzy metric (M, N).*

Proof We prove only the sufficiency part since the necessity part is similar. Let $\epsilon > 0$ and $t > 0$. Put $r = \frac{\epsilon}{t+\epsilon}$. Since $\{f_n\}$ converges uniformly to f with respect to the intuitionistic fuzzy metric (M, N), there exists $k \in \mathbf{N}$ such that $M(f_n(x), f(x), t) > 1-r$ for all $n \ge k$ and for all $x \in X$ and hence by Theorem 2.10, $d(f_n(x), f(x)) < \epsilon$ for all $n \ge k$ and for all $x \in X$. Hence $\{f_n\}$ converges uniformly to f with respect to the metric d.

Lemma 3.10 *Let (X, d) and (Y, d') be metric spaces. Let $(X, M, N, *, \diamond)$ and $(Y, M', N', *', \diamond')$ be the corresponding induced intuitionistic fuzzy metric spaces.*

Then a family $\mathcal{F}$ of functions from X to Y is equicontinuous with respect to the metric if and only if $\mathcal{F}$ is equicontinuous with respect to the intuitionistic fuzzy metric.

Proof We prove only the necessity part since the sufficiency part is similar. Let $r \in (0,1)$ and $t > 0$. Put $\epsilon = \frac{rt}{1-r}$. Since $\mathcal{F}$ is equicontinuous with respect to the metric, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x), f(y)) < \epsilon$ for all $f \in \mathcal{F}$. Put $t_0 = t$ and $r_0 = \frac{\delta}{t_0+\delta}$. Then $M(x,y,t_0) > 1 - r_0 \Rightarrow M(f(x), f(y), t) > 1 - r$ for all $f \in \mathcal{F}$. Hence $\mathcal{F}$ is equicontinuous with respect to the intuitionistic fuzzy metric.

Corollary 3.11 *Let X be a compact metric space and Y be a complete metric space. Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y. If $\{f_n\}_{n\in\mathbf{N}}$ is a sequence in $\mathcal{F}$ such that $\overline{\{f_n(x) : n \in \mathbf{N}\}}$ is a compact subset of Y for each $x \in X$, then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.*

Proof Let (Y,d) be the given metric space and $(Y, M, N, *, \diamond)$ be the induced intuitionistic fuzzy metric space. Then $(Y, M, N, *, \diamond)$ is complete if and only if (Y,d) is complete. Hence by Lemmas 3.9 and 3.10 and Theorem 3.8, we get the required result.

Now, we shall prove that every intuitionistic fuzzy metric space is metrizable.

Lemma 3.12 *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space. If $\mathcal{A}$ is an open covering of X, then there is an open covering $\mathcal{B}$ of X such that $\mathcal{B}$ is a countably locally finite refinement of $\mathcal{A}$.*

Proof Since $\mathcal{A}$ is open covering of X, by well-ordering theorem we can choose a well ordering $<$ for $\mathcal{A}$. For each $n \in \mathbf{N}$ and $U \in \mathcal{A}$, define $S_n(U) = \{x \in X : B(x, \frac{1}{n}, \frac{1}{n}) \subset U\}$ and $R_n(U) = S_n(U) - \bigcup_{V<U} V$. If $V, W \in \mathcal{A}$ with $V < W$ and if $x \in R_n(V)$ and $y \in R_n(W)$, we show that $M(x, y, \frac{1}{n}) \le 1 - \frac{1}{n}$. Since $x \in R_n(V)$, we have $x \in S_n(V)$. Since $y \in R_n(W)$ and $V < W$, $y \notin V$ and hence $M(x, y, \frac{1}{n}) \le 1 - \frac{1}{n}$. For given $n \in \mathbf{N}$, we can find $s \in (0, \frac{1}{n})$ such that $(1-s)*(1-s)*(1-s) > 1 - \frac{1}{n}$. Let $E_n(U) = \bigcup\{B(x, s, \frac{1}{3n}) : x \in R_n(U)\}$. Then clearly $E_n(U)$'s are open [17]. We claim that $E_n(U)$'s are disjoint. Let $V, W \in \mathcal{A}$ with $V < W$ and let $x \in E_n(V)$ and $y \in E_n(W)$. Then we show that $M(x, y, \frac{1}{3n}) \le 1 - s$. If $M(x, y, \frac{1}{3n}) > 1 - s$, since $x \in E_n(V)$ and $y \in E_n(W)$, there exist $x_0 \in R_n(V)$ and $y_0 \in R_n(W)$ such that $x \in B(x_0, s, \frac{1}{3n})$ and $y \in B(y_0, s, \frac{1}{3n})$. Also since $V < W$, we have $M(x_0, y_0, \frac{1}{n}) \le 1 - \frac{1}{n}$. But

$$\begin{aligned} 1 - \frac{1}{n} &\ge M(x_0, y_0, \frac{1}{n}) \\ &\ge M(x, x_0, \frac{1}{3n}) * M(x, y, \frac{1}{3n}) * M(y, y_0, \frac{1}{3n}) \\ &\ge (1-s)*(1-s)*(1-s) > 1 - \frac{1}{n}, \end{aligned}$$

which is a contradiction and hence $M(x, y, \frac{1}{3n}) \leq 1 - s$.

Let $\mathcal{E}_n = \{E_n(U) : U \in \mathcal{A}\}$. We claim that $\mathcal{E}_n$ refines $\mathcal{A}$. If $y \in E_n(U)$, then there exists $x \in R_n(U)$ such that $y \in B(x, s, \frac{1}{3n})$. Since $s < \frac{1}{n}$, we have $y \in B(x, s, \frac{1}{3n}) \subset B(x, \frac{1}{n}, \frac{1}{n}) \subset U$. Since $E_n(U) \subset U$ for all $U \in \mathcal{A}$, $\mathcal{E}_n$ refines $\mathcal{A}$. We claim that $\mathcal{E}_n$ is locally finite. Since $s \in (0, 1)$, we can find $r_0 \in (0, 1)$ such that $(1 - r_0) * (1 - r_0) > 1 - s$. For each $x \in X$, $B(x, r_0, \frac{1}{6n})$ intersects at most one element of $\mathcal{E}_n$. For, if $B(x, r_0, \frac{1}{6n})$ intersect $E_n(U)$ and $E_n(V)$ with $U < V$, then there exist $y \in E_n(U)$ and $z \in E_n(V)$ such that $M(x, y, \frac{1}{6n}) > 1 - r_0$ and $M(x, z, \frac{1}{6n}) > 1 - r_0$. Since $U < V$, we have $M(y, z, \frac{1}{3n}) \leq 1 - s$. But

$$\begin{aligned} M(y, z, \frac{1}{3n}) &\geq M(x, y, \frac{1}{6n}) * M(x, z, \frac{1}{6n}) \\ &\geq (1 - r_0) * (1 - r_0) > 1 - s, \end{aligned}$$

which is a contradiction. Hence $\mathcal{E}_n$ is locally finite. Now, we consider the family $\mathcal{B} = \bigcup_{n \in \mathbf{N}} \mathcal{E}_n$. Let $x \in X$. Since $\mathcal{A}$ is cover of X, there exists a $U \in \mathcal{A}$ such that U is the first element of $\mathcal{A}$ that contains x. Since U is open, there exists $n \in \mathbf{N}$ such that $B(x, \frac{1}{n}, \frac{1}{n}) \subset U$. Then $x \in S_n(U)$ and since U is the first element of $\mathcal{A}$ that contains x, $x \in R_n(U)$ and thus $x \in E_n(U)$. Hence $\mathcal{B}$ is an open covering of X such that $\mathcal{B}$ is countably locally finite refinement of $\mathcal{A}$.

Theorem 3.13 *Every intuitionistic fuzzy metric space has a countably locally finite basis.*

Proof For each $n \in \mathbf{N}$, let $\mathcal{A}_n = \{B(x, \frac{1}{n}, \frac{1}{n}) : x \in X\}$. Then $\mathcal{A}_n$ covers X for each $n \in \mathbf{N}$. By Lemma 3.12, there exists an open covering $\mathcal{B}_n$ of X which is a countably locally finite refinement of $\mathcal{A}_n$. Let $\mathcal{B} = \bigcup_{n \in \mathbf{N}} \mathcal{B}_n$. Then $\mathcal{B}$ is countably locally finite. We claim that $\mathcal{B}$ is a basis for X. Let $x \in X$. Given $r \in (0, 1)$ and $t > 0$, we can find $n_0 \in \mathbf{N}$ such that $(1 - \frac{1}{n_0}) * (1 - \frac{1}{n_0}) > 1 - r$ and $t > \frac{2}{n_0}$. Let $B \in \mathcal{B}_{n_0}$ with $x \in B$. Since $\mathcal{B}_{n_0}$ refines $\mathcal{A}_{n_0}$, there exists a $x_0 \in X$ such that $B \subset B(x_0, \frac{1}{n_0}, \frac{1}{n_0})$. For any $y \in B$, we have

$$\begin{aligned} M(x, y, t) &> M(x, y, \frac{2}{n_0}) \geq M(x, x_0, \frac{1}{n_0}) * M(y, x_0, \frac{1}{n_0}) \\ &\geq (1 - \frac{1}{n_0}) * (1 - \frac{1}{n_0}) > 1 - r. \end{aligned}$$

Then $y \in B(x, r, t)$ and thus $B \subset B(x, r, t)$.

Corollary 3.14 *Every intuitionistic fuzzy metric space is metrizable.*

Proof By Theorem 2.12 and 3.13, every intuitionistic fuzzy metric space is regular and has a basis that is countably locally finite. Hence by Nagata-Smirnov metrization theorem ([16], Theorem 40.3, p. 250), it is metrizable.

Remark 3.15 The topologies induced by a metric and the corresponding intuitionistic fuzzy metric are same. Hence by Theorem 3.13, we can deduce that every metric space has a basis that is countably locally finite. Since intuitionistic fuzzy metric is a generalization of fuzzy metric, the results for fuzzy metric space are particular cases of the above theorem.

4 Conclusions

As a generalization of fuzzy metric, the notion of intuitionistic fuzzy metric was introduced by Park [17]. Because, in intuitionistic fuzzy metric space, the degree of nearness and the degree of non-nearness between two points with respect to some value are allowed, there is room for more flexibility. In this paper, we prove some results of metric spaces including Uniform continuity theorem and Ascoli-Arzela theorem for intuitionistic fuzzy metric spaces and also prove that every intuitionistic fuzzy metric space is metrizable. Applications of intuitionistic fuzzy metric may have been done in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory [9, 10].

Acknowledgments

This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD) (KRF-2006- 521-C00017).

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,453-464,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

A Preconditioned Linear Sampling Method in Inverse Acoustic Obstacle Scattering

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Abstract — The problem of determining the shape of an obstacle from far-field measurements is considered. It is well known that Kirsch's $(F^\star F)^{1/4}$-method has been widely used for shape reconstructions obtained via the singular system of the ill conditioned discretized far-field operator F. In this work we present a preconditioned version of this method. In particular, the singular system of an appropriate preconditioner constructed via the Algebraic Multigrid Method (AMG) is used for the reconstructions since it exploits stable eigenvalues, in contrast to the ones of the disretized far field operator.

Keywords:AMG Preconditioning, Inverse Scattering, Linear Sampling Method.

1. INTRODUCTION

For inverse acoustic scattering the original linear sampling method was introduced by Colton and Kirsch [7], and mathematically clarified in [8]. Kirsch [9] improved the original version of the linear sampling method, leading to the so-called $(F^\star F)^{1/4}$ -method. Recently, Arens [2] presented a proof of convergence for this method for the case of acoustic scattering by sound soft obstacles. Linear sampling methods involve the solution of a linear Fredholm equation of the first kind, the far-field equation, which is written for each point x_0 inside the scatterer and whose integral kernel is the far-field pattern, i.e. far-field data that are usually contaminated with significant noise. Its right-hand side is an exactly known analytic function. Consequently, in the second version of the linear sampling method, Kirsch, characterized the obstacle in terms of the spectral data of the far field operator F. Two of the attractive futures of this method is its computational speed, and the very low amount of *a priori* information on the scatterers. In other words, it is not necessary to *a priori* know the number of scatterers or the kind of the boundary condition that is satisfied by the total field. One of its disadvantages though is that it only gives an explicit characterization of the scattering obstacle (i.e. it only determines the support of the scatterer).

It is well known that every numerical implementation of an inverse scattering method requires at some point regularization in order to cope with the ill-posedness of the problem, and the linear sampling method is not an exception. In most numerical applications of the linear sampling method, Tikhonov regularization has been

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employed and the regularization constant was computed via Morozov's discrepancy principle [8], which involved the computation of the zeros of the discrepancy function at each point of the grid, a process that is time-consuming. In addition, the noise level in the data should be known *a priori*, something that in real life applications is not the case in general.

In this work we propose a characterization of the object via a preconditioned $(F^\star F)^{1/4}$ -method. This paper attempts to alleviate the ill-posedness of this problem via the construction of an appropriate preconditioner. The preconditioner is effectively constructed via the Algebraic Multigrid Method (AMG) from the descretized form of the operator $(F^\star F)^{1/2}$ and is symmetric and positive definite. It is very important to point out that in our approach we are not dealing with the eigenvalues of the ill possed matrix that corresponds to the discretized form of the operator F, but we rather dealing with the more stable eigenvalues of the matrix that is constructed via the AMG.

We organize our paper as follows. Section 2 will be devoted to the formulation of the problem. Consequently, section 3 will deal with the idea behind the selection of our preconditioner as well as with its implementation within the framework of the linear sampling method. In section 4 we will describe the construction of the preconditioner via the Algebraic Multigrid Method (AMG). Finally, in section 5 we will show the effectiveness of the preconditioner via numerical examples for the case of inpenetrable scatterers. For the reconstructions we will use simulated data obtained by means of the Nyström method [6].

2. FORMULATION OF THE PROBLEM

It is well known that the propagation of time-harmonic acoustic fields in a homogeneous medium, in the presence of a sound soft obstacle D, is modeled by the exterior boundary value problem (direct obstacle scattering problem)

$$\triangle_2 u(x) + k^2 u(x) = 0, \quad x \in \Re^2 \setminus \bar{D} \tag{2.1}$$

$$u(x) + u^i(x) = 0, \quad x \in \partial D \tag{2.2}$$

where k is a real positive wavenumber and u^i is a given incident field, that in the presence of D will generate the scattered field u.
In addition, the scattered field u will satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \tag{2.3}$$

$|r| = |x|$, $x \in \Re^2 \setminus \bar{D}$, and the limit is taken uniformly for all directions $\hat{x} = x/|x|$.

The Green formula implies that the solution u of the direct obstacle scattering problem above has the asymptotic behaviour

$$u(x) = u_\infty(\hat{x}) \frac{e^{ikr}}{\sqrt{r}} + O(r^{-3/2}) \tag{2.4}$$

for some analytic function u_∞, called the far field-pattern of u, and defined on the unit sphere Ω. In the case of the inverse problem, it represents the measured data. In particular, the inverse problem that will be considered here, is the problem of finding the shape of D from a complete knowledge of the far-field pattern.

We now define the far-field equation

$$(Fg_z)(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z} \tag{2.5}$$

where the right hand side is the far-field pattern of the fundamental solution of the Helmholtz equation, $z \in \Re^2$ and $F : L^2(\Omega) \to L^2(\Omega)$ is given by

$$(Fg)(\hat{x}) = \int_\Omega u_\infty(\hat{x}; \hat{d})\, g(\hat{d})\; ds(\hat{d}), \quad d \in \Omega \tag{2.6}$$

It is well known that the first version of the linear sampling method [7] solves the linear operator equation (2.5) based on the numerical observation that its solution will have a large norm outside and close to ∂D. Hence, reconstructions are obtained by plotting the norm of the solution. However, the problem is that the right-hand side does not in general belong to the range of the operator F. Kirsch [9] was able to overcome this difficulty with the introduction of a new version of the linear sampling method based on appropriate factorization of the far-field operator F. In this method, Kirsch is elegantly using the spectral properties of the operator F to characterize the obstacle. In particular, the following far-field equation is now used in place of equation (2.5)

$$(F^\star F)^{1/4} g_z = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z} \tag{2.7}$$

and the spectral properties of F are used for the reconstructions. However, due to noisy data, the discretized version of the far field operator F is characterized by numerical instability which may result to false information about its singular system. In the next section, we will show that we can overcome this difficulty via the construction of an appropriate preconditioner with eigenvalues far away from zero (in contrast to the eigenvalues of F that cluster around zero) and whose singular system will be used for the reconstructions. That way, Tikhonov regularization can be avoided and the noise in the data does not need to be known in advance.

3. THE PRECONDITIONED LINEAR SAMPLING METHOD

The far field operator F maps a density g onto the far-field pattern of the scattered field generated from the incident field $u^i = v_g$ via the Herglotz wave operator $H : L^2(\Omega) \to L^2(\partial D)$ given by

$$Hg(x) = \int_\Omega e^{ikx\cdot d}\, g(d)\, ds(d), \qquad x \in \partial D \tag{3.1}$$

As indicated in [9], closely related to F, is the operator $G : L^2(\partial D) \to L^2(\Omega)$ defined by $Gh = u_\infty$, where $u_\infty \in L^2(\Omega)$ is the far field pattern of the solution u of the exterior Dirichlet problem with boundary data $h \in L^2(\partial D)$.

The far-field operators H and G are related through the single layer boundary operator $S : L^2(\partial D) \to L^2(\partial D)$, defined by

$$S\phi(x) = \int_{\partial D} \Phi(x,y)\, \phi(y)\, ds(y), \quad \phi \in L^2(\partial D) \tag{3.2}$$

where Φ denotes the fundamental solution of the Helmholtz equation in two dimensions and defined by

$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|) \quad x \neq y \tag{3.3}$$

It is not difficult to show that

$$H = \zeta\, S^\star\, G^\star \quad \text{on} \quad L^2(\partial D) \tag{3.4}$$

and

$$F = -GH \tag{3.5}$$

with $\zeta = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}}$ in two dimensions [9]. Using (3.4) and (3.5), we arrive to the factorization [9]

$$F = -\frac{1}{\zeta} G S^\star G^\star \tag{3.6}$$

We now consider a singular system $(\mu_j, \psi_j, \tilde{\psi}_j)$ of the operator F in $L^2(\Omega)$. Since F is normal [9], we have $\mu_j = |\lambda_j|$ and $\tilde{\psi} = \mathrm{sign}(\lambda_j)\psi$, where λ_j are the eigenvalues of F. Using that system, Kirsch showed that the sequence defined by

$$\phi_j = \frac{1}{i\sqrt{\lambda_j}} G^\star \psi_j \tag{3.7}$$

forms a Riesz basis in $H^{-1/2}(\partial D)$ [9].

The above result combined with Picard's theorem yields a characterization of the range of G [9]:

$$\mathrm{Range}(G) = \{\sum_{j=1}^{\infty} \rho_j \psi_j : \sum_{j=1}^{\infty} \frac{|\rho_j|}{\mu_j} < \infty\} = \mathrm{Range}((F^\star F)^{1/4}) \tag{3.8}$$

Our approach from the other hand is based on the operator $\mathcal{F} = (F^\star F)^{-1/2}$. Since $\mathrm{Range}((F^\star F)^{1/2}) \subseteq \mathrm{Range}((F^\star F)^{1/4})$ we define the operator $G_1 : M \to \mathrm{Range}(G_1)$ as a restriction of G i.e.

$$G_1 = G\,|_M \text{ with } M = \{\phi \in L^2(\partial D) \mid G\phi \in \mathrm{Range}((F^\star F)^{1/2})\}$$

Applying the operator $\mathcal{F}$ on both sides of (2.7) we obtain

$$(\mathcal{F}^\star\mathcal{F})^{1/4}g_z = \mathcal{F}\zeta e^{-ik\hat{x}\cdot z}, \tag{3.9}$$

with the understanding that the right hand side of the equation (2.7) is an element of Range($(F^\star F)^{1/2}$). Comparing now equations (2.7) and (3.9), it is easy to see that solving the latter is a matter of using the singular system $(\tilde{\mu}_j, \chi_j, \tilde{\chi}_j)$ of the operator $\mathcal{F}$ instead of F. Furthermore, all theoretical results presented above, and with details in [9], remain valid with just minor modifications. In particular, it is easy to show that equation (3.8) can now take the form

$$\mathrm{Range}((\mathcal{F}^\star\mathcal{F})^{1/4}) = \{\sum_{j=1}^{\infty}\tilde{\rho}_j\chi_j : \sum_{j=1}^{\infty}\frac{|\tilde{\rho}_j|}{\tilde{\mu}_j} < \infty\} = \mathrm{Range}((\mathcal{F}G_1) \tag{3.10}$$

where the $\tilde{\mu}_j$'s denote the eigenvalues of $\mathcal{F}$.

The discretized form of $\mathcal{F}$ is constructed using the AMG, and numerical experiments presented in section 5 show that it is better conditioned than F, hence its use will provide reliable results even without the use of Tikhonov regularization. The following section describes the way the authors constructed an effective preconditioner for the solution of the discretized form of equation (2.7).

4. CONSTRUCTION OF THE AMG PRECONDITIONER

Frequently, the solution of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ requires multiplication of both of its sides by a nonsingular $n \times n$ matrix M^{-1}, called preconditioner, i.e.

$$M^{-1}A\mathbf{x} = M^{-1}\mathbf{b} \tag{4.1}$$

The role of the preconditioner is to make the matrix $M^{-1}A$ better conditioned than the original matrix A and hence to accelerate the convergence of the iterative method [13, 14] used. Developing efficient preconditioners has been one of the major research interests in computational electromagnetics [10, 12].

AMG methods have been developed for solving linear systems posed on large, unstructured grids because they do not require geometric grid information. Classical AMG was originally used for the solution of linear systems with symmetric and positive definite M-matrices. Recently, many new AMG approaches are developed to solve more general linear systems [3, 5].

As mentioned in section 3, we are trying to solve the discretized form of the preconditioned problem (3.9). Notice that the preconditioner M^{-1} introduced in (4.1) corresponds to the discretized form of $\mathcal{F}$. This preconditioner can be constructed by solving the following systems:

$$M\mathbf{x} = \mathbf{e}, \tag{4.2}$$

where the right-hand side vector $\mathbf{e} = \mathbf{e}^{(k)}$, $k = 1, \cdots, n$, and with $\mathbf{e}^{(k)}$ being the kth canonical vector. Hence, the solution $\mathbf{x}$ corresponds to the kth column of M^{-1}.

In this paper, we suggest to solve (4.2) via the AMG. We begin our discussion by introducing the basic framework of AMG. AMG does not require to access the physical grids of problems. With "grids" we mean sets of indices of the unknown variables. Hence the grid set for (4.2) is $\Omega = \{1, 2, \cdots, n\}$, since the unknown vector $\mathbf{x}$ in (4.2) has components $x_1, x_2, ..., x_n$. The main idea of AMG is to remove the smooth error by coarse grid correction, where smooth error is the error not eliminated by relaxation on the fine grid, which also can be characterized by small residuals [4]. In order to develop the multi-grid algorithm, we consider the sets of grids in each level. The number 0 stands for the finest-grid level. Then the numbers $1, 2, \cdots, l_{max}$ represent the corresponding coarse-grid levels. A direct solver was used at the coarsest-grid level l_{max}. Hence, equation (4.2) can be written as $M_0\mathbf{x}_0 = \mathbf{e}_0$ and the set of finest grid set is $\Omega_0 = \Omega$. Assume now that we have defined a relaxation scheme, a set of coarse-grid points Ω_l, a coarse-grid operator A_l, where $l = 1, 2, \cdots, l_{max}$, and intergrid transfer operators I_l^{l+1} (restriction) and I_{l+1}^l (interpolation), where $l = 0, 1, \cdots, l_{max} - 1$. For the relaxation scheme we employ Gauss-Seidel iterations.

We are now ready to perform an AMG V-cycle as follows:

Algorithm 4.1 AMG V-Cycle

$$\mathbf{x}_l \leftarrow \mathbf{AMGV}(M_l, \mathbf{x}_l, \mathbf{e}_l)$$

if Ω_l = coarsest grid, **then**

$\mathbf{x}_l \leftarrow$ DIRECTSOLVE $(M_l\mathbf{x}_l = \mathbf{e}_l)$

else

$\mathbf{x}_l \leftarrow$ Relax ν_1 times on $M_l\mathbf{x}_l = \mathbf{e}_l$ on Ω_l with initial guess $\mathbf{x}_l$

$\mathbf{e}_{l+1} \leftarrow I_l^{l+1}(\mathbf{e}_l - M_l\mathbf{x}_l)$

$\mathbf{x}_{l+1} \leftarrow 0,$

$\mathbf{x}_{l+1} \leftarrow \mathbf{AMGV}(M_{l+1}, \mathbf{x}_{l+1}, \mathbf{e}_{l+1}).$

Correct $\mathbf{x}_l = \mathbf{x}_l + I_{l+1}^l\mathbf{x}_{l+1}.$

$\mathbf{x}_l \leftarrow$ Relax ν_1 times on $M_l\mathbf{x}_l = \mathbf{e}_l$ on Ω_l with $\mathbf{x}_l$.

endif

In general, the restriction operator I_l^{l+1} is defined by the transpose of the interpolation operator I_{l+1}^l, i.e., $I_l^{l+1} = (I_{l+1}^l)^T$ and the coarse grid operator M_{l+1} is constructed from the fine grid operator M_l by the Galerkin approach:

$$M_{l+1} = I_l^{l+1}\, M_l\, I_{l+1}^l, \tag{4.3}$$

so that AMG satisfies the principle that the coarse-grid problem needs to provide a good approximation to fine-grid error in the range of interpolation [4]. Hence, to

set up an AMG preconditioner, we need to find a suitable coarsening strategy and an effective interpolation operator. The creation of the coarse-grid sets Ω_l, where $l = 1, 2, \cdots, l_{max}$ is based on a combinatorial clustering algorithm developed by Vaněk, Mandel and Brezina in [15] with normalized edge weights:

$$\overline{\omega}_{ij} = |M_{ij}|/\sqrt{|M_{ii}| \cdot |M_{jj}|}. \tag{4.4}$$

The first step of their coarsening algorithm iterates through the nodes $\Omega_l = \{C_1^l, C_2^l, \cdots, C_{n_l}^l\}$ creating clusters $\{j \mid \overline{\omega}_{ij} \geq \eta_1\}$ for a given tolerance $\eta_1 > 0$, provided no node in $\{j \mid \overline{\omega}_{ij} \geq \eta_1\}$ is already a cluster. Two nodes i and j are said to be strongly connected if $\overline{\omega}_{ij} \geq \eta_1$. In the second step, unassigned nodes are assigned to a cluster from step one to which the node is strongly connected, if any. In the last step, the remaining nodes are assigned to clusters consisting of strong neighborhoods intersecting with the set of remaining nodes. As explained earlier, the next important phase in AMG is to construct an interpolation operator. In smoothed aggregation AMG [11], we solve a local linear system to obtain an interpolation vector that interpolates a value for a coarse-grid cluster onto its neighborhood. In other words, we consider one cluster C_k^{l+1}, called C_k, from the set of coarse-grid points Ω_{l+1}. For each cluster C_k, we define a neighborhood N_k as

$$N_k = \{j \notin C_k \mid \overline{\omega}_{ij} \geq \eta_2, i \in C_k\},$$

and also have an associated interpolation vector p_k, which is the kth column of the interpolation operator $I_{l+1}^l = [p_1, p_2, \cdots, p_{n_{l+1}}]$. Denote $M_{IJ} = [\ a_{ij} \mid i \in I, j \in J\ ]$ to be an $|I| \times |J|$ matrix where I and J are sets of nodes. For a cluster C_k, its local matrix L_{C_k} is given by

$$L_{C_k} = \begin{bmatrix} M_{C_kC_k} & M_{C_kN_k} \\ M_{N_kC_k} & M_{N_kN_k} \end{bmatrix}. \tag{4.5}$$

Then its corresponding interpolation vector p_k, the kth column of I_{l+1}^l, can be obtained by solving the local linear system below:

$$L_{C_k}\, p_k = \epsilon_{C_k}, \tag{4.6}$$

where ϵ_{C_k} is the vector given by

$$(\epsilon_{C_k})_i = \begin{cases} 1 & \text{if } i \in C_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Using $I_l^{l+1} = (I_{l+1}^l)^T$ and $M_{l+1} = I_l^{l+1} M_l I_{l+1}^l$, we have defined all components of the AMG method.

After the discussion above it becomes apparent that solving the system (4.2) via an AMG V-cycle (Algorithm 4.1) for $k = 1, \cdots, n$, we obtain the desired preconditioner M^{-1}.

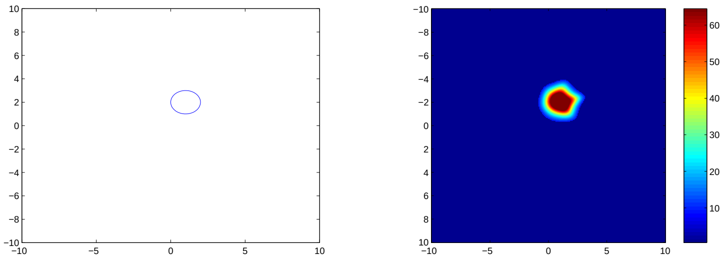

Figure 1: Visualization of a circle (left). Profile reconstruction via the AMG precondtitioner (right).

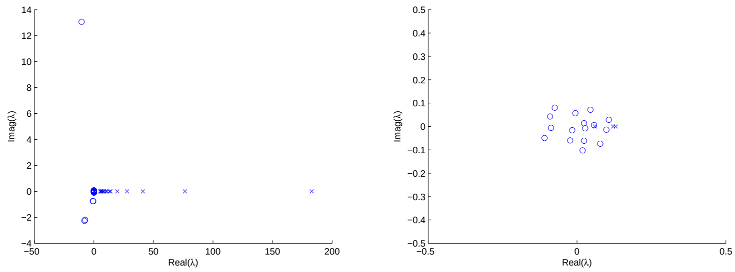

Figure 2: Eigenvalue distribution of $\mathcal{F}$ and F (left). Eigenvalue distribution of $\mathcal{F}$ and F around zero (right).

5. NUMERICAL APPLICATIONS

In this section we present reconstructions of impenetrable objects from a finite number of u_∞ measurements. In particular, synthetic data produced using Nyström method are used. Visualizations follow from equation (3.10) in a very natural way. The key fact in our approach is the utilization of the singular system of the numerically stable discretized form of the operator $\mathcal{F}$, in place of the singular system of F. In the sequel, along with domain reconstructions, we provide some eigenvalue plots for the discretized operators $\mathcal{F}$ and F respectively. In the numerical experiments below, a pointwise random noise is imposed on F in a similar way as in [1]. In particular, let N_R and N_I be random matrices such that the noisy far-field matrix F_δ is defined by

$$F_\delta = F + \epsilon\,(N_R + iN_I)F,$$

where ϵ is given, and the corresponding noise level is computed in the matrix spectral norm, $\|F_\delta - F\|_2 \leq \delta$.

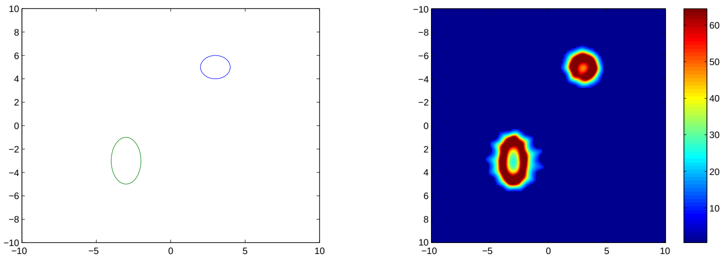

Figure 3: Visualization of a circle and a ellipse (left). Profile reconstruction via the AMG precondtitioner (right).

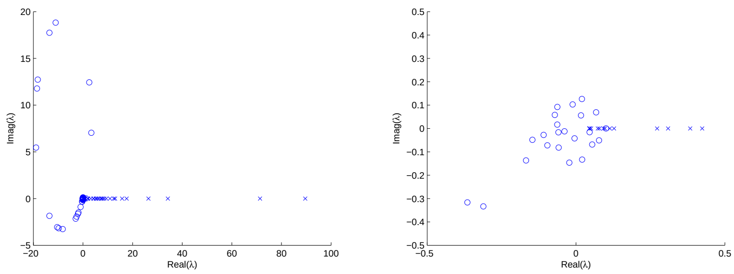

Figure 4: Eigenvalue distribution of $\mathcal{F}$ and F (left). Eigenvalue distribution of $\mathcal{F}$ and F around zero (right).

In the figures of eigenvalue distribustions, $'\times'$ and $'\circ'$ represent the eigenvalues of the discretized forms of the operators $\mathcal{F}$ and F respectively. It is observed that even in the presence of noisy far-field measurements the eigenvalues of $\mathcal{F}$ are located far away from zero, in contrast to the eigenvalues of F that are clustered around it.

In our first numerical experiment we are reconstructing the profile of a circle of radius 1. The far-field pattern has been synthetically produced via Nyström method in the case of 21 incident and observed directions with $k = 1$. The object is located in a grid of 63×63 points. Furthermore, a 1% noise is pointwise added to each element of the far-field matrix as described above. The left image in figure 1 shows the obstacle that we are about to reconstruct, whereas the right one shows the profile reconstructed via the AMG. Furthermore, the left image in figure 2 shows the eigenvalue distributions for the operators $\mathcal{F}$ and F respectively, and the right one captures their eigenvalues around zero. Notice that very few eigenvalues of $\mathcal{F}$ appear within a "small window" around zero, in contrast to the ones of F. This fact suggests that costly identification of the optimal value of the parameter

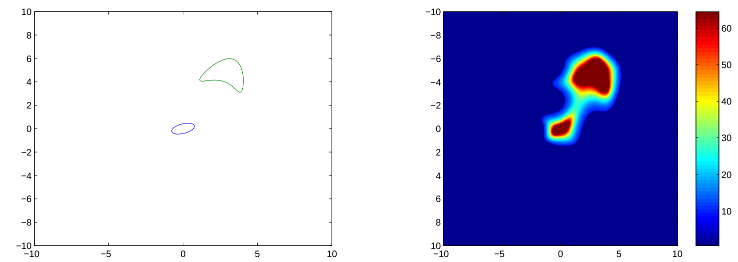

Figure 5: Visualization of a kite and a ellipse (left). Profile reconstruction via the AMG precondtitioner (right).

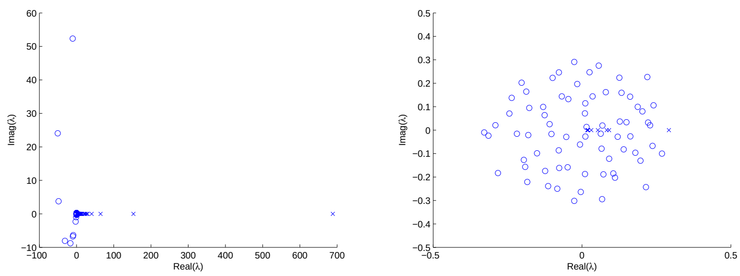

Figure 6: Eigenvalue distribution of $\mathcal{F}$ and F (left). Eigenvalue distribution of $\mathcal{F}$ and F around zero (right).

in Tikhonov's regularization can be avoided. In addition, the amount of noise in the data doesn't need to be known *a priori*.

In our second numerical experiment D is given by two disjoint obstacles $D = D_1 \cup D_2$, where D_1 is the vertical ellipse with axes 1 and 2 and D_2 is the circle of radius 1 and center at the point $(4, 5)$. The objects are located in a grid of 40×40 points. We took $k = 2$ and computed the far field pattern for 40 incident and observed directions. In addition 2% noise has been added to the far-field matrix. Figure 3 shows the original(left) and the reconstructed profile (right). The last image of figure 4 shows the eigenvalue distribution for the the operators $\mathcal{F}$ and F, respectively. Similarly as in the previous example the eigenvalues of $\mathcal{F}$ are moving far away from zero and the method yielded reliable reconstructions.

In our last numerical experiment we are dealing with an example considered in [9], were the profiles of a kite and an ellipse are considered. The ellipse has axes 0.8, 0.4 and center $(0, 0)$, while the kite is parametrized by $x(t) = (\cos t + 0.65\cos(2t) - 0.65, 1.5\sin t)$, $t \in [0, \pi]$, then rotated by 70 degrees and shifted to

center $(3,5)$. In this case the objects are located in a grid of 80×80 points, $k = 1$, and the far field pattern is computed for 80 incident and observed directions. In this example 5% noise has been added to the data. Original profiles as well as reconstructions and eigenvalue distibutions are shown in figures 5 and 6. It turns out that our reconstructions are comparable to the ones of [9], that obtained via Tikhonov regularization.

In all three numerical experiments the construction of the preconditioner took just a few seconds on a Pentium 4 at 3.4GHz CPU with 3 Gb of memory.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,465-482,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Asymptotic distribution of the sample average value-at-risk

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*Rachev gratefully acknowledges research support by grants from Division of Mathematical, Life and Physical Sciences, College of Letters and Science, University of California, Santa Barbara, the Deutschen Forschungsgemeinschaft and the Deutscher Akademischer Austausch Dienst.

Abstract

In this paper, we prove a result for the asymptotic distribution of the sample average value-at-risk (AVaR) under certain regularity assumptions. The asymptotic distribution can be used to derive asymptotic confidence intervals when $AVaR_\epsilon(X)$ is calculated by the Monte Carlo method which is adopted in many risk management systems. We study the effect of the tail behavior of the random variable X on the convergence rate and the improvement of a tail truncation method.

Keywords average value-at-risk, risk measures, heavy-tails, asymptotic distribution, Monte Carlo

1 Introduction

The average value-at-risk (AVaR) risk measure has been proposed in the literature as a coherent alternative to the industry standard Value-at-Risk (VaR), see Artzner et al. (1998) and Pflug (2000). It has been demonstrated that it has better properties than VaR for the purposes of risk management and, being a downside risk-measure, it is superior to the classical standard deviation and can be adopted in a portfolio optimization framework, see Rachev et al. (2006), Stoyanov et al. (2007), Biglova et al. (2004), and Rachev et al. (2008).

The AVaR of a random variable X at tail probability ϵ is defined as

$$AVaR_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F^{-1}(p)dp.$$

where $F^{-1}(x)$ is the inverse of the cumulative distribution function (c.d.f.) of the random variable X. The random variable may describe the return of stock, for example. A practical problem of computing portfolio AVaR is that usually we do not know explicitly the c.d.f. of portfolio returns. In order to solve this practical problem, the Monte Carlo method is employed. The returns of the portfolio constituents are simulated and then the returns of the portfolio are calculated. In effect, we have a sample from the portfolio return distribution which we can use to estimate AVaR. The larger the sample, the closer the estimate to the true value. However, with any finite sample, the sample AVaR will fluctuate about the true value and, having only a sample estimate, we have to know the probability distribution of the sample AVaR in order to build a confidence interval for the true value. The sample AVaR equals,

$$\widehat{AVaR}_\epsilon(X) = -\frac{1}{\epsilon}\int_0^\epsilon F_n^{-1}(p)dp.$$

where $F_n^{-1}(p)$ denotes the inverse of the sample c.d.f. $F_n(x) = \frac{1}{n}\sum_{i=1}^n I\{X_i \leq x\}$ in which $I\{A\}$ denotes the indicator function of the event A.

The problem of computing the distribution of the sample AVaR is a complicated one even if we know the distribution of X. From a practical viewpoint, X describes portfolio returns which can be a complicated function of the joint distribution of the risk drivers. Therefore, we can only rely on large sample theory to gain insight into the fluctuations of sample AVaR. That is, for a large n, we can use a limiting distribution to calculate a confidence interval.

In this paper, first we prove a limit theorem for the sample AVaR in Section 2 . The limit theorem does not give answers to the question of how many simulations are necessary in order for the limiting distribution to be acceptable as a model for practical purposes. This number depends also on the distribution of X. A major factor is the tail behavior of X and, more precisely, how heavy the left tail of the distribution is. We study this problem in Section 3.1 assuming that X has Student's t distribution. Finally, we illustrate the impact of a tail truncation method in a finite and infinite variance case.

2 A limit theorem

In this section, we prove the following limit theorem.

Theorem 1. *Suppose that X is random variable with finite second moment $EX^2 < \infty$. Furthermore, suppose that the c.d.f. of X is differentiable at $x = q_\epsilon$, where q_ϵ is the ϵ-quantile of X. Then, as $n \to \infty$,*

$$\frac{\sqrt{n}}{\sigma_\epsilon}\left(\widehat{AVaR}_\epsilon(X) - AVaR_\epsilon(X)\right) \xrightarrow{w} N(0,1) \tag{1}$$

where $\xrightarrow{w}$ denotes weak limit and

$$\sigma_\epsilon^2 = \frac{1}{\epsilon^2}D(\max(q_\epsilon - X, 0)). \tag{2}$$

Proof. We apply the following more general result,

$$\phi(F_n) - \phi(F) \xrightarrow{w} N(0, \lambda^2)$$

where ϕ is a differentiable functional, F_n is the empirical c.d.f., F is the c.d.f. of X, and

$$\lambda^2 = D(\phi'(\delta_{X_i} - F)) = \int_R (IF_\phi(x))^2 dF(x) < \infty$$

in which IF_ϕ stands for the influence function of the functional ϕ, δ_{X_i} is the cdf of the observation X_i. By the definition of the influence function,

$$\phi'(\delta_{X_i} - F) = \frac{d}{dt}(\phi((1-t)F + t\delta_{X_i}))|_{t=0} = \frac{d}{dt}(\phi(F_t))|_{t=0}.$$

The proof of the main result reduces to calculating the influence function of $\phi(F)$ and then calculating the variance λ^2. We need the assumed properties of the c.d.f. for the calculation of the influence function. In our case, from the definition of AVaR,

$$\begin{aligned}\phi(F) &= -\frac{1}{\epsilon}\int_0^\epsilon F^{-1}(p)dp \\ &= -F^{-1}(\epsilon) + \frac{1}{\epsilon}\int_{-\infty}^{F^{-1}(\epsilon)} F(p)dp.\end{aligned} \tag{3}$$

The influence function can be directly calculated,

$$\begin{aligned}IF_\phi(x) &= \frac{d}{dt}(\phi(F_t))|_{t=0} \\ &= -\frac{d}{dt}(F_t^{-1}(\epsilon))|_{t=0} + \frac{1}{\epsilon}\frac{d}{dt}\left(\int_{-\infty}^{F_t^{-1}(\epsilon)} F_t(p)dp\right)\Bigg|_{t=0}\end{aligned}$$

The second term is differentiated separately below

$$\frac{d}{dt}\left(\int_{-\infty}^{F_t^{-1}(\epsilon)} F_t(p)dp\right)\Bigg|_{t=0} = \epsilon\frac{d}{dt}(F_t^{-1}(\epsilon))|_{t=0} + \max(q_\epsilon - x, 0) - \int_{-\infty}^{q_\epsilon} F(y)dy$$

where where q_ϵ stands for the ϵ-quantile of X and we take advantage of the chain rule

$$\frac{d}{dt}\left(\int_a^{f(t)} G(t,y)dy\right) = G(t, f(t))f'(t) + \int_a^{f(t)} G_t(t,y)dy$$

in which $f(x)$ is a monotonically increasing function. In computing the derivative we used that $F(x)$ is differentiable at $x = q_\epsilon$. Finally, for the influence function we obtain

$$IF_\phi(x) = \frac{1}{\epsilon}\max(q_\epsilon - x, 0) - \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} F(y)dy$$

Now we can calculate the variance,

$$\lambda^2 = D(IF_\phi(X)) = \frac{1}{\epsilon^2}D(\max(q_\epsilon - X, 0)).$$

It is also straightforward to check that $E(IF_\phi(X)) = 0$,

$$E(IF_\phi(X)) = \frac{1}{\epsilon}E\max(q_\epsilon - X, 0) - \frac{1}{\epsilon}\int_0^\epsilon p dF^{-1}(p) = 0$$

follows after integration by parts. □

The variance of the asymptotic normal distribution is not possible to calculate if we do not know the cdf $F(x)$ of X. Therefore, if we have only a sample of i.i.d. observations, the variance σ^2 has to be estimated. To this end, expressing the variance in terms of conditional moments may be more useful. The variance of the asymptotic normal distribution given in (2) equals

$$\sigma_\epsilon^2 = \frac{q_\epsilon^2}{\epsilon} - \frac{2q_\epsilon}{\epsilon}E(X|X \leq q_\epsilon) + \frac{1}{\epsilon}E(X^2|X \leq q_\epsilon) - (q_\epsilon - E(X|X \leq q_\epsilon))^2 \quad (4)$$

An estimate of σ_ϵ^2 can be obtained by estimating the conditional moments and the corresponding quantile from the sample.

Furthermore, we would like to remark on a consistency with the classical theory behind constructing confidence intervals for the mean of a random variable. Suppose that the tail probability approaches one. In this case, the AVaR turns into the mean of X,

$$\lim_{\epsilon \to 1} AVaR_\epsilon(X) = EX,$$

the sample AVaR turns into the sample average,

$$\lim_{\epsilon \to 1} \widehat{AVaR}_\epsilon(X) = \frac{1}{n}\sum_{i=1}^n X_i,$$

where $X_1, \ldots, X_n$ is a sample if i.i.d. observations, and the variance of the asymptotic normal distribution becomes the variance of X,

$$\lim_{\epsilon \to 1} \sigma_\epsilon = DX.$$

Therefore, we obtain as a special case the classical CLT

$$\frac{\sqrt{n}}{\sqrt{DX}} \left(\frac{1}{n} \sum_{i=1}^{n} X_i - EX \right) \xrightarrow{w} N(0,1).$$

3 Monte Carlo experiments

In this section, our goal is to investigate the effect of the tail behavior on the rate of convergence in (1). We are also interested in the question if tail truncation improves the convergence and by how much. Generally, the tail truncation method consists in “replacing” the tails of X with the tails of a thin-tailed distribution “far away” from the center of the distribution of X, for example beyond the 0.1% and 99.9% quantiles. The tail truncation method has applications in finance for modeling the distribution of stock returns, a practical reason being that stock exchanges close if a severe market crash occurs. This method also has application in derivatives pricing with a heavy-tailed distributional assumption for the return of the underlying, see Rachev et al. (2005) and the references therein.

In the following sections, we start with Student's t distribution and investigate the convergence rate in the limit relation (1) as degrees of freedom increase. We address the same questions with a truncated Student's t distribution in which the truncation is done in the simplest possible way — we set the the values of the random variable which are beyond the 0.1% and 99.9% quantiles to be equal to the corresponding quantiles. As a result, small point masses appear at the 0.1% and 99.9% quantiles. We also focus on the class of stable distributions and truncated stable distributions in which the same truncation technique is adopted as in the case of Student's t distribution.

3.1 The effect of tail thickness

The impact of the tail behavior on the rate of convergence in Theorem 1 is first studied when X has Student's t distribution, $X \in t(\nu)$, with $\nu \geq 3$. We need the condition on the degrees of freedom in order for the random variable to have finite variance. Taking advantage of the expression for the density,

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\sqrt{\nu\pi}} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \ x \in \mathbb{R},$$

it is possible to compute explicitly the variance in equation (2). In fact, for this purpose the expression in (4) is more appropriate. As a first step, we calculate the two conditional expectations.

$$\begin{aligned}
E(X|X \leq q_\epsilon) &= \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} x f_X(x)dx \\
&= \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} x\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\
&= \frac{1}{\epsilon}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{\nu}{2}\int_{-\infty}^{q_\epsilon}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} d\left(1+\frac{x^2}{\nu}\right) \\
&= -\frac{1}{\epsilon}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{\sqrt{\nu}}{(\nu-1)\sqrt{\pi}}\left(1+\frac{q_\epsilon^2}{\nu}\right)^{\frac{1-\nu}{2}}, \text{ if } \nu > 1.
\end{aligned} \tag{5}$$

$$\begin{aligned}
E(X^2|X \leq q_\epsilon) &= \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} x^2 f_X(x)dx \\
&= \frac{1}{\epsilon}\int_{-\infty}^{q_\epsilon} x^2\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \\
&= \frac{1}{\epsilon}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\frac{1}{\sqrt{\nu\pi}}\frac{\nu}{1-\nu}\int_{-\infty}^{q_\epsilon} x d\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}+1} \\
&= q_\epsilon E(X|X \leq q_\epsilon) + \frac{\nu}{\epsilon(\nu-2)}F_{\nu-2}\left(q_\epsilon\sqrt{\frac{\nu-2}{\nu}}\right), \text{ if } \nu > 2.
\end{aligned} \tag{6}$$

where the last equality follows by integration by parts and $F_\nu(x)$ is the c.d.f. of Student's t distribution with ν degrees of freedom. Plugging these expressions in (4), we obtain the expression for the variance σ_ϵ^2.

Note that, besides an equation for σ_ϵ^2, we can explicitly calculate the AVaR of X since in the case of Student's t distribution we can express AVaR as a conditional expectation,

$$AVaR_\epsilon(X) = -E(X|X \leq q_\epsilon).$$

Having an expression for the variance allows us to use the test of Kolmogorov and address the question oh how many simulations are needed in order to accept the hypothesis that the distribution of the random variable in the left-hand side of the limit relation (1),

ν	$\epsilon = 0.01$	$\epsilon = 0.05$
3	70000	17000
4	60000	9000
5	50000	7000
6	23000	4500
7	14000	4200
8	13000	4100
9	12000	4000
10	12000	3900
15	11000	3850
25	10000	3800
50	10000	3750
∞	10000	3300

Table I: The number of observations sufficient to accept the normal distribution as an approximate model for different values of ν and ϵ.

$$\frac{\sqrt{n}}{\sigma_\epsilon}\left(\widehat{AVaR}_\epsilon(X) - AVaR_\epsilon(X)\right), \tag{7}$$

is standard normal. If we accept the null hypothesis for a given value of n, then the standard normal distribution can be used as an approximate model and we can calculate not only confidence intervals but also other characteristics based on it.

Table I shows the values of n sufficient to accept the null hypothesis in the test of Kolmogorov for different degrees of freedom and tail probabilities. We chose $\epsilon = 0.01$ and $\epsilon = 0.05$ since these values are frequently used in financial industry in value-at-risk estimation. The numbers in the table are calculated by generating independently 2000 samples of a given size and then from each sample (7) is estimated. In effect, we obtain 2000 observations from the distribution of (7).

In line with intuition, the numbers Table I indicate that when the tail is heavier, we need a larger sample in order for the asymptotic law to be sufficiently close to the distribution of (7) in terms of the Kolmogorov metric. Another expected conclusion is that as the tail probability increases, a smaller sample turns out to be sufficient.

In Table II, we calculated the 95% confidence interval for AVaR when the sample size changes from 250 to 10000 observations. We generated 2000 independent samples and then computed the quantity in equation (7). Thus, the 95% confidence intervals are obtained from 2000 observations of the random variable in (7). As n increases, the two quantiles approach the corresponding quantiles of the standard normal distribution. Note that the largest

	$n=250$		$n=500$		$n=1000$		$n=5000$		$n=10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.110	2.011	-1.257	2.173	-1.352	2.202	-1.633	2.037	-1.664	2.007
4	-1.337	2.144	-1.442	2.229	-1.543	2.082	-1.744	2.230	-1.756	2.176
5	-1.441	2.153	-1.529	2.224	-1.728	2.190	-1.843	2.060	-1.807	2.009
6	-1.522	2.134	-1.618	2.033	-1.701	2.115	-1.848	1.987	-1.955	1.982
7	-1.627	2.050	-1.668	1.975	-1.827	2.043	-1.841	2.048	-1.913	2.014
8	-1.655	2.028	-1.760	2.145	-1.836	2.032	-1.898	2.034	-1.866	1.939
9	-1.720	1.938	-1.753	2.146	-1.798	2.075	-1.866	2.005	-1.905	2.007
10	-1.747	1.925	-1.809	1.980	-1.762	2.078	-1.822	1.950	-1.962	2.000
15	-1.813	1.751	-1.848	1.896	-1.891	1.956	-1.969	1.941	-1.968	1.873
25	-1.848	1.760	-1.933	2.028	-1.897	1.950	-1.939	1.957	-1.899	1.923
50	-1.898	1.948	-1.962	1.900	-1.971	1.973	-1.961	1.914	-1.895	1.948
∞	-1.921	1.761	-1.976	1.920	-1.964	1.822	-1.869	1.907	-2.004	1.937

Table II: The 95% confidence bounds generated from 2000 simulations from the distribution of (7) with $\epsilon = 0.01$. The corresponding quantiles of $N(0,1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

$n = 10000$ is generally below the sample sizes for $\epsilon = 0.01$ given in Table I. Nevertheless, the relative discrepancies between the quantiles given in Table II and the corresponding standard normal distribution quantiles are less than 5% for $\nu \geq 6$.[1] The relative discrepancies between the quantiles given in Table III the corresponding standard normal distribution quantiles for $n = 10000$ have the same magnitude. However, in this case $n = 10000$ is well above the sample sizes given in Table I for $\epsilon = 0.05$. As a result, we can conclude that even smaller samples than the ones given in Table I can lead to 95% confidence intervals obtained via resampling from (7) being close to the corresponding 95% confidence interval obtained from the limit distribution even though the Kolmogorov test fails for such samples. For instance, the relative deviation between the quantiles given in Table II for $n = 5000$ and the corresponding standard normal distribution quantiles are below 7% for $n \geq 6$, which is a small deviation for all practical purposes.

As a result of this analysis, we can conclude that for the purposes of building confidence intervals for $AVaR_\epsilon(X)$ when $X \in t(\nu)$, with $\nu \geq 6$ and $\epsilon = 0.01, 0.05$, we can safely employ the asymptotic law when the sample size we use for AVaR estimation contains more than 5000 observations. If Student's t distribution is fitted on daily stock-returns time series, such values

[1]If we generate a sample of 2000 observations from the standard normal distribution, a relative deviation below 6% between the estimated quantile $q_{2.5\%}$ and the corresponding standard normal quantile happens with about 95% probability, and below 7.7% with about 99% probability.

	$n = 250$		$n = 500$		$n = 1000$		$n = 5000$		$n = 10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.422	2.110	-1.543	2.016	-1.549	1.981	-1.725	1.947	-1.883	1.987
4	-1.647	2.169	-1.737	2.235	-1.787	2.171	-1.900	2.226	-1.849	2.115
5	-1.749	2.081	-1.811	2.096	-1.757	2.148	-1.868	2.015	-1.937	2.100
6	-1.810	2.071	-1.896	2.030	-1.921	1.941	-1.958	1.998	-1.886	2.032
7	-1.786	2.215	-1.824	1.990	-1.809	2.086	-1.986	2.030	-1.916	2.015
8	-1.932	2.131	-1.870	2.058	-1.755	2.090	-1.937	2.014	-1.915	1.952
9	-1.848	2.139	-1.884	2.081	-1.930	2.023	-1.995	1.964	-1.863	2.048
10	-1.906	2.103	-2.021	1.966	-1.839	2.087	-2.009	1.930	-1.989	1.995
15	-1.797	1.905	-1.929	2.056	-1.944	1.952	-1.924	1.973	-1.947	1.979
25	-1.958	1.950	-1.994	1.956	-1.939	1.968	-2.085	1.993	-1.894	1.944
50	-1.986	1.927	-1.980	1.823	-1.962	1.883	-1.911	1.969	-2.002	1.935
∞	-2.013	1.828	-1.953	1.869	-1.975	1.893	-2.034	1.958	-1.903	1.944

Table III: The 95% confidence bounds generated from 2000 simulations from the distribution of (7) with $\epsilon = 0.05$. The corresponding quantiles of $N(0, 1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

for ν are very common.

Figure 1 illustrates the differences in the convergence rate when X has Student's t distribution with $\nu = 3$, which corresponds to heavier tails, and $\nu = 10$. Since high degrees of freedom imply more light tails, smaller samples are sufficient for the density of (7) to be closer to the standard normal density.

3.2 The effect of tail truncation

The stochastic stability of sample AVaR increases dramatically after tail truncation. In this section, we repeat the calculations from the previous section but when X has Student's t distribution with the tails truncated at $q_{0.1\%}$ and $q_{99.9\%}$ quantiles. The random variable Y is said to have a truncated distribution at these quantiles if it has the representation

$$Y = XI\{q_{0.1\%} \leq X \leq q_{99.9\%}\} + q_{0.1\%}I\{X < q_{0.1\%}\} + q_{99.9\%}I\{X > q_{99.9\%}\}$$

in which $X \in t(\nu)$, $I\{A\}$ denotes the indicator of the event A, and $q_{0.1\%}$, $q_{99.9\%}$ are the corresponding quantiles of X. The tail truncation introduces small point masses at the two quantile levels.

The two conditional expectations in (4) can be related to the corresponding conditional expectations of X. In the following, we assume that the tail probability ϵ is larger from the tail probability of the left truncation point,

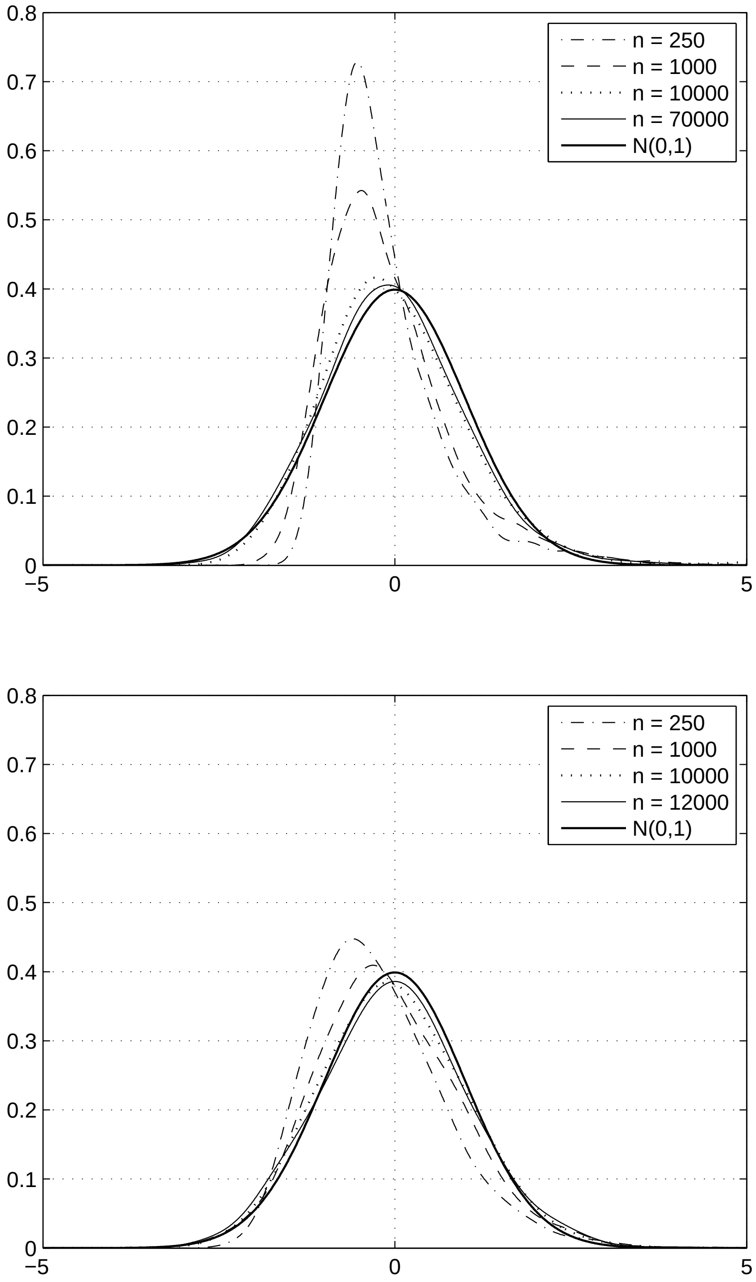

Figure 1: The density of (7) approaching the $N(0,1)$ density as the sample size increases with $\nu = 3$ (top) and $\nu = 10$ (bottom).

ν	$\epsilon = 0.01$	$\epsilon = 0.05$
3	12000	4000
4	11500	3600
5	11000	3300
6	11000	3200
7	10500	3100
8	10000	3000
9	10000	3000
10	10000	3000
15	10000	2950
25	10000	2900
50	10000	2900
∞	10000	2900

Table IV: The number of observations sufficient to accept the normal distribution as an approximate model for different values of ν and ϵ.

$\epsilon > 0.001$. Under this assumption, the ϵ-quantile of X is the same as the ϵ-quantile of Y.

$$E(Y|Y \leq q_\epsilon) = E(X|X \leq q_\epsilon) - \frac{0.001}{\epsilon} E(X|X \leq q_{0.1\%}) + \frac{0.001 q_\epsilon}{\epsilon}$$

$$E(Y^2|Y \leq q_\epsilon) = E(X^2|X \leq q_\epsilon) - \frac{0.001}{\epsilon} E(X^2|X \leq q_{0.1\%}) + \frac{0.001 q_\epsilon^2}{\epsilon}$$

in which the conditional expectations of X can be computed according to formulae (5) and (6). Plugging the expressions for the conditional expectations of Y in the expression for σ_ϵ^2, we obtain the variance of the asymptotic distribution. Furthermore, the tail truncation does not break the link between AVaR and the conditional expectation, therefore

$$AVaR_\epsilon(Y) = -E(Y|Y \leq q_\epsilon).$$

In the following, we investigate the convergence rate of

$$\frac{\sqrt{n}}{\sigma_\epsilon}\left(\widehat{AVaR}_\epsilon(Y) - AVaR_\epsilon(Y)\right), \tag{8}$$

for different degrees of freedom to the standard normal distribution and we compare the results to the ones in the previous section.

Table IV is the counterpart of Table I for the truncated distribution. It is impressive how the sample size sufficient to accept the null hypothesis in the Kolmogorov test decreases after tail truncation. The most dramatic change

	$n = 250$		$n = 500$		$n = 1000$		$n = 5000$		$n = 10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.723	1.699	-1.847	1.932	-1.850	1.958	-1.966	1.921	-1.860	1.936
4	-1.759	1.694	-1.863	1.819	-1.903	1.860	-1.989	1.942	-1.964	1.886
5	-1.808	1.536	-1.884	1.871	-1.926	1.932	-1.961	1.964	-1.782	2.066
6	-1.947	1.565	-1.937	1.759	-2.002	1.734	-2.057	1.946	-1.981	1.958
7	-1.960	1.524	-1.960	1.666	-1.965	1.844	-2.101	1.932	-1.981	1.927
8	-2.002	1.567	-2.015	1.693	-1.903	1.802	-1.952	1.856	-1.917	1.928
9	-1.963	1.552	-2.030	1.748	-2.106	1.779	-1.965	2.026	-1.932	1.938
10	-2.003	1.596	-2.119	1.709	-2.034	1.850	-1.925	1.813	-1.990	1.952
15	-2.090	1.485	-2.159	1.650	-2.065	1.786	-1.983	1.847	-2.035	1.855
25	-2.183	1.502	-2.084	1.578	-2.093	1.747	-2.016	1.806	-1.954	1.877
50	-2.272	1.509	-2.089	1.632	-2.042	1.726	-1.938	1.914	-2.056	1.970

Table V: The 95% confidence bounds generated from 2000 simulations from the distribution of (8) with $\epsilon = 0.01$. The corresponding quantiles of $N(0, 1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

is in the case $\nu = 3$. Now we need only 12000 observations compared to 70000 in the non-truncated case.

Tables V and VI are the counterparts of Tables II and III. The relative deviation of the quantiles $q_{2.5\%}$ and $q_{97.5\%}$ of the random variable in (8) from those of the standard normal distribution are below 7% for all degrees of freedom and $n = 10000$, and, with a few exceptions, for $n = 5000$. Compare Figure 2 and the top plot in Figure 1 for an illustration of the improvement in the convergence rate. These results indicate that the asymptotic distribution can be used to obtain a 95% confidence bound for the sample AVaR for all degrees of freedom if the sample size contains more than 5000 observations.

3.3 Infinite variance distributions

A critical assumption behind the limit result in Theorem 1 is the finite variance of X. To be more precise, the condition of finite variance can be loosened to finite downside semi-variance,

$$D \max(-X, 0) < \infty,$$

because it is the behavior of the left tail which is important. As a consequence, the sample AVaR of distributions with infinite variance, but finite downside semi-variance, may still follow Theorem 1.

However, there are infinite variance distributions for which

$$D \max(-X, 0) = \infty$$

	$n = 250$		$n = 500$		$n = 1000$		$n = 5000$		$n = 10000$	
ν	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$	$q_{2.5\%}$	$q_{97.5\%}$
3	-1.815	2.116	-1.866	2.041	-1.939	2.018	-1.944	1.975	-2.045	1.874
4	-1.756	2.150	-1.811	2.073	-2.052	2.060	-1.923	1.973	-1.922	1.854
5	-1.820	1.954	-1.971	2.032	-1.916	2.036	-1.826	1.960	-1.941	1.883
6	-1.899	2.089	-1.981	2.036	-2.012	2.012	-1.955	1.933	-1.921	2.011
7	-2.001	2.032	-1.921	1.997	-1.949	1.980	-1.980	1.936	-2.016	1.915
8	-1.888	1.995	-1.922	2.050	-1.907	1.917	-1.942	1.911	-1.910	1.903
9	-2.017	2.003	-1.892	1.918	-1.899	2.017	-1.931	2.001	-2.009	1.967
10	-1.928	1.814	-1.992	1.960	-1.870	1.949	-1.845	2.076	-1.992	1.898
15	-2.059	1.983	-2.020	2.007	-1.961	1.922	-1.953	1.870	-1.936	1.874
25	-1.999	1.854	-2.038	1.945	-1.889	2.028	-2.031	1.916	-1.975	1.890
50	-1.960	1.898	-2.028	1.898	-1.947	1.906	-2.015	2.002	-1.959	1.911

Table VI: The 95% confidence bounds generated from 2000 simulations from the distribution of (8) with $\epsilon = 0.05$. The corresponding quantiles of $N(0,1)$ are $q_{2.5\%} = -1.96$ and $q_{2.5\%} = 1.96$.

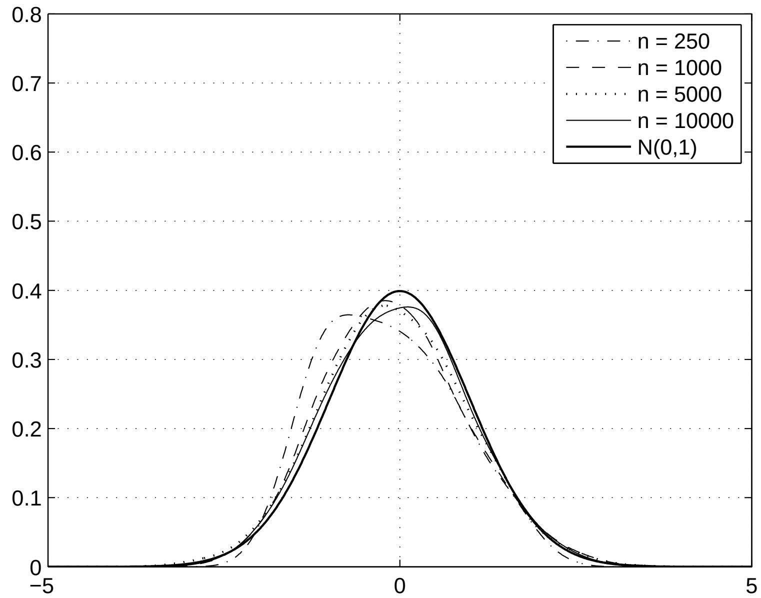

Figure 2: The density of (8) approaching the $N(0,1)$ density as the sample size increases with $\nu = 3$ and $\epsilon = 0.01$.

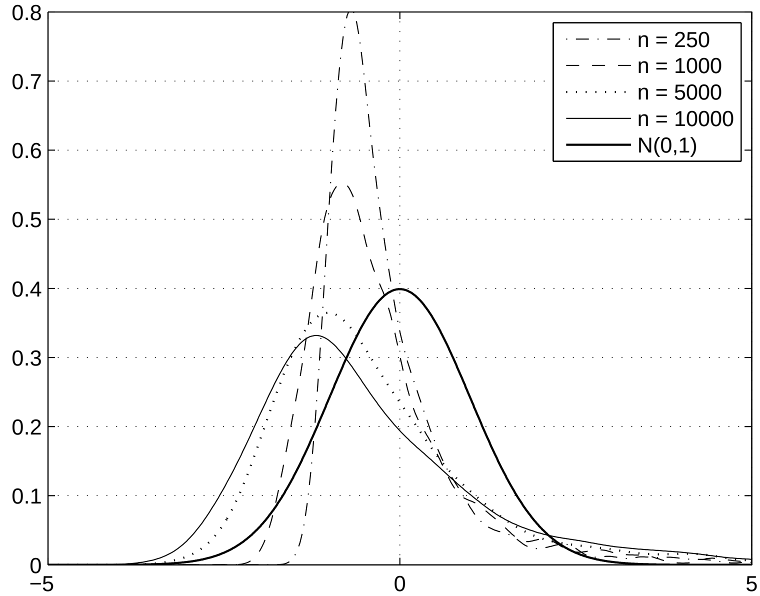

Figure 3: Lack of convergence, X has a stable distribution with $X \in S_{1.5}(1, 0, 0)$ and $\epsilon = 0.05$.

and, therefore, the limit result in Theorem 1 does not hold for them. Such is the class of stable distributions which arises from generalizations of the Central Limit Theorem and has been proposed as a model for stock return distributions, see Rachev and Mittnik (2000).

Stable distributions are introduced by their characteristic functions. X is said to have a stable distribution if its characteristic function is

$$\varphi(t) = Ee^{itX} = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta \frac{t}{|t|} \tan(\frac{\pi\alpha}{2})) + i\mu t\}, & \alpha \neq 1 \\ \exp\{-\sigma |t| (1 + i\beta \frac{2}{\pi} \frac{t}{|t|} \ln(|t|)) + i\mu t\}, & \alpha = 1 \end{cases}$$

Except for a couple of representatives, generally no closed-form expressions for their densities and c.d.f.s are known. If $\alpha < 2$, then X has infinite variance. If $1 < \alpha \leq 2$, then X has finite mean and the AVaR of X can be calculated. In our calculations, we will use the semi-analytic formula in Stoyanov et al. (2006).

Even though we know that Theorem 1 does not hold for a stable distribution with $\alpha < 2$, we simulate 2000 draws from the random variable

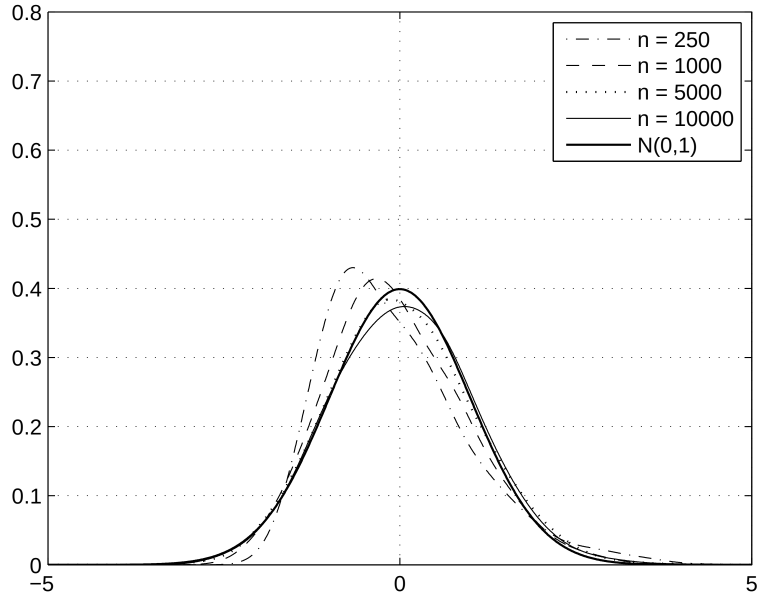

Figure 4: After tail truncation at $q_{0.1\%}$ and $q_{99.9\%}$, there is a fast convergence to $N(0,1)$, $\alpha = 1.5$ and $\epsilon = 0.05$.

in equation (7) in which σ_ϵ is estimated from a generated sample by estimating the corresponding conditional moments. In theory these the second conditional moment explodes but for any finite sample its estimate is a finite number. Our goal is to see what happens when Theorem 1 does not hold. Figure 3 illustrates such a divergent case in which $\alpha = 1.5$ and $\epsilon = 0.05$. The lack of convergence is quite obvious.

Stable distributions with $\alpha < 2$ in combination with a tail truncation method have been proposed as a model for the returns of the underlying in derivatives pricing. It is interesting to see how much the simple truncation technique we applied in the previous section can change Figure 3. With its tails truncated according to our simple method, the random variable becomes with a bounded support and, therefore, it has finite variance. As a consequence, Theorem 1 holds. Figure 4 illustrates this change. We observe a quick convergence rate, similar to the one illustrated in Figure 2 for Student's t distribution.

4 Conclusion

In this paper, we study the asymptotic distribution of sample AVaR. Under certain regularity conditions, we prove a limit theorem in which the limiting distribution is the normal distribution. We study how the convergence rate in the limit theorem is influenced by the tail behavior of the random variable. An expected result is that, other things equal, more observations are needed when the tail is heavier. We find out that a simple tail truncation method improves dramatically the convergence rate. As a consequence, the asymptotic distribution is reliable for confidence interval calculations when the number of simulations is more than 5000 if the random variable has a truncated Student's t distribution.

We also consider an infinite variance case in which the random variable as a stable distribution with finite mean. We illustrate the lack of convergence and demonstrate the improvement due to tail truncation at high quantiles.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,483-493,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

A note on generalized twisted q-Euler numbers and polynomials

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Abstract In this paper we construct a new generalized twisted q-Euler polynomials and generalized twisted q-Euler numbers attached to χ. We investigate some properties which are related to the generalized twisted q-Euler Polynomials. We also derive the existence of a specific interpolation function which interpolate the generalized twisted q- Euler polynomials at negative integer.

2000 Mathematics Subject Classification - 11B68, 11S40

Key words- Euler numbers, Euler polynomials, Generalized Euler numbers, Generalized Euler polynomials, Euler numbers, Euler polynomials, Generalized twisted q-Euler numbers, Generalized twisted q-Euler polynomials

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

$$[x]_q = [x : q] = \frac{1-q^x}{1-q}, \text{ cf. } [1, 2, 3, 4, 5, 9, 11] .$$

*This work was supported by Hannam University Research Fund, 2007

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. For $g \in UD(\mathbb{Z}_p, \mathbb{C}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function$\}$, the p-adic q-integral (or q-Volkenborn integration) was defined by [3]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x g(x).$$

Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned} X &= X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on X, cf.[1, 2, 3, 4, 6, 7, 8, 9]. For $g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \int_X g(x) d\mu_1(x), \text{ cf. [4, 5].}$$

The Euler numbers E_n are usually defined by means of the following generating function:

$$e^{Et} = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \text{ cf. [8, 9, 10]}$$

where the symbol E_n is interpreted to mean that E^n must be replaced by E_n when we expand the one on the left. These numbers are classical and important in mathematics and in various places like analysis, number theory. Frobenius extended such numbers as E_n to the so-called Frobenius-Euler numbers $H_n(u)$ belonging to an algebraic number u with $|u| > 1$. Let u be an algebraic number. For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler numbers $H_n(u)$ belonging to u are defined by the generating function

$$e^{H(u)t} = \frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \text{ cf. [1,4,9,10],}$$

with the usual convention of symbolically replacing H^n by H_n. The Euler polynomials $E_n(x)$ are defined by

$$e^{E(x)t} = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \text{ cf. [3,4,5,8,9,10].}$$

For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler polynomials $H_n(u,x)$ belonging to u are defined by

$$e^{H(u,x)t} = \frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(u,x) \frac{t^n}{n!}, \text{ cf. [8, 9, 10].}$$

T. Kim[2] gave relation between $B_{n,w}$ and $H_n(u)$, nth Euler numbers as follows:

$$B_{n,w} = \frac{n}{w-1} H_{n-1}(w^{-1}) \text{ if } w \neq 1.$$

In [2], Kim defined the locally constant function as follows: Let

$$T_p = \cup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},$$

where $C_{p^m} = \{w | w^{p^m} = 1\}$ is the cyclic group of order p^m. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$. If we take $g(x) = \phi_w(x) e^{tx}$, then we easily see that

$$\int_{\mathbb{Z}_p} \phi_w(x) e^{tx} d\mu_1(x) = \frac{t}{we^t - 1}.$$

T.Kim [2] treated analogue of Bernoulli numbers, which is called twisted Bernoulli numbers. We define the twisted Bernoulli polynomials $B_{n,w}(x)$

$$e^{xt} \frac{t}{we^t - 1} = \sum_{n=0}^{\infty} B_{n,w}(x) \frac{t^n}{n!}.$$

By using Taylor series of e^{tx} in the above equation, we obtain

$$\int_{\mathbb{Z}_p} x^n \phi_w(x) d\mu_1(x) = B_{n,w},$$

where $B_{n,w} = B_{n,w}(0)$.

Now, we consider the case $q \in (-1, 0)$ corresponding to q-deformed fermionic certain and annihilation operators and the literature given therein [3,4,5,7,8]. The expression for the $I_q(g)$ remains same, so it is tempting to consider the limit $q \to -1$. That is,

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \leq x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

From (1.2), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l),$$

where $g_n(x) = g(x+n)$. Ryoo and Kim[9] treated analogue of Euler numbers, which is called twisted Euler numbers. We can consider twisted Euler numbers. If we take $g(z) = \phi_w(z)e^{tz}, (w \in T_p)$ in (1.2), then we have

$$I_{-1}(\phi_w(z)e^{tz}) = \frac{2}{we^t+1} = \sum_{n=0}^{\infty} E_{n,w}\frac{t^n}{n!}.$$

We define twisted Euler numbers $E_{n,w}$ as follows:

$$F_w(t) = \frac{2}{we^t+1} = \sum_{n=0}^{\infty} E_{n,w}\frac{t^n}{n!}.$$

Twisted Euler polynomials $E_{n,w}(z)$ are defined by means of the generating function

$$F_w(t,z) = \frac{2}{we^t+1}e^{zt} = I_{-1}(\phi_w(x)e^{t(z+x)}) = \sum_{n=0}^{\infty} E_{n,w}(z)\frac{t^n}{n!},$$

where $E_{n,w}(0) = E_{n,w}$. Let χ be the Dirichlet character with conductor $f(\ =\text{odd}) \in \mathbb{N}$. Ryoo, Kim and Jang[8] studied on the generalized Euler numbers and polynomials. The generalized Euler numbers associated with χ, $E_{n,\chi}$, was defined by means of the generating function

$$F_\chi(t) = \frac{2\sum_{a=0}^{f-1}\chi(a)(-1)^a e^{at}}{e^{ft}+1} = \sum_{n=0}^{\infty} E_{n,\chi}\frac{t^n}{n!}.$$

Generalized Euler polynomials $E_{n,\chi}(x)$, was also defined by means of the generating function

$$F_\chi(t,z) = \frac{2\sum_{a=0}^{f-1}\chi(a)(-1)^a e^{at}}{e^{ft}+1}e^{zt} = \sum_{n=0}^{\infty} E_{n,\chi}(z)\frac{t^n}{n!}.$$

Substituting $g(x) = \chi(x)\phi_w(x)e^{tx}$ into (1.2), then the generalized twisted Euler numbers $E_{n,\chi,w}$ are defined by means of the generating functions

$$\begin{aligned} F_{\chi,w}(t) &= \int_X \phi_w(x)e^{tx}\chi(x)d\mu_{-1}(x) \\ &= \frac{2\sum_{a=0}^{f-1} e^{ta}(-1)^a\chi(a)\phi_w(a)}{\phi_w(f)e^{ft}+1} = \sum_{n=0}^{\infty} E_{n,\chi,w}\frac{t^n}{n!} \end{aligned}$$

By using the above equation, $E_{n,\chi,w}$ are defined by

$$E_{n,\chi,w} = \int_X \phi_w(x)x^n\chi(x)d\mu_{-1}(x).$$

Generalized twisted Euler polynomials $E_{n,\chi,w}(z)$ are defined by

$$\begin{aligned} F_{\chi,w}(t,z) = F_{\chi,w}(t)e^{zt} &= \int_X \phi_w(x)e^{tx}\chi(x)d\mu_{-1}(x)e^{tz} \\ &= \left(\frac{2\sum_{a=0}^{f-1} e^{ta}(-1)^a\chi(a)\phi_w(a)}{\phi_w(f)e^{ft}+1}\right)e^{zt} = \sum_{n=0}^{\infty} E_{n,\chi,w}(z)\frac{t^n}{n!}. \end{aligned}$$

We set

$$F_{\chi,w}(t,z) = \frac{2\sum_{a=0}^{f-1}(-1)^a\chi(a)\phi_w(a)e^{(a+z)t}}{\phi_w(f)e^{ft}+1}.$$

Observe that if $w \to 1$, then $F_{\chi,w}(t,z) \to F_\chi(t,z)$ and $E_{n,\chi,w}(z) \to E_{n,\chi}(z)$. Ryoo and Kim [9]studied on the twisted Euler zeta function and twisted Hurwitz Euler zeta function. We gave the relation between twisted Euler numbers and twisted l-functions at non-positive integers. Observe that if $\chi = \chi^0$, then $F_{\chi,w}(t,z) \to F_w(t,z)$ and $E_{n,\chi,w}(z) \to E_{n,w}(z)$. T. Kim defined the generalized q-Euler numbers as follows: The generalized q-Euler numbers associated with χ, $E_{n,\chi,q}$, are defined by

$$E_{n,\chi,q} = \int_X \chi(x)[x]_q^n d\mu_{-q}(x).$$

We give the generating function of q-Euler numbers as follows:

$$F_{\chi,q}(t) = [2]_q \sum_{l=0}^{\infty} \chi(l)q^l(-1)^l e^{[l]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!}. \tag{1.5}$$

C.S.Ryoo [10] introduced the generalized q-Euler polynomials associated with χ. The generalized q-Euler polynomials associated with χ, $E_{n,\chi,q}(z)$, are defined by

$$E_{n,\chi,q}(z) = \int_X \chi(x)[z+x]_q^n d\mu_{-q}(x).$$

Hence we give the generating function of q-Euler polynomials as follows:

$$F_{\chi,q}(t,z) = [2]_q \sum_{l=0}^{\infty} \chi(l)q^l(-1)^l e^{[z+l]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,q}(z)\frac{t^n}{n!}.$$

Observe that if $q \to 1$, then $F_{\chi,q}(t,z) \to F_\chi(t,z)$ and $E_{n,\chi,q}(z) \to E_{n,\chi}(z)$. T. Kim and C.S. Ryoo studied the generalized q-Euler numbers and q-Euler polynomials and derived a Dirichlet's type l-series which interpolates the generalized q-Euler polynomials $E_{n,\chi,q}(z)$.

The purpose of this paper is to construct the generalized twisted q-Euler polynomials $E_{n,\chi,w,q}(z)$ attached to χ and derive a new Dirichlet's type l-series which interpolates the generalized twisted q-Euler polynomials $E_{n,\chi,w,q}(z)$.

2. Generalized twisted q-Euler numbers and polynomials

In this section, we introduce the generalized twisted q-Euler numbers and q-Euler polynomials. These numbers will be used to prove the analytic continuation of the q-l-series. Let χ be the Dirichlet character with conductor $f(=\text{odd}) \in \mathbb{N}$. Then the generalized twisted q-Euler numbers associated with χ, $E_{n,\chi,w,q}$, are defined by

$$E_{n,\chi,w,q} = \int_X \chi(x)\phi_w(x)[x]_q^n d\mu_{-q}(x). \tag{2.1}$$

By using p-adic q-integral, we obtain,

$$E_{n,\chi,w,q} = \frac{[f]_q^n}{[f]_{-q}} \sum_{a=0}^{f-1} \chi(a)(-1)^a q^a \phi_w(a) E_{n,w^f,q^f}\left(\frac{a}{f}\right).$$

Note that

$$\begin{aligned}
&\int_X \chi(x)\phi_w(x)[x]_q^n d\mu_{-q}(x) \\
&= [2]_q \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{al} \frac{1}{1+\phi_w(f)q^{f(1+l)}}.
\end{aligned}$$

Hence we have

$$E_{n,\chi,w,q} = [2]_q \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{al} \frac{1}{1+\phi_w(f)q^{f(1+l)}}.$$

By simple calculation, we obtain

$$\begin{aligned}
&[2]_q \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{al} \frac{1}{1+\phi_w(f)q^{f(1+l)}} \\
&= [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(l)(-1)^l [l]_q^n.
\end{aligned}$$

We have the following theorem.

Theorem 2. Let χ be a primitive Dirichlet character of conductor $f(=\text{odd}) \in \mathbb{N}$, we have

$$E_{n,\chi,w,q} = [2]_q \sum_{l=1}^{\infty} \chi(l) q^l \phi_w(l)(-1)^l [l]_q^n.$$

We set

$$F_{\chi,w,q}(t) = \sum_{n=0}^{\infty} E_{n,\chi,w,q} \frac{t^n}{n!}.$$

By using Theorem 2, we obtain

$$\begin{aligned}
F_{\chi,w,q}(t) &= \sum_{n=0}^{\infty} E_{n,\chi,w,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(l)(-1)^l [l]_q^n \frac{t^n}{n!} \\
&= [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(l)(-1)^l \sum_{n=0}^{\infty} [l]_q^n \frac{t^n}{n!} \\
&= [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(l)(-1)^l e^{[l]_q t}.
\end{aligned}$$

Hence we give the generating function of q-Euler numbers as follows:

$$F_{\chi,w,q}(t) = [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(l)(-1)^l e^{[l]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,w,q} \frac{t^n}{n!}.$$

The generalized twisted q-Euler polynomials associated with χ, $E_{n,\chi,w,q}(z)$, are defined by

$$E_{n,\chi,w,q}(z) = \int_X \chi(x)\phi_w(x)[z+x]_q^n d\mu_{-q}(x). \tag{2.2}$$

By using p-adic q-integral, we obtain,

$$E_{n,\chi,w,q}(z) = \frac{[f]_q^n}{[f]_{-q}} \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a E_{n,w^f,q^f}\left(\frac{a+z}{f}\right).$$

Since

$$\begin{aligned}
&\int_X \chi(x)\phi_w(x)[z+x]_q^n d\mu_{-q}(x) \\
&= [2]_q \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{al} q^{zl} \frac{1}{1+\phi_w(f) q^{f(1+l)}},
\end{aligned}$$

we have

$$\begin{aligned}
&E_{n,\chi,w,q}(z) \\
&= [2]_q \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(a+z)l} \frac{1}{1+\phi_w(f) q^{f(1+l)}}.
\end{aligned}$$

By simple calculation, we obtain

$$\begin{aligned}
&[2]_q \sum_{a=0}^{f-1} \chi(a)(-1)^a \phi_w(a) q^a \left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{(a+z)l} \frac{1}{1+\phi_w(f) q^{f(1+l)}} \\
&= [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(a)(-1)^l [z+l]_q^n.
\end{aligned} \tag{2.3}$$

We set

$$F_{\chi,w,q}(t,z) = \sum_{n=0}^{\infty} E_{n,\chi,w,q}(z) \frac{t^n}{n!}.$$

By using (2.3), we obtain

$$F_{\chi,w,q}(t) = [2]_q \sum_{l=0}^{\infty} \chi(l) q^l \phi_w(a)(-1)^l e^{[z+l]_q t}.$$

Hence we give the generating function of the generalized twisted q-Euler polynomials as follows:

$$F_{\chi,w,q}(t,z)=[2]_q\sum_{l=0}^{\infty}\chi(l)q^l\phi_w(a)(-1)^le^{[z+l]_qt}=\sum_{n=0}^{\infty}E_{n,\chi,w,q}(z)\frac{t^n}{n!}.$$

Observe that

$$F_{\chi,w,q}(t,z)=\int_X\chi(x)\phi_w(x)e^{[x+z]_qt}d\mu_{-q}(x).$$

By using the above equation, we have

$$\begin{aligned}\sum_{n=0}^{\infty}E_{n,\chi,w,q}(z)\frac{t^n}{n!}&=\int_X\chi(x)\phi_w(x)e^{[z+x]_qt}d\mu_{-q}(x)\\&=\sum_{n=0}^{\infty}\left(\int_X\chi(x)\phi_w(x)[z+x]_q^nd\mu_{-q}(x)\right)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}q^{zl}[z]_q^{n-k}\int_X\chi(x)\phi_w(x)[x]_q^kd\mu_{-q}(x)\right)\frac{t^n}{n!}\end{aligned}$$

By using comparing coefficients $\frac{t^n}{n!}$, we have the following theorem.

Theorem 3. For any positive integer n, we have

$$E_{n,\chi,w,q}(z)=\sum_{k=0}^{n}\binom{n}{k}q^{zl}[z]_q^{n-k}E_{k,\chi,w,q}.$$

We have the following remark.

Remark 1. Note that

(1) $E_{n,\chi,w,q}(0)=E_{n,\chi,w,q}$,
(2) If $\chi=\chi^0$, then $E_{n,\chi,w,q}(z)=E_{n,w,q}(z),E_{n,\chi,w,q}=E_{n,w,q}$,
(3) If $q\to 1$, then $E_{n,\chi,w,q}(z)=E_{n,\chi,w}(z),E_{n,\chi,w,q}=E_{n,\chi,w}$,
(4) If $q\to 1$, then $F_{\chi,w,q}(t,z)=F_{\chi,w}(t,z)$,
(5) If $\chi=\chi^0$, then $F_{\chi,w,q}(t)=F_{w,q}(t),F_{\chi,w,q}(t,z)=F_{w,q}(t,z)$,
(6) If $\chi=\chi^0,q\to 1$ then $E_{n,\chi,w,q}(z)=E_{n,w}(z),E_{n,\chi,w,q}=E_{n,w}$.
(7) If $\chi=\chi^0,q\to 1,w\to 1$ then $E_{n,\chi,w,q}(z)=E_n(z),E_{n,\chi,w,q}=E_n$.

In complex case, the generating function of generalized twisted q-Euler numbers is given by

$$F_{\chi,w,q}(t)=[2]_q\sum_{l=0}^{\infty}\chi(l)q^lw^l(-1)^le^{[l]_qt}=\sum_{n=0}^{\infty}E_{n,\chi,w,q}\frac{t^n}{n!},\tag{2.4}$$

where $q \in \mathbb{C}$ with $|q| < 1$ and w is the rth root of unity. Generalized twisted q-Euler polynomials $E_{n,\chi,w,q}(z)$, was also defined by means of the generating function

$$F_{\chi,w,q}(t,z) = F_{\chi,w,q}(t)e^{zt} = [2]_q \sum_{l=0}^{\infty} \chi(l) q^l w^l (-1)^l e^{[z+l]_q t} = \sum_{n=0}^{\infty} E_{n,\chi,w,q}(z) \frac{t^n}{n!}. \tag{2.5}$$

We define interpolation functions of the generalized twisted q-Euler numbers and polynomials. Thus we need the following relations. By using (2.4), we have

$$\left(\frac{d}{dt}\right)^k F_{\chi,w,q}(t)\big|_{t=0} = [2]_q \sum_{l=1}^{\infty} \chi(l) q^l w^l (-1)^l [l]_q^k, \tag{2.6}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,\chi,w,q} \frac{t^n}{n!}\right) \big|_{t=0} = E_{k,\chi,w,q}, \text{ for } k \in \mathbb{N}. \tag{2.7}$$

By (2.6), (2.7), we have the following theorem.

Theorem 4. For any positive integer k, we have

$$E_{k,\chi,w,q} = [2]_q \sum_{n=1}^{\infty} \chi(n) w^n q^n (-1)^n [n]_q^k.$$

The above generating function is used to construct a q-Dirichlet series. We define q-analogue Dirichlet's type l-function as follows:

Definition 1. Let $s \in \mathbb{C}$.

$$l_{w,q}(s,\chi) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n \chi(n) w^n q^n}{[n]_q^s}.$$

By using Definition 1 and Theorem 4, we have the following theorem.

Theorem 5. For any positive integer k, we have

$$E_{k,\chi,w,q} = l_{w,q}(-k,\chi).$$

Observe that

$$\lim_{w\to 1} l_{w,q}(s,\chi) = l_q(s,\chi),$$

which is a q-analogue Dirichlet's type l-function ([10]).

By using (2.5), we have

$$\left(\frac{d}{dt}\right)^k F_{\chi,w,q}(t,z)\big|_{t=0} = [2]_q \sum_{l=1}^{\infty} \chi(l) w^l q^l (-1)^l [z+l]_q^k, \tag{2.8}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_{n,\chi,w,q}(z)\frac{t^n}{n!}\right)\Big|_{t=0} = E_{k,\chi,w,q}(z), \text{ for } k \in \mathbb{N}. \tag{2.9}$$

By (2.8), (2.9), we have the following theorem.

Theorem 6. For any positive integer k, we have

$$E_{k,\chi,w,q}(z) = [2]_q \sum_{n=0}^{\infty} \chi(n) w^n q^n (-1)^n [n+z]_q^k.$$

By using the above theorem, we define the two-variable q-analogue l-series.

Definition 2. Let $s \in \mathbb{C}$.

$$l_{w,q}(s,z|\chi) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n \chi(n) w^n q^n}{[n+z]_q^s}.$$

Note that $l_{w,q}(s,z|\chi)$ is analytic continuation in $\mathbb{C}$ with only simple pole at $s = 1$, and $l_{w,q}(s,1|\chi) = l_{w,q}(s|\chi)$. By using Definition 2 and Theorem 6, we have the following theorem.

Theorem 7. For any positive integer k, we have

$$E_{k,\chi,w,q}(z) = l_{w,q}(-k,z|\chi).$$

If $w \to 1$, then $l_{w,q}(s,z|\chi) \to l_q(s,z|\chi)$, where $l_q(s,z|\chi)$ is the two-variable l-series cf [10].

In general, how many roots does $E_{n,\chi,w,q}(z)$ have ? Find the numbers of complex zeros $c_{E_{n,\chi,w,q}(z)}$ of the $E_{n,\chi,w,q}(z), Im(z) \neq 0$. Using numerical experiments, we hope to investigate the structure of the complex roots of the generalized q-Euler polynomials $E_{n,\chi,w,q}(z)$. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the generalized q-Euler polynomials $E_{n,\chi,w,q}(z)$ to appear in mathematics and physics. For related topics the interested reader is referred to [7, 8].

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,495-508,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Padé Spline Functions

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Abstract

We present here the definition of Padé spline functions, their expressions, and the estimate of the remainders of padé spline expansions. Some algorithms are also given.

AMS Subject Classification: 41A15, 65D07, 65D15.

Key Words and Phrases: Padé two-point approximation, Padé spline, Rational Hermite interpolation, padé spline expansion.

1 Introduction

Padé approximation is derived by expanding a function as a ratio of two power series and both the numerator and denominator coefficients are thus determined (*cf.* Baker [1-2], Baker and Graves-Morris [3], and Brent, Gustavson, and Yun [4]). In this paper, we shall use two points Padé approximation to construct Padé spline functions. The main idea initially came from the author's talk at the Joint U.S.- China Workshop on Approximation Theory that took place in April, 1985, Hangzhou, China ([5]).

Let

*The research of this author was partially supported by ASD Grant and sabbatical leave of the Illinois Wesleyan University.

$$\triangle : a = x_0 < x_1 < \cdots < x_n = b$$

be an arbitrary partition on the interval $[a, b]$, and let f be a kth differentiable function defined on $[a, b]$ with function value and dervatives at each node x_i $(i = 0, 1, \cdots, n)$

$$y_i^{(m)} = f^{(m)}(x_i), \quad m = 0, 1, \cdots, k-1; i = 0, 1, \cdots, n.$$

Denote by π_k the collection of all polynomials of degree less than or equal to k. We now give the definition of Padé spline functions.

Definition 1.1 *We call $R_{r,\ell}^{(k)}(\triangle)$ the set of Padé spline function of order k with nodes x_i $(i = 0, 1, \ldots, n)$, if any function $R(x) \in R_{r,\ell}^{(k)}$ satisfies the following coditions for all $x \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$,*

(i) $R(x) = \frac{P_i(x)}{Q_i(x)}$, $P_i(x) \in \pi_r$, $Q_i(x) \in \pi_\ell$,

(ii) $\sum_{m=0}^{k-1} y_{i-1}^{(m)} \frac{(x-x_{i-1})^m}{m!} - \frac{P_i(x)}{Q_i(x)} = O\left((x - x_{i-1})^k\right)$,

(iii) $\sum_{m=0}^{k-1} y_i^{(m)} \frac{(x-x_i)^m}{m!} - \frac{P_i(x)}{Q_i(x)} = O\left((x - x_i)^k\right)$,

(iv) $r + \ell = 2k - 1$.

From Definition 1.1, we immediately know $R_{r,\ell}^{(k)}(\triangle) \in C^k$. In addition, the rational Hermite interpolation and rational contact interpolation can be easily obtained by using the padé spline functions.

Although Definition 1.1 only gives the piecewise expression of Padé spline functions, we might discuss its globe expression as follows.

Suppose $Q_i(x_i) \neq 0$. Denote

$$G_i(x) = \sum_{m=0}^{k-1} R^{(m)}(x_i) \frac{(x - x_i)^m}{m!}.$$

Thus

$$R(x) - G_i(x) = \frac{P_i(x) - G_i(x)Q_i(x)}{Q_i(x)}$$

has multiple roots at x_i of order k; i.e.,

$$P_i(x) - G_i(x)Q_i(x) = (x - x_i)^k F_i(x),$$

where $deg\ F_i(x) \le \max\{r, k+\ell\} - k = \max\{r-k, \ell\}$. Thus, the expressions of $R(x)$ on $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ are respectively

$$\frac{P_i(x)}{Q_i(x)} = \sum_{m=0}^{k-1} R^{(m)}(x_i)\frac{(x-x_i)^m}{m!} + (x-x_i)^k \frac{F_i(x)}{Q_i(x)} \tag{1.1}$$

and

$$\frac{P_{i+1}(x)}{Q_{i+1}(x)} = \sum_{m=0}^{k-1} R^{(m)}(x_i)\frac{(x-x_i)^m}{m!} + (x-x_i)^k \frac{F_{i+1}(x)}{Q_{i+1}(x)}. \tag{1.2}$$

Consequently,

$$\begin{aligned} \frac{P_{i+1}(x)}{Q_{i+1}(x)} - \frac{P_i(x)}{Q_i(x)} &= \left[\frac{F_{i+1}(x)}{Q_{i+1}(x)} - \frac{F_i(x)}{Q_i(x)}\right](x-x_i)^k \\ &= \frac{M_i(x)}{Q_i(x)Q_{i+1}(x)}(x-x_i)^k, \end{aligned}$$

where $M_i(x) = Q_i(x)F_{i+1}(x) - F_i(x)Q_{i+1}(x)$ is in $\pi_{r+\ell-k}$. Therefore, for $x \in [x_i, x_{i+1}]$, we have

$$\begin{aligned} \frac{P_{i+1}(x)}{Q_{i+1}(x)} - \frac{P_1(x)}{Q_1(x)} &= \sum_{j=1}^{i}\left(\frac{P_{j+1}(x)}{Q_{j+1}(x)} - \frac{P_j(x)}{Q_j(x)}\right) \\ &= \sum_{j=1}^{i} \frac{M_j(x)}{Q_j(x)Q_{j+1}(x)}(x-x_j)_+^k, \end{aligned} \tag{1.3}$$

where $M_j(x) \in \pi_{r+\ell-k}$ and

$$(x-x_j)_+ := \begin{cases} x - x_j & \text{if } x \ge x_j \\ 0 & \text{if } x < x_j. \end{cases}$$

From Eq. (1.3) we obtain the globe expression of Padé spline function $R(x)$ as follows.

$$R(x) = \frac{P_1(x)}{Q_1(x)} + \sum_{j=1}^{n-1} \frac{M_j(x)}{Q_j(x)Q_{j+1}(x)}(x-x_j)_+^k, \tag{1.4}$$

where $M_j(x) \in \pi_{r+\ell-k}$ is completely determined by $P_1(x)$, $Q_j(x)$ $(j = 1, 2, \ldots, n)$ as well as the values and the first k derivatives of $R(x)$ at x_j $(j = 1, 2, \ldots, n)$.

We can also show that if any real-valued function $R(x)$ defined on $[a, b]$ can be written as in Eq. (1.4) with $M_j(x) \in \pi_{r+\ell-k}$, then $R(x) \in R_{r,\ell}^{(k)}(\triangle)$; i.e., $R(x)$ is a Padé spline function defined as in Definition 1.1. Indeed, assume that $R(x)$ shown as in (1.4) is given, where $M_j(x) \in \pi_{r+\ell-k}$, $P_1(x) \in \pi_r$, $Q_i(x) \in \pi_\ell$ $(i = 1, 2, \ldots, n)$, and the greatest common divisor $(P_1(x), Q_1(x)) = 1$, there exist $p(x)$ and $q(x)$ such that

$$p(x)Q_1(x) + P_1(x)q(x) \equiv 1.$$

By multiplying $\phi_1(x) = (x-x_1)^k M_1(x)$ on the both sides of the last equation, we obtain

$$p(x)\phi(x)Q_1(x) + P_1(x)q(x)\phi(x) = \phi(x). \tag{1.5}$$

Since we can write

$$p(x)\phi(x) = P_1(x)r_1(x) + s_1(x)$$

and

$$q(x)\phi(x) = Q_1(x)r_2(x) + s_2(x),$$

Eq. (1.5) can be changed to

$$\left[P_1(x)r_1(x) + s_1(x)\right] Q_1(x) + P_1(x)\left[Q_1(x)r_2(x) + s_2(x)\right] = \phi(x). \tag{1.6}$$

If $(x - x_1) \not| s_2(x)$, we set $Q_2(x) = -s_2(x)$ and

$$P_2(x) = P_1(x)\left[r_1(x) + r_2(x)\right] + s_1(x).$$

If $(x - x_1) | s_2(x)$, then $(x - x_1) \not| Q_1(x)$ because of $(P_1(x), Q_1(x)) = 1$. We thus denote $Q_1(x) = -s_2(x) - Q_1(x)$ and

$$P_2(x) = P_1(x)\left[r_1(x) + r_2(x) - 1\right] + s_1(x).$$

Therefore in either case, we can write Eq. (1.6) as

$$P_2(x)Q_1(x) - P_1(x)Q_2(x) = (x - x_1)^k M_1(x),$$

where $(x - x_1) \not| Q_2(x)$.

Similarly, we cane decompose $(x - x_j)^k M_j(x)$ into

$$(x - x_j)^k M_j(x) = P_{j+1}(x)Q_j(x) - P_j(x)Q_{j+1}(x), \tag{1.7}$$

where $Q_j(x_j) \neq 0$, $Q_{j+1}(x_j) \neq 0$, and $j = 1, 2, \ldots, n-1$. Since $M_j(x) \in \pi_{r+\ell-k}$, $P_j(x) \in \pi_r$ for all $j = 1, 2, \ldots, n-1$.

For $x \in [x_i, x_{i+1}]$ $(j = 0, 1, \ldots, n-1)$, from Eqs. (1.4) and (1.7), we have

$$R(x) = \frac{P_1(x)}{Q_1(x)} + \sum_{j=1}^{i} \left(\frac{P_{j+1}(x)}{Q_{j+1}(x)} - \frac{P_j(x)}{Q_j(x)} \right) = \frac{P_{i+1}(x)}{Q_{i+1}(x)}.$$

In addition, since

$$\left[M_i(x)(x - x_i)^k \right]^{(m)} \Big|_{x=x_i} = 0$$

for $i = 1, 2, \ldots, n-1$ and $m = 0, 1, \ldots, k-1$, by using the Lemma shown as in [6], we obtain

$$\left[\frac{(x - x_i)^k M_i(x)}{Q_i(x)Q_{i+1}(x)} \right]^{(m)} \Bigg|_{x=x_i} = 0.$$

Consquently,

$$\left[\frac{P_{i+1}(x)}{Q_{i+1}(x)} \right]^{(m)} \Bigg|_{x=x_i} = \left[\frac{P_i(x)}{Q_i(x)} \right]^{(m)} \Bigg|_{x=x_i}.$$

It follows that $R(x) \in R^{(k)}_{r,\ell}$. Thus we have established the following result.

Theorem 1.2 *Function $R(x)$ defined on $[a, b]$ and shown as in Eq. (1.4) is in $R^{(k)}_{r,\ell}(\triangle)$ if and only if $M_j(x) \in \pi_{r+\ell-k}$.*

2 Algorithm

In this section, we will give two algorithms for constructing Padé spline functions. Our first algorithm is to construct the functions piece by piece by using continued fractions. The second algorithm is based on the general expression of Padé spline functions shown as in (1.4). To describe the algorithms clearly, we only consider the Padé spline function set $R^{(k)}_{k-1,k}$, which is the most important set in the Padé approximation. The first algorithms is also an improvement of [7-8].

For $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, write $R(x) = \frac{P_i(x)}{Q_i(x)}$ as its continued fraction form:

$$\begin{aligned} \frac{P_i(x)}{Q_i(x)} &= a_{i,0} + \frac{x-x_i|}{|a_{i,1}} + \cdots + \frac{x-x_i|}{|a_{i,k-1}} + \frac{x-x_i|}{|a_{i+1,0}} \\ &\quad + \frac{x-x_{i+1}|}{|a_{i+1,1}} + \cdots + \frac{x-x_{i+1}|}{|a_{i+1,k-1}}. \end{aligned} \tag{2.1}$$

Denote

$$\frac{S_{i,0}(x)}{T_{i,0}(x)} = a_{i,0} + \frac{x-x_i|}{|a_{i,1}} + \cdots + \frac{x-x_i|}{|a_{i,k-1}}. \tag{2.2}$$

It is easy to find that

$$\left[\frac{S_{i,0}(x)}{T_{i,0}(x)}\right]^{(m)}\Bigg|_{x=x_i} = \left[\frac{P_i(x)}{Q_i(x)}\right]^{(m)}\Bigg|_{x=x_i} = y_i^{(m)}$$

for $m = 0, 1, \ldots, k-1$, which implies by the Lemma in [6]

$$S^{(m)}_{i,0}(x_i) = [f(x)T_{i,0}(x)]^{(m)}\big|_{x=x_i}, \tag{2.3}$$

where $m = 0, 1, \ldots, k-1$. From Eq. (2.3) we can find the coefficients of $S_{i,0}(x)$ and $T_{i,0}(x)$. Then, by using the following relations, (2.4) and (2.5), we can determine the coefficient set $\{a_{i,0}, a_{i,1}, \ldots, a_{i,k-1}\}$ of the continued fraction (2.1).

$$S_{i,0} = \Pi_{j=0}^{k-1} a_{i,j} \left(1 + \sum_{j=0}^{k-2} \frac{x-x_i}{a_{i,j}a_{i,j+1}} + \sum_{0\le j<\ell\le k-3} \frac{(x-x_i)^2}{a_{i,j}a_{i,j+1}a_{i,\ell+1}a_{i,\ell+2}}\right.$$

$$+\sum_{0\le j<\ell<m\le k-4}\frac{(x-x_i)^3}{a_{i,j}a_{i,j+1}a_{i,\ell+1}a_{i,\ell+2}a_{i,m+2}a_{i,m+3}}+\cdots\Bigg), \tag{2.4}$$

$$\begin{aligned} T_{i,0}(x) &= \Pi_{j=1}^{k-1}a_{i,j}\left(1+\sum_{j=1}^{k-2}\frac{x-x_i}{a_{i,j}a_{i,j+1}}\right.\\ &\qquad \left.+\sum_{1\le j<\ell\le k-3}\frac{(x-x_i)^2}{a_{i,j}a_{i,j+1}a_{i,\ell+1}a_{i,\ell+2}}+\cdots\right). \end{aligned} \tag{2.5}$$

Denote

$$\frac{S_{i+1,0}(x)}{T_{i+1,0}(x)}=a_{i+1,0}+\frac{x-x_{i+1}|}{|a_{i+1,1}}+\cdots+\frac{x-x_{i+1}|}{|a_{i+1,k-1}} \tag{2.6}$$

and

$$\frac{S_{i,-1}(x)}{T_{i,-1}(x)}=a_{i,0}+\frac{x-x_i|}{|a_{i,1}}+\cdots+\frac{x-x_i|}{|a_{i,k-2}}. \tag{2.7}$$

Then,

$$\frac{P_i(x)}{Q_i(x)}=\frac{S_{i,0}S_{i+1,0}+(x-x_i)S_{i,-1}T_{i+1,0}}{T_{i,0}S_{i+1,0}+(x-x_i)T_{i,-1}T_{i+1,0}}.$$

Similar to Eq. (2.3), from [6] we have

$$\begin{aligned} &[S_{i,0}S_{i+1,0}+(x-x_i)S_{i,-1}T_{i+1,0}]^{(m)}\Big|_{x=x_{i+1}}\\ =\ &\{f(x)\,[T_{i,0}S_{i+1,0}+(x-x_i)T_{i,-1}T_{i+1,0}]\}^{(m)}\Big|_{x=x_{i+1}}. \end{aligned} \tag{2.8}$$

In Eq. (2.8), since $S_{i,-1}$ and $T_{i,-1}$ have been determined from (2.3), we thus find $S_{i+1,0}(x)$ and $T_{i+1,0}(x)$. We can also establish the relations between the functions $S_{i+1,0}(x)$ and $T_{i+1,0}(x)$ and the coefficient set $\{a_{i+1,j}: j=0,1,\ldots,k-1\}$, which is as the same as Eqs. (2.4) and (2.5) except an index change of $i\to i+1$. From the relations we finally determine the set $\{a_{i+1,j}: j=0,1,\ldots,k-1\}$.

Example 2.1. As an example, we now consider the case of $k=2$. Obviously, we have

$$S_{i,0}(x) = a_{i,0}a_{i,1} + x - x_i,\ T_{i,0}(x) = a_{i,1},$$
$$S_{i+1,0}(x) = a_{i+1,0}a_{i+1,1} + x - x_{i+1},\ T_{i+1,0} = a_{i+1,1},$$
$$S_{i,-1}(x) = a_{i,0},\ T_{i,-1}(x) = 1.$$

Thus, (2.3) is reduced to

$$[a_{i,0}a_{i,1} + x - x_i]^{(m)}\Big|_{x=x_i} = a_{i,1}y_i^{(m)}, \quad m = 0, 1.$$

Assume that $y_i' \neq 0$, we solve $a_{i,0} = y_i$ and $a_{i,1} = 1/y_i'$.

From Eq. (2.8) we have

$$\left[\left(\frac{y_i}{y_i'} + x - x_i\right)(a_{i+1,0}a_{i+1,1} + x - x_{i+1}) + y_i a_{i+1,1}(x - x_i)\right]^{(m)}\Bigg|_{x=x_{i+1}}$$
$$= \left\{f(x)\left[\frac{1}{y_i'}(a_{i+1,0}a_{i+1,1} + x - x_{i+1}) + a_{i+1,1}(x - x_i)\right]\right\}^{(m)}\Bigg|_{x=x_{i+1}}$$

for $m = 0, 1$. From the last equation it can be found that

$$\begin{aligned} a_{i+1,0} &= \frac{y_i'(x_{i+1} - x_i)(y_{i+1} - y_i)}{y_i'(x_{i+1} - x_i) - (y_{i+1} - y_i)}, \\ a_{i+1,1} &= \frac{[y_{i+1} - y_i - y_i'(x_{i+1} - x_i)]^2}{y_i'\left[y_i'y_{i+1}'(x_{i+1} - x_i)^2 - (y_{i+1} - y_i)^2\right]}. \end{aligned}$$

Substituting the obtained coefficient set $\{a_{i,0}, a_{i,1}, a_{i+1,0}, a_{i+1,1}\}$ into the expression of the Padé spline function $R(x) \in R_{2,1}^{(2)}$

$$R(x) = a_{i,0} + \frac{x - x_i|}{|a_{i,1}} + \frac{x - x_i|}{|a_{i+1,0}} + \frac{x - x_{i+1}|}{|a_{i+1,1}}$$

yields

$$\begin{aligned} R(x) = {} & [(x - x_i)(x - x_{i+1}) + a_{i,0}a_{i,1}(x - x_{i+1}) + a_{i+1,0}a_{i+1,1}(x - x_i) \\ & + a_{i,0}a_{i+1,1}(x - x_i) + a_{i,0}a_{i,1}a_{i+1,0}a_{i+1,1}] \,/\, [a_{i,1}(x - x_{i+1}) \\ & + a_{i+1,1}(x - x_i) + a_{i,1}a_{i+1,0}a_{i+1,1}] \end{aligned}$$

for $i = 0, 1, \ldots, n-1$.

We now discuss the second algorithm. Denote $a_m^{(i)} = y_i^{(m)}/m!$ and $\ell_i = x - x_i$ $(i = 0, 1, \ldots, n)$, and write

$$
\begin{aligned}
P_i(x) &= \alpha_0^{(i)} + \alpha_1^{(i)}\ell_i + \cdots + \alpha_{k-1}^{(i)}\ell_i^{k-1} \\
&= \bar{\alpha}_0^{(i)} + \bar{\alpha}_1^{(i)}\ell_{i+1} + \cdots + \bar{\alpha}_{k-1}^{(i)}\ell_{i+1}^{k-1} \qquad (2.9)\\
Q_i(x) &= \beta_0^{(i)} + \beta_1^{(i)}\ell_i + \cdots + \beta_k^{(i)}\ell_i^{k} \\
&= \bar{\beta}_0^{(i)} + \bar{\beta}_1^{(i)}\ell_{i+1} + \cdots + \bar{\beta}_k^{(i)}\ell_{i+1}^{k} \qquad (2.10)
\end{aligned}
$$

From conditions (ii) and (iii) in Definition 1.1, we have

$$
\begin{aligned}
\sum_{m=0}^{k-1} y_i^{(m)}\frac{(x-x_i)^m}{m!} - \frac{P_i(x)}{Q_i(x)} &= (x-x_i)^k\sum_{j=0}^{\infty} c_j(x-x_i)^j \\
\sum_{m=0}^{k-1} y_{i+1}^{(m)}\frac{(x-x_{i+1})^m}{m!} - \frac{P_i(x)}{Q_i(x)} &= (x-x_{i+1})^k\sum_{j=0}^{\infty} d_j(x-x_{i+1})^j
\end{aligned}
$$

Substituting expressions (2.9) and (2.10) into the last two equtions yields

$$
\sum_{m=0}^{k-1}\sum_{j=0}^{k} a_m^{(i)}\beta_j^{(i)}\ell_i^{m+j} - \sum_{j=0}^{k-1}\alpha_j^{(i)}\ell_i^j = \sum_{j=k}^{2k-1} r_j^{(i)}\ell_i^j \qquad (2.11)
$$

$$
\sum_{m=0}^{k-1}\sum_{j=0}^{k} a_m^{(i+1)}\bar{\beta}_j^{(i)}\ell_{i+1}^{m+j} - \sum_{j=0}^{k-1}\bar{\alpha}_j^{(i)}\ell_{i+1}^j = \sum_{j=k}^{2k-1} \bar{r}_j^{(i)}\ell_{i+1}^j. \qquad (2.12)
$$

Therefore we obtain

$$
\alpha_j^{(i)} = \sum_{\mu=0}^{j} a_{j-\mu}^{(i)}\beta_\mu^{(i)} \qquad (2.13)
$$

and

$$
\bar{\alpha}_j^{(i)} = \sum_{\mu=0}^{j} a_{j-\mu}^{(i+1)}\bar{\beta}_\mu^{(i)} \qquad (2.14)
$$

for $j = 0, 1, \ldots, k-1$.

Denote $h_i = x_{i+1} - x_i$. From Eq. (2.9) we have

$$
\begin{aligned}
\bar{\alpha}_j^{(i)} &= \left.\frac{\partial^j P_i(x)}{j!\partial x^j}\right|_{x=x_{i+1}} \\
&= \frac{1}{j!}\left(j!\alpha_j^{(i)} + \frac{(j+1)!}{1!}\ \alpha_{j+1}^{(i)}\ell_i\Big|_{x=x_{i+1}} + \frac{(j+2)!}{2!}\ \alpha_{j+2}^{(i)}\ell_i^2\Big|_{x=x_{i+1}}\right. \\
&\quad \left. + \cdots + \frac{(k-1)!}{(k-j-1)!}\ \alpha_{k-1}^{(i)}\ell_i^{k-j-1}\Big|_{x=x_{i+1}}\right) \\
&= \sum_{\nu=0}^{k-j-1}\binom{j+\nu}{\nu}\alpha_{j+\nu}^{(i)}h_i^{\nu} \qquad (2.15)
\end{aligned}
$$

for $j = 0, 1, \ldots, k-1$. Similarly, from Eq. (2.10) we obtain

$$
\bar{\beta}_j^{(i)} = \sum_{\nu=0}^{k-j}\binom{j+\nu}{\nu}\beta_{j+\nu}^{(i)}h_i^{\nu} \qquad (2.16)
$$

for $j = 0, 1, \ldots, k$. Substituting (2.15), (2.16), and (2.13) into (2.14) yields

$$
\sum_{\mu=0}^{k-j-1}\binom{j+\mu}{\mu}\sum_{\nu=0}^{j+\mu}\alpha_{j+\mu-\nu}^{(i)}\beta_{\nu}^{(i)}h_i^{\mu} = \sum_{\mu=0}^{j}a_{j-\mu}^{(i+1)}\sum_{\nu=0}^{k-\nu}\binom{\mu+\nu}{\nu}\beta_{\mu+\nu}^{(i)}h_i^{\nu}, \qquad (2.17)
$$

where $j = 0, 1 \ldots, k-1$. We separate the left-hand side of Eq. (2.17) into two parts and write them as

$$
\begin{aligned}
&\sum_{\nu=0}^{j}\beta_{\nu}^{(i)}\left[\sum_{\mu=0}^{k-j-1}\binom{j+\mu}{\mu}a_{j+\mu-\nu}^{(i)}h_i^{\mu}\right] + \sum_{\nu=1}^{k-j-1}q_{\nu+j}^{(i)}\left[\sum_{\mu=\nu}^{k-j-1}\binom{j+\mu}{\mu}a_{\mu-\nu}^{(i)}h_i^{\mu}\right] \\
= &\sum_{\nu=0}^{j}\beta_{\nu}^{(i)}\left[\sum_{\mu=0}^{k-j-1}\binom{j+\mu}{\mu}a_{j+\mu-\nu}^{(i)}h_i^{\mu}\right] \\
&+ \sum_{\nu=j+1}^{k-1}q_{\nu}^{(i)}\left[\sum_{\mu=\nu-j}^{k-j-1}\binom{j+\mu}{\mu}a_{j+\mu-\nu}^{(i)}h_i^{\mu}\right]. \qquad (2.18)
\end{aligned}
$$

Similarly, we can change the right-hand side of Eq. (2.17) to

$$
\sum_{\nu=0}^{j}\beta_{\nu}^{(i)}\left[\sum_{\mu=0}^{\nu}\binom{\nu}{\nu-\mu}a_{j-\mu}^{(i+1)}h_i^{\nu-\mu}\right] + \sum_{\nu=j+1}^{k}q_{\nu}^{(i)}\left[\sum_{\mu=0}^{j}\binom{\nu}{\nu-\mu}a_{j-\mu}^{(i+1)}h_i^{\nu-\mu}\right]. \qquad (2.19)
$$

Substituting expressions (2.18) and (2.19) into (2.17) yields the following equations for $j = 0, 1, \ldots, k-1$:

$$\begin{aligned}
&\sum_{\nu=0}^{j} \beta_\nu^{(i)} \left[\sum_{\mu=0}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu - \sum_{\mu=0}^{\nu} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} \right] \\
&+ \sum_{\nu=j+1}^{k-1} q_\nu^{(i)} \left[\sum_{\mu=\nu-j}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu - \sum_{\mu=0}^{j} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} \right] \\
&-\beta_k^{(i)} \left[\sum_{\mu=0}^{j} \binom{k}{k-\mu} a_{j-\mu}^{(i+1)} h_i^{k-\mu} \right] = 0. \qquad (2.20)
\end{aligned}$$

Eqs. (2.20) is a homogeneous system of $k+1$ unknowns, $\beta_0^{(i)}$, $\beta_1^{(i)}$, ..., $\beta_k^{(i)}$, consisting of k equations. Hence, it has nontrivial solution. To simplyfy the expression of (2.20), we denote

$$b_{j,\nu}^{(i)} := \begin{cases} \sum_{\mu=0}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu & \\ \qquad - \sum_{\mu=0}^{\nu} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} & \text{if } 0 \le \nu \le j, \\ \sum_{\mu=\nu-j}^{k-j-1} \binom{j+\mu}{\mu} a_{j+\mu-\nu}^{(i)} h_i^\mu & \\ \qquad - \sum_{\mu=0}^{j} \binom{\nu}{\nu-\mu} a_{j-\mu}^{(i+1)} h_i^{\nu-\mu} & \text{if } j+1 \le \nu \le k-1, \\ -\sum_{\mu=0}^{j} \binom{k}{k-\mu} a_{j-\mu}^{(i+1)} h_i^{k-\mu} & \text{if } \nu = k \end{cases} \qquad (2.21)$$

and rewrite (2.20) as

$$\sum_{\nu=0}^{k} b_{j,\nu}^{(i)} \beta_\nu^{(i)} = 0, \quad j = 0, 1, \ldots, k-1. \qquad (2.22)$$

After finding

$$\begin{aligned}
Q_i(x) &= \sum_{j=0}^{k} \beta_j^{(i)} (x - x_i)^j \\
&= \begin{vmatrix} 1 & x - x_i & (x-x_i)^2 & \cdots & (x-x_i)^k \\ b_{0,0}^{(i)} & b_{0,1}^{(i)} & b_{0,2}^{(i)} & \cdots & b_{0,k}^{(i)} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & b_{k-1,2}^{(i)} & \cdots & b_{k-1,k}^{(i)} \end{vmatrix}, \qquad (2.23)
\end{aligned}$$

by (2.13) we have

$$P_i(x) = \sum_{\nu=0}^{k-1} \beta_\nu^{(i)} \left(\sum_{j=\nu}^{k-1} a_{j-\nu}^{(i)} (x-x_i)^j \right) + \beta_k^{(i)} \cdot 0 =$$

$$\begin{vmatrix} \sum_{j=0}^{k-1} a_j^{(i)} t_i(x)^j & \sum_{j=1}^{k-1} a_{j-1}^{(i)} t_i(x)^j & \cdots & \sum_{j=k-1}^{k-1} a_{j-k+1}^{(i)} t_i(x)^j & 0 \\ b_{0,0}^{(i)} & b_{0,1}^{(i)} & \cdots & b_{0,k-1}^{(i)} & b_{0,k}^{(i)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & \cdots & b_{k-1,k-1}^{(i)} & b_{k-1,k}^{(i)} \end{vmatrix},$$

where $t_i(x) = x - x_i$. We now calculate $r_j^{(i)}$ and $\bar{r}_j^{(i)}$ in Eqs. (2.11) and (2.12). First, from (2.11) we obtain

$$r_\mu^{(i)} = \sum_{j=0}^{k} a_{\mu-j}^{(i)} \beta_j^{(i)}, \tag{2.24}$$

where $\mu = k, k+1, \ldots, 2k-1$, and $a_\nu^{(i)} = 0$ for all $\nu \geq k$. Comparing the last equation with (2.23) yields

$$r_\mu^{(i)} = \begin{vmatrix} a_\mu^{(i)} & a_{\mu-1}^{(i)} & a_{\mu-2}^{(i)} & \cdots & a_{\mu-k}^{(i)} \\ b_{0,0}^{(i)} & b_{0,1}^{(i)} & b_{0,2}^{(i)} & \cdots & b_{0,k}^{(i)} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & b_{k-1,2}^{(i)} & \cdots & b_{k-1,k}^{(i)} \end{vmatrix}. \tag{2.25}$$

Secondly, from (2.12) we have

$$\bar{r}_\mu^{(i)} = \sum_{j=0}^{k} a_{\mu-j}^{(i+1)} \bar{\beta}_j^{(i)}, \tag{2.26}$$

where $\mu = k, k+1, \ldots, 2k-1$, and $a_\nu^{(i)} = 0$ for all $\nu \geq k$. Substituting (2.16) into (2.26) yields

$$\begin{aligned} \bar{r}_\mu^{(i)} &= \sum_{j=0}^{k} \left[\sum_{\nu=0}^{k-j} \binom{j+\nu}{\nu} \beta_{j+\nu}^{(i)} h_i^\nu \right] a_{\mu-j}^{(i+1)} \\ &= \sum_{\nu=0}^{k} \left[\sum_{j=0}^{\nu} \binom{\nu}{\nu-j} a_{\mu-j}^{(i+1)} h_i^{\nu-j} \right] \beta_\nu^{(i)}. \end{aligned} \tag{2.27}$$

Denoting $c_{\mu,\nu}^{(i+1)} = \sum_{j=0}^{\nu} \binom{\nu}{\nu-j} a_{\mu-j}^{(i+1)} h_i^{\nu-j}$ in (2.27) and using (2.23), we obtain

$$\bar{r}_\mu^{(i)} = \sum_{\nu=0}^{k} c_{\mu,\nu}^{(i+1)} \beta_\mu^{(i)} = \begin{vmatrix} c_{\mu,0}^{(i+1)} & c_{\mu,1}^{(i+1)} & c_{\mu,2}^{(i+1)} & \cdots & c_{\mu,k}^{(i+1)} \\ b_{0,0}^{(i)} & b_{0,1}^{(i)} & b_{0,2}^{(i)} & \cdots & b_{0,k}^{(i)} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k-1,0}^{(i)} & b_{k-1,1}^{(i)} & b_{k-1,2}^{(i)} & \cdots & b_{k-1,k}^{(i)} \end{vmatrix}. \tag{2.28}$$

Therefore, Eqs. (2.11) and (2.11) are evantually obtained as

$$\sum_{m=0}^{k-1} y_i^{(m)} \frac{(x-x_i)^m}{m!} - \frac{P_i}{Q_i} = \sum_{j=k}^{2k-1} \frac{r_j^{(i)}}{Q_i}(x-x_i)^j,$$
$$\sum_{m=0}^{k-1} y_{i+1}^{(m)} \frac{(x-x_{i+1})^m}{m!} - \frac{P_i}{Q_i} = \sum_{j=k}^{2k-1} \frac{\bar{r}_j^{(i)}}{Q_i}(x-x_{i+1})^j,$$

from which we have the Padé spline function defined on $[a, b]$ with the form

$$R(x) = \frac{P_0}{Q_0} + \sum_{i=0}^{n-2} \left[\sum_{\mu=k}^{2k-1} \left(\frac{\bar{r}_\mu^{(i)}}{Q_i} - \frac{r_\mu^{(i)}}{Q_{i+1}} \right) \right] (x - x_{i+1})_+^\mu, \tag{2.29}$$

where $r_\mu^{(i+1)}$, $\bar{r}_\mu^{(i)}$, and Q_i are given by Eqs. (2.25), (2.28), and (2.23), respectively.

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Null Controllability Of Nonlinear Infinite Neutral Systems With Multiple Delays In Control

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Abstract

Sufficient conditions are developed for the null controllability of nonlinear infinite neutral systems with time varying multiple delays in control. It is shown that if the uncontrolled system is uniformly asymptotically stable, and if the linear system is controllable, then the nonlinear infinite neutral system is null controllable. An example is provided to illustrate the obtained results.

Keywords

Controllability, infinite neutral systems, multiple delays, uniform asymptotic stability.

1. Introduction

Neutral functional differential equations are characterized by a delay in the derivative. Equations of this type have applications in the study of electrical networks containing lossless transmission lines. It is well known that the mixed initial-boundary hyperbolic partial differential equation which arises in the study of lossless transmission lines can be replaced by an associated neutral functional differential equation [1]. The aim of this paper is to study the null controllability of such systems by introducing multiple delays in control. For motivation of time varying multiple delays in control variables refer to the book by Klamka [13].

The problem of controllability and null controllability of functional differential equations in finite dimensional space has been studied by several authors. These include Chukwu [7] Balachandran et al [2], Umana [16], Umana and Nse [17], Iheagwam and Onwuatu [11], and Fu [8]. Several authors have extended the controllability and null controllability concepts to infinite dimensional systems. These include Balachandran and Dauer [3], Sinha [15], Onwuatu [14], Balachandran and Anandhi [5], Iyai [12], and Balachandran and Leelamani [6].

It is well known [10] that if the linear control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1}$$

is proper and if the free system

$$\dot{x}(t) = A(t)x(t) \tag{1.2}$$

is uniformly asymptotically stable, then (1.1) is null controllable with constraints. Chukwu [7] obtained an analogous result for the delay system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + f(t, x_t, u(t)) \tag{1.3}$$

where

$$L(t,\phi) = \sum_{k=0}^{\infty} A_k(t)\phi(-t_k) + \int_{-\gamma}^{0} A(t,s)\phi(s)ds\,. \tag{1.4}$$

Sinha [15] studied the nonlinear infinite delay system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta + f(t, x_t, u(t)) \tag{1.5}$$

and showed that (1.5) is Euclidean null controllable if the linear base system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) \tag{1.6}$$

is proper and if the free system

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta \tag{1.7}$$

is uniformly asymptotically stable, provided that f satisfies some growth conditions. Balachandran and Dauer [3] studied this problem for system (1.5) with distributed delays in the control. This problem for system (1.5) with time varying multiple delays has also been studied by the same authors [4]. Onwuatu [14] extended Sinha's results to neutral systems.

In this paper, nonlinear infinite neutral control systems with time varying multiple delays of the following form are considered:

$$\frac{d}{dt}D(t,x_t) = L(t,x_t) + \sum_{i=0}^{p} B_i(t)u(t-h_i) + \int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta + f(t,x_t,u(t)) \tag{1.8}$$

$$x(t) = \phi(t), \qquad t \in (-\infty, 0]$$

where $L(t,\phi)$ is as defined in (1.4), $A(\theta)$ is an $n\times n$ continuous matrix and $f(t,x_t,u(t))$ is a nonlinear continuous matrix function.

Our results extend those of [3,15,4] to neutral systems and those of [14,12] to neural systems with multiple delays in the control.

2. Preliminaries

Let n and m be positive integers, R the real line $(-\infty,\infty)$. Denote by R^n, the space of real n-tuples with the Euclidean norm defined by $|\cdot|$. If $J=[t_0,t_1]$ is any interval of R, the usual Lebesgue space of square integrable functions from J to R^m will be denoted by $L_2(J,R^m)$.

Let $\gamma \geq h > 0$ be a given real number and let $B = B([-\gamma,0],R^n)$ be the Banach space of functions which are continuous on $[-\gamma,0]$ with $\|\phi\| = \sup_{-\gamma\leq s<0}|\phi(s)|$, $\phi\in B([-\gamma,0],R^n)$. If x is a function from $[t_0-\gamma,\infty)$ to R^n, let x_t, $t\in[0,\infty)$, be a function from $[-\gamma,0]$ to R^n, defined by $x_t(s)=x(t+s)$, $s\in[-\gamma,0]$. Similarly, if u is a function from $[t_0-\gamma,\infty)$ to R^m, let u_t, $t\in[0,\infty)$

be a function from $[-\gamma,0]$ to R^m, defined by $u_t(s)=u(t+s)$, $s\in[-\gamma,0]$. In system (1.8), assume that $D(\cdot,\cdot):R\times B\to R^n$ is defined by

$$D(t,x_t)=x(t)-g(t,x_t)$$

where

$$g(t,\phi)=\sum_{n=1}^{\infty}A_n(t)\phi(-w_n(t))+\int_{-\gamma}^{0}A(t,s)\phi(s)ds$$

and where $0<w_n(t)\le\gamma$ and $A_n(t)$ and $A(t,s)$ are $n\times n$ matrix functions.

A nonautonomous linear homogeneous neutral differential equation is defined to be

$$\frac{d}{dt}D(t,x_t)=L(t,x_t) \tag{2.1}$$

Our objective is to study the null controllability of system (1.8) through its linear control base system

$$\frac{d}{dt}D(t,x_t)=L(t,x_t)+\sum_{i=0}^{p}B_i(t)u(t-h_i) \tag{2.2}$$

and its free system

$$\frac{d}{dt}D(t,x_t)=L(t,x_t)+\int_{-\infty}^{0}A(\theta)x(t+\theta)d\theta\ . \tag{2.3}$$

Here B_i are continuous $n\times m$ matrix functions, each A_k is a continuous $n\times n$ matrix function for $0\le h_k\le\gamma$, $A(\theta)$ is an $n\times n$ matrix whose elements are square integrable on $(-\infty,0]$. D,L,f satisfy enough smoothness conditions to ensure that a solution of (1.8) exists through each (t_0,ϕ) in $J\times B$, is unique, depends continuously on (t_0,ϕ) and can be continued to the right as the trajectory remains in a bounded set in $J\times B$.

If $T(t,t_0):B \to B$, $t \geq t_0$ is defined by $T(t,t_0)\phi = x_t(t_0,\phi)$, where $x(t_0,\phi)$ is the solution of (2.1), then there exists an $n \times n$ matrix function $X(t,s)$ defined for $0 \leq s \leq \gamma$, $t \in [0,\infty)$, continuous in s from the right, of bounded variation in s, $X(t,s) = 0,\ t < s \leq t+\gamma$, such that the solution $x(t_0,\phi)$ of (2.2) is given by

$$x(t,t_0,\phi,u) = T(t,t_0)\phi(0) + \int_{t_0}^{t} X(t,s)\sum_{i=0}^{p} B_i(s)u(s-h_i)ds \tag{2.4}$$

The corresponding solution of (1.8) is given by

$$\begin{aligned} x(t,t_0,\phi,u,f) &= T(t,t_0)\phi(0) + \int_{t_0}^{t} X(t,s)\sum_{i=0}^{p} B_i(s)u(s-h_i)ds \\ &\quad + \int_{t_0}^{t} X(t,s)\left(\int_{-\gamma}^{0} A(\theta)x(t+\theta)d\theta\right)ds + \int_{t_0}^{t} X(t,s)f(s,x_s,u(s))ds . \end{aligned} \tag{2.5}$$

Observe that the uniqueness of solution of (2.1) implies that

$$T(t_2,t_1)T(t_1,t_0) = T(t_2,t_0), \quad t_1,t_2 \geq t_0 \geq 0 .$$

If we let

$$X_0(s) = \begin{cases} 0, & -\gamma \leq s < 0 \\ I, & s = 0 \end{cases}$$

then $T(t,t_0)X_0(s) = X(t+s,t_0) = X_t(\cdot,t_0)(s)$. Therefore, $T(t,t_0)I = X(t,t_0)$.

Now, let us assume that the functions $h_i(t):[t_0,t_1] \to R,\ i = 0,1,2,...,p$ are twice continuously differentiable and strictly increasing functions in the time interval $[t_0,t_1]$. Moreover, $h_i(t) \leq t$ for $t \in [t_0,t_1]$, and $i = 0,1,2,...,p$. Let us introduce the time-lead functions

$$r_i(t):[h_i(t_0),\ h_i(t_1)] \to [t_0,t_1],\ i = 0,1,2,...,p ,$$

such that $r_i(h_i(t)) = t$ for $t \in [t_0,t_1]$.

Furthermore, only for simplicity and compactness of notation let us assume that $h_0(t) = t$ and for $t = t_1$ the functions $h_i(t)$ satisfy the following inequalities

$$h = h_p(t_1) \le h_{p-1}(t_1) \le \ldots \le h_{m+1}(t_1) \le t_0 =$$

$$h_m(t_1) \le h_{m-1}(t_1) \le \ldots \le h_1(t_1) \le h_0(t_1) = t_1 \tag{2.6}$$

Using the time-lead function and the inequalities (2.6) we have

$$\begin{aligned} x(t_1, t_0, \phi, u, f) = T(t_1, t_0)\phi(0) &+ \sum_{i=0}^{m} \int_{t_0 - h_i}^{t_0} X(t_1, s + r_i) B_i(s + r_i)(s + \dot{r}_i)\eta(s) ds \\ &+ \sum_{i=m+1}^{p} \int_{t_0 - h_i}^{t_1 - h_i} X(t_1, s + r_i) B_i(s + r_i)(s + \dot{r}_i)\eta(s) ds \\ &+ \sum_{i=0}^{m} \int_{t_0}^{t_1} X(t_t, s + r_i) B_i(s + r_i)(s + \dot{r}_i) u(s) ds \\ &+ \int_{t_0}^{t_1} X(t_1, s) \left(\int_{-r}^{0} A(\theta) x(s + \theta) d\theta \right) ds \\ &+ \int_{t_0}^{t_1} X(t_1, s) f\left(s, x_s, u(s)\right) ds \end{aligned} \tag{2.7}$$

where $u(s) = \eta(s)$ for $s \in [t_0 - r, t_0]$.

For brevity, we introduce as in [4], the following notations:

$$\begin{aligned} H(t_1, \eta) = &\sum_{i=0}^{m} \int_{t_0 - h_i}^{t_0} X(t_1, s + r_i) B_i(s + r_i)(s + \dot{r}_i)\eta(s) ds \\ &+ \sum_{i=m+1}^{p} \int_{t_0 - h_i}^{t_1 - h_i} X(t_1, s + r_i) B_i(s + r_i)(s + \dot{r}_i)\eta(s) ds \end{aligned}$$

$$\begin{aligned} q(t_1, \eta) = T(t_1, t_0)\phi(0) + H(t_1, \eta) &+ \int_{t_0}^{t_1} X(t_1, s) f\left(s, x_s, u(s)\right) ds \\ &+ \int_{t_0}^{t_1} X(t_1, s) \left(\int_{-r}^{0} A(\theta) x(s + \theta) d\theta \right) ds, \end{aligned}$$

$$G_i(t,s) = \sum_{j=0}^{i} X(t, s+r_j) B_j(s+r_j)(s+\dot{r}_j).$$

Define the controllability matrix of (2.2) at time t by

$$W(t_0,t) = \int_{t_0}^{t} G_m(t,s) G_m^{\mathrm{T}}(t,s) ds$$

where • denotes matrix transpose.

Definition 2.1: The system (1.8) is said to be null controllable on $[t_0, t_1]$ if for each $\phi \in B\left([-\gamma, 0], R^n\right)$, there exists a $t_1 \geq t_0$, $u \in L_2\left([t_0, t_1], IU\right)$, IU a compact convex subset of R^m, such that the solution $x(t, t_0, \phi, u, f)$ of (1.8) satisfies $x_{t_0}(t_0, \phi, u, f) = \phi$ and $x(t_1, t_0, \phi, u, f) = 0$.

3. Main Results

Theorem 3.1: Assume for system (1.8) that

(i) the constraint set IU is an arbitrary compact subset of R^m,

(ii) the system (2.3) is uniformly asymptotically stable so that the solution of (2.3) satisfies

$$\left\| x(t, t_0, \phi, 0, 0) \right\| \leq M e^{-\alpha(t-t_0)} \left\| \phi \right\| \text{ for some } \alpha > 0,\ M > 0,$$

(iii) the linear control system (2.2) is controllable in R^n,

(iv) the continuous function f satisfies $\left| f\left(t, x(\cdot), u(\cdot)\right) \right| \leq \exp(-\beta t) \pi\left(x(\cdot), u(\cdot)\right)$, for all

$\left(t, x(\cdot), u(\cdot)\right) \in [t_0, \infty) \times B\left([-\gamma, 0], R^n\right) \times L_2\left([t_0, t_1], IU\right)$, where

$$\int_{t_0}^{\infty} \pi\left(x(\cdot), u(\cdot)\right) ds \leq K < \infty \quad \text{and} \quad \beta - \alpha \geq 0,$$

then system (1.8) is Euclidean null controllable.

Proof: By (iii), $W^{-1}(t_0, t_1)$ exists for each $t_1 > t_0$. Suppose the pair of functions x, u form a solution pair to the set of integral equations

$$u(t) = -G_m^{\mathrm{T}}(t_1, t) W^{-1}(t_0, t_1) q(t_1, \eta) \tag{3.1}$$

for some suitably chosen $t_1 \geq t \geq t_0$, $u(t) = \eta(t)$, $t \in [t_0 - \gamma, t_0]$ and

$$x(t) = T(t,t_0)\phi(0) + H(t,\eta) + \int_{t_0}^{t} G_m(t,s)u(s)ds$$

$$+\int_{t_0}^{t} X(t,s)\left(\int_{-\gamma}^{0} A(\theta)x(t+\theta)d\theta\right)ds + \int_{t_0}^{t} X(t,s)f\left(s,x_s,u(s)\right)ds \tag{3.2}$$

$$x(t) = \phi(t), \qquad t \in [t_0 - \gamma, t_0].$$

Then u is square integrable on $[t_0 - \gamma, t_1]$ and x is a solution of (1.8) corresponding to u with initial state $z(t_0) = \left(x(t_0), \phi, \eta\right)$.

Also

$$x(t_1) = T(t_1,t_0)\phi(0) - \int_{t_0}^{t_1} G_m(t_1,s)G_m^T(t_1,s)W^{-1}(t_0,t_1)[T(t_1,t_0)\phi(0) + H(t_1,\eta)$$

$$+\int_{t_0}^{t_1} X(t_1,s)\left(f\left(s,x_s,u(s)\right) + \int_{-\gamma}^{0} A(\theta)x(t+\theta)d\theta\right)ds]ds$$

$$+\int_{t_0}^{t_1} X(t,s)\left(f\left(s,x_s,u(s)\right) + \int_{-\gamma}^{0} A(\theta)x(t+\theta)d\theta\right)ds = 0. \tag{3.3}$$

We now show that $u:[t_0,t_1] \to IU$ is a compact constraint subset of R^m, that is $|u| \leq a$ for some constant $a > 0$. Since (2.3) is uniformly asymptotically stable and B_i are continuous in t, it follows that

$$\left|G_m^{\mathrm{T}}(t_1,t_0)W^{-1}(t_0,t_1)\right| \leq k_1, \text{ for some } k_1 > 0,$$

$$\left|T(t_1,t_0)\phi(0)\right| \leq k_2 \exp\left(-\alpha(t_1 - t_0)\right), \text{ for some } k_2 > 0,$$

$$\left|H(t,\eta)\right| \leq k_3 \exp\left(-\alpha(t_1 - t_0)\right), \text{ for some } k_3 > 0.$$

Hence,

$$|u(t)| \leq k_1\left[k_2 \exp\left(-\alpha(t_1 - t_0)\right) + k_3 + \int_{t_0}^{t_1} M \exp\left(-\alpha(t_1 - s)\right)\exp(-\beta s)\pi\left(x(\cdot),u(\cdot)\right)ds\right],$$

and therefore

$$\left|u(t)\right| \le k_1\left[(k_2+k_3)\exp\left(-\alpha(t_1-t_0)\right)+KM\exp(-\alpha t_1)\right] \tag{3.4}$$

since $\beta-\alpha\ge 0$ and $s\ge t_0\ge 0$. Hence, by taking t_1 sufficiently large, we have $\left|u(t)\right|\le a$, $t\in[t_0,t_1]$ which proves that u is an admissible control.

We now prove the existence of a solution pair of the integral equations (3.1) and (3.2). Let B be the Banach space of all functions

$$(x,u):[t_0-h,t_1]\times[t_0-h,t_1]\to R^n\times R^m$$

where $x\in B\left([t_0-h,t_1],R^n\right)$ and $u\in L_2\left([t_0-h,t_1],R^m\right)$ with the norm defined by

$$\left\|(x,u)\right\|\le\left\|x\right\|_2+\left\|u\right\|_2,$$

where $\left\|x\right\|_2=\left\{\int_{t_0-h}^{t_1}\left|x(s)\right|^2 ds\right\}^{\frac{1}{2}}$, and $\left\|u\right\|_2=\left\{\int_{t_0-h}^{t_1}\left|u(s)\right|^2 ds\right\}^{\frac{1}{2}}$.

Define the operator $T:B\to B$ by $T(x,u)=(y,v)$,

where

$$v(t)=-G_m^{\mathrm{T}}(t_1,t)W^{-1}(t_0,t_1)q(t_1,\eta) \quad \text{for } t\in[t_0,t_1] \tag{3.5}$$

and

$$v(t)=\eta(t) \quad \text{for } t\in[t_0-\gamma,t_0];$$

$$y(t)=T(t,t_0)\phi(0)+H(t,\eta)+\int_{t_0}^{t}G_m(t,s)v(s)ds+\int_{t_0}^{t}X(t,s)\left(\int_{-\gamma}^{0}A(\theta)x(t+\theta)d\theta\right)ds$$

$$+\int_{t_0}^{t}X(t,s)f\left(s,x_s,u(s)\right)ds \quad \text{for } t\in J \tag{3.6}$$

and $y(t)=\phi(t)$ for $t\in[t_0-\gamma,t_0]$. From equation (3.4), we have shown that $\left|v(t)\right|\le a$, $t\in J$ and also $v:[t_0-h,t_0]\to IU$, so $\left|v(t)\right|\le a$. Hence,

$$\|v(t)\|_2 \le a(t_1 + h - t_0)^{\frac{1}{2}} = \beta_0 .$$

Also

$$|y(t)| \le k_2 + k_3 \exp\left(-\alpha(t - t_0)\right) + k_4 \int_{t_0}^{t} |v(s)|\, ds + KM \exp(-\alpha t_1)$$

where $k_4 = \sup|G_m(t,s)|$. Since $\alpha > 0$, $t \ge t_0 \ge 0$, it follows that

$$|y(t)| \le k_2 + k_3 + k_4 a(t_1 - t_0) + KM \equiv \beta, \quad t \in J$$

and

$$|y(t)| \le \sup|\phi(t)| \equiv \delta, \quad t \in [t_0 - h, t_0].$$

Hence, if $\lambda = \max\{\beta, \delta\}$, then

$$\|y\|_2 \le \lambda(t_1 + h - t_0)^{\frac{1}{2}} \equiv \beta_1 < \infty .$$

Let $r = \max\{\beta_0, \beta_1\}$. Then letting

$$Q(r) = \left\{(x,u) \in B : \|x\|_2 \le r, \|u\|_2 \le r\right\},$$

it follows that $T : Q(r) \to Q(r)$. Since $Q(r)$ is closed, bounded and convex, by Riesz's theorem, it is relatively compact under the transformation T. The Schauder fixed point theorem implies that T has a fixed point $(x,u) \in Q(r)$. This fixed point (x,u) of T is a solution pair of the integral equations (3.5), (3.6). Hence, the system (1.8) is Euclidean null controllable.

4. Applications

If we now specialize to the constant systems with multiple delays in the control defined by

$$\frac{d}{dt}\left(x(t) - A_{-1}x(t-h)\right) = A_0 x(t) + A_1 x(t-h) + B_0 u(t) + B_1 u(t-h) \tag{4.1}$$

$$\frac{d}{dt}\left(x(t) - A_{-1}x(t-h)\right) = A_0 x(t) + A_1 x(t-h) + B_0 u(t) + B_1 (t-h) + C_0 \int_{-\infty}^{0} \exp(\eta\theta) x(t+\theta)\, d\theta \tag{4.2}$$

$$\frac{d}{dt}\left(x(t)-A_{-1}x(t-h)\right)=A_0x(t)+A_1x(t-h)+B_0u(t)+B_1u(t-h)$$

$$+C_0\int_{-\infty}^{0}\exp(\eta\theta)x(t+\theta)d\theta+f\left(t,x(t),x(t-h),u(t)\right) \qquad (4.3)$$

then the following results follow:

Theorem 4.1: If $\operatorname{rank}[B_0, A_0B_0]=n$, then system (4.1) is completely controllable on $[t_0,t_1]$.

Proof: This is equivalent to Theorem 2 of Gahl [9].

Theorem 4.2: In system (4.2), assume that

(i) (4.2) with $u=0$ is uniformly asymptotically stable,

(ii) $\operatorname{rank}[B_0, A_0B_0]=n$,

then system (4.2) is null controllable with constraints.

Proof: By (ii), (4.1) is completely controllable. Hence (i) and (ii) satisfy the requirements of Theorem 3.1 and the proof is complete.

Theorem 4.3: For system (4.3), assume that

(i) f satisfies all smoothness conditions for the existence and uniqueness of solutions,

(ii) the zero solution of (4.2) with $u=0$ is uniformly asymptotically stable,

(iii) $\operatorname{rank}[B_0, A_0B_0]=n$,

(iv) $f(t,0,0,0)=0$,

then system (4.3) is null controllable with constraints.

Proof: Immediate from Theorems 3.1 and 4.1.

Example

Consider the system

$$\frac{d}{dt}\left(x(t)-A_{-1}x(t-h)\right)=A_0x(t)+A_1x(t-h)+B_0u(t)+B_1u(t-h)$$

$$+C_0\int_{-\infty}^{0}\exp(\eta\theta)x(t+\theta)d\theta + f\left(t, x(t), x(t-h), u(t)\right) \tag{4.4}$$

where

$$A_{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ A_0=\begin{pmatrix}-1 & 1\\ 1 & -2\end{pmatrix},\ A_1=\begin{pmatrix}0 & 3\\ 0 & -1\end{pmatrix},\ B_0=\begin{pmatrix}0\\ 1\end{pmatrix},\ B_1=\begin{pmatrix}-1\\ 0\end{pmatrix},\ C_0=\begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix}$$

with

$$f\left(t, x(t), x(t-h), u(t)\right)=\begin{pmatrix}0\\ e^{-t}\sin\left(x(t)+x(t-h)\right)\cos u(t)\end{pmatrix}.$$

The characteristic roots of the homogeneous equation

$$\frac{d}{dt}\left(x(t)-A_{-1}x(t-h)\right)=A_0x(t)+A_1x(t-h)+\int_{-\infty}^{0}\exp(\eta\theta)x(t+\theta)d\theta \tag{4.5}$$

is

$$\lambda^2+3\lambda+1+(3\lambda-\lambda^2)e^{-2\lambda}+(2-3\lambda)e^{-\lambda}+(\lambda+1)\int_{-\infty}^{0}\exp[(\lambda+\eta)\theta]d\theta=0 \tag{4.6}$$

Every root of (4.6) has negative real part. Hence, by Theorem 1 of Sinha [15], system (4.5) is uniformly asymptotically stable.

We now show that the linear base system

$$\frac{d}{dt}\left(x(t)-A_{-1}x(t-h)\right)=A_0x(t)+A_1x(t-h)+B_0u(t)+B_1u(t-h) \tag{4.7}$$

is controllable on any interval $[0,t]$, $t>0$. By Theorem 4.1, we show that rank$[B_0, A_0B_0]=n$.

But

$$\text{rank}\,[B_0, A_0B_0]=\text{rank}\begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}=\text{rank}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}=2=n\,.$$

Since rank$[B_0, A_0B_0]=2$ for each $t>0$, the system (4.7) is controllable on each $[0,t]$, $t>0$ on R^n. We conclude that system (4.4) is null controllable, by Theorem 3.1, since

$$\left|f\left(t,x(t),x(t-h),0\right)\right|\leq\left|e^{-t}\sin\left(x(t)+x(t-h)\right)\right|\leq e^{-t}\equiv\pi(t).$$

Conclusion

We have derived sufficient conditions for the null controllability of nonlinear infinite neutral systems with time varying multiple delays in control. These conditions are given with respect to the uniform asymptotic stability of the free linear base system and the controllability of the linear controllable base system, with the assumption that the perturbation function f satisfies some smoothness and growth conditions.

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,523-525,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

Moments of the Scaled Burr Type X Distribution

by

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Abstract: A recent paper by Surles and Padgett proposed the scaled Burr type X distribution and discussed various properties, including moments. The paper claimed that closed-form expressions for $E(X^k)$ are possible only for certain special cases: when the parameter of the distribution is assumed to be an integer or when $k = 2$ (the latter represented as an infinite sum). In this note, we show that one can derive simple expressions for $E(X^k)$ for all even $k \geq 2$ without any restriction on the parameter of the distribution. The expressions only involve the gamma function and its derivatives.

1 Introduction

Surles and Padgett (2005) defined the scaled Burr X distribution with shape parameter θ and scale parameter σ by the cdf

$$F(x) = \left[1 - \exp \left\{ - \left(\frac{x}{\sigma} \right)^2 \right\} \right]^\theta \tag{1}$$

for $x > 0$, $\theta > 0$ and $\sigma > 0$. Surles and Padgett (2005) discussed various properties of this distribution, including moments and their approximations, maximum likelihood estimators and their asymptotic properties as well as types I and II censoring. This distribution is a particular case of the exponentiated Weibull distribution introduced by Mudholkar *et al.* (1995); see Mudholkar and Hutson (1996), Nassar and Eissa (2003) and Nadarajah and Gupta (2005) for more recent developments.

This note concerns the moment properties of a random variable X having the cdf (1). Surles and Padgett (2005) claim that 'closed-form expressions for the moments only exist for certain special cases ...." In particular, two closed-form expressions are given:

$$E\left(X^k\right) = \sigma^k \theta \Gamma \left(\frac{k}{2} + 1 \right) \sum_{j=0}^{\theta-1} (-1)^j \binom{\theta-1}{j} \frac{1}{(j+1)^{k/2+1}}$$

applicable when $\theta \geq 1$ is an integer; and,

$$E\left(X^2\right) = \theta \sigma^2 \sum_{i=0}^{\infty} \frac{1}{i(\theta+i)}$$

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applicable for any $\theta > 0$. In this note, we show that one can derive simple expressions for $E(X^k)$ for any even $k \geq 2$ and for any $\theta > 0$. The expressions only involve the gamma function and its derivatives.

2 Moments

Theorem 1 derives the expression for the kth moment for any even $k \geq 2$ and for any $\theta > 0$.

Theorem 1 *If X is a random variable with the cdf (1) then*

$$E\left(X^k\right) = \theta\Gamma(\theta)\sigma^k(-1)^{k/2}\frac{\partial^{k/2}}{\partial\beta^{k/2}}\left\{\frac{\Gamma(\beta+1)}{\Gamma(\theta+\beta+1)}\right\}\Bigg|_{\beta=0} \tag{2}$$

for any even $k \geq 2$ and for any $\theta > 0$.

Proof: The pdf corresponding to (1) is:

$$f(x) = \frac{2\theta x}{\sigma^2}\exp\left\{-\left(\frac{x}{\sigma}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{x}{\sigma}\right)^2\right\}\right]^{\theta-1}$$

and so one can express $E(X^k)$ as

$$\begin{aligned} E\left(X^k\right) &= \frac{2\theta}{\sigma^2}\int_0^\infty x^{k+1}\exp\left\{-\left(\frac{x}{\sigma}\right)^2\right\}\left[1-\exp\left\{-\left(\frac{x}{\sigma}\right)^2\right\}\right]^{\theta-1}dx \\ &= (-1)^{k/2}\sigma^k\theta\int_0^1 y^{\theta-1}\left\{\log(1-y)\right\}^{k/2}dy, \end{aligned} \tag{3}$$

which follows by substituting $y = 1-\exp\{-(x/\theta)^2\}$. The result in (2) follows by applying equation (2.6.9.5) in Prudnikov *et al* (1986, volume 1) to calculate the integral in (3). ■

The result in (2) can be used to derive simple expressions for moments of even-order. Corollary 1 illustrates this for the first five even-order moments.

Corollary 1 *If X is a random variable with the cdf (1) then the first five even-order moments are given by*

$$\begin{aligned} E\left(X^2\right) &= \sigma^2\Big[\gamma+\Psi(\theta+1)\Big], \\ E\left(X^4\right) &= (1/6)\sigma^4\Big[\pi^2-6\Psi'(\theta+1)+6\gamma^2+12\gamma\Psi(\theta+1)+6\Psi^2(\theta+1)\Big], \\ E\left(X^6\right) &= (1/2)\sigma^6\Big[4\zeta(3)+2\Psi''(\theta+1)+\pi^2\gamma+\pi^2\Psi(\theta+1)-6\Psi'(\theta+1)\gamma \\ &\quad -6\Psi'(\theta+1)\Psi(\theta+1)+2\gamma^3+6\gamma^2\Psi(\theta+1)+6\gamma\Psi^2(\theta+1)+2\Psi^3(\theta+1)\Big], \\ E\left(X^8\right) &= (1/20)\sigma^8\Big[3\pi^4-20\Psi(3\theta+1)+160\zeta(3)\gamma+160\zeta(3)\Psi(\theta+1)+80\Psi''(\theta+1)\gamma \end{aligned}$$

$$
\begin{aligned}
& +80\Psi^{''}(\theta+1)\Psi(\theta+1) - 20\pi^2\Psi^{'}(\theta+1) + 60\left\{\Psi^{'}(\theta+1)\right\}^2 + 20\pi^2\gamma^2 \\
& +40\pi^2\gamma\Psi(\theta+1) + 20\pi^2\Psi^2(\theta+1) - 120\Psi^{'}(\theta+1)\gamma^2 \\
& -240\Psi^{'}(\theta+1)\gamma\Psi(\theta+1) - 120\Psi^{'}(\theta+1)\Psi^2(\theta+1) + 20\gamma^4 + 80\gamma^3\Psi(\theta+1) \\
& \left. +120\gamma^2\Psi^2(\theta+1) + 80\gamma\Psi^3(\theta+1) + 20\Psi^4(\theta+1)\right],
\end{aligned}
$$

$$
\begin{aligned}
E\left(X^{10}\right) = (1/12)\sigma^{10} & \left[60\pi^2\gamma^2\Psi(\theta+1) - 120\Psi^{''}(\theta+1)\Psi^{'}(\theta+1) + 240\zeta(3)\gamma^2\right. \\
& -60\pi^2\Psi^{'}(\theta+1)\Psi(\theta+1) + 60\pi^2\gamma\Psi^2(\theta+1) + 180\left\{\Psi^{'}(\theta+1)\right\}^2\Psi(\theta+1) \\
& +20\pi^2\gamma^3 - 60\pi^2\Psi^{'}(\theta+1)\gamma + 120\Psi^{''}(\theta+1)\Psi^2(\theta+1) + 180\left\{\Psi^{'}(\theta+1)\right\}^2\gamma \\
& +480\zeta(3)\gamma\Psi(\theta+1) + 60\gamma^4\Psi(\theta+1) + 120\gamma^3\Psi^2(\theta+1) + 120\gamma^2\Psi^3(\theta+1) \\
& +60\gamma\Psi^4(\theta+1) + 9\pi^4\gamma + 9\pi^4\Psi(\theta+1) + 40\zeta(3)\pi^2 - 240\zeta(3)\Psi^{'}(\theta+1) \\
& -360\Psi^{'}(\theta+1)\gamma^2\Psi(\theta+1) + 240\zeta(3)\Psi^2(\theta+1) + 120\Psi^{''}(\theta+1)\gamma^2 \\
& +240\Psi^{''}(\theta+1)\gamma\Psi(\theta+1) + 20\pi^2\Psi^3(\theta+1) - 120\Psi^{'}(\theta+1)\gamma^3 \\
& -120\Psi^{'}(\theta+1)\Psi^3(\theta+1) + 12\Psi^5(\theta+1) + 12\gamma^5 - 60\Psi^{'''}(\theta+1)\gamma \\
& -60\Psi^{'''}(\theta+1)\Psi(\theta+1) + 20\Psi^{''}(\theta+1)\pi^2 - 360\Psi^{'}(\theta+1)\gamma\Psi^2(\theta+1) \\
& \left. +12\Psi^{''''}(\theta+1) + 288\zeta(5)\right],
\end{aligned}
$$

where $\gamma = 0.5772156649\cdots$ *is Euler's constant,* $\Psi(x) = d\log\Gamma(x)/dx$ *is the digamma function and* $\zeta(x) = \sum_{k=1}^{\infty} k^{-x}$ *is the zeta function.*

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JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.10,NO.4,527-535,2008, COPYRIGHT 2008 EUDOXUS PRESS, LLC

On the Rate of Approximation of Meyer-König and Zeller Operators

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Abstract

In this paper the asymptotic property of Meyer-König and Zeller operators M_n for bounded functions on $[0,1]$ is studied. An asymptotic convergence theorem of this type approximation is established by means of some probabilistic methods and results and accurate estimate technique to the basis functions of the operators M_n. The main result of this paper subsumes the approximation of the operators M_n for functions of bounded variation as a special case.

2000 **Mathematics Subject Classification:** 41A10, 41A36, 41A25.

Keywords: Rate of approximation, Meyer-König and Zeller operators, Probability distribution, Basis functions.

1 INTRODUCTION

For a function f defined on [0, 1], Meyer-König and Zeller operator M_n [10] is defined by

$$M_n(f,x) = \sum_{k=0}^{\infty} f(\frac{k}{n+k}) m_{n,k}(x), \quad 0 \le x < 1,$$

$$M_n(f,1) = f(1), \quad m_{n,k}(x) = \binom{n+k-1}{k} x^k (1-x)^n. \tag{1}$$

If replacing the basis function $m_{n,k}(x)$ in definition (1) with the new basis function $\hat{m}_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}$, one get a modified version of M_n, which belongs to Cheney and Sharma [4]. The asymptotic convergence properties of Bernstein type operators for bounded functions have been studied in [11]. In

this paper we study the asymptotic convergence property of the operators M_n for bounded functions on $[0,1]$. By means of some probabilistic methods and results and accurate estimate technique to the basis functions $m_{n,k}$, we establish an asymptotic convergence theorem of this type approximation. Our investigation subsumes the approximation of operators M_n for functions of bounded variation as a special case.

The following three quantities were first introduced in [11]. For their basic properties, one refers to [11].

$$\Omega_{x-}(f,\delta_1)=\sup_{t\in[x-\delta_1,x]}|f(t)-f(x)|,\qquad \Omega_{x+}(f,\delta_2)=\sup_{t\in[x,x+\delta_2]}|f(t)-f(x)|,$$
$$\Omega(x,f,\lambda)=\sup_{t\in[x-x/\lambda,x+(1-x)/\lambda]}|f(t)-f(x)|,$$

where $f\in I$, $x\in[0,1]$ is fixed, $0\le\delta_1\le x$, $0\le\delta_2\le 1-x$, and $\lambda\ge 1$.

The following example shows that in the case of approximation of functions of bounded variation, the above quantities may give better asymptotic estimate than using the total variation of function of bounded variation.

Example 1. Consider the function $f_0(x)=\begin{cases} x^2sin(\pi/x), & x\in(0,1]\\ 0, & x=0\end{cases}$.
$f_0(x)$ is bounded variation on [0, 1] by the boundedness of $f_0'(x)$. On the interval $[0,n^{-1}]$ taking points:

$$\frac{1}{n}>\frac{1}{n+1/2}>\frac{1}{n+1}>\frac{1}{(n+1)+1/2}>...>\frac{1}{n+n}>\frac{1}{(n+n)+1/2}>0.$$

It is easy to observe that

$$\bigvee_0^{1/n}(f_0)\ge\left(\frac{1}{n+1/2}\right)^2+\left(\frac{1}{(n+1)+1/2}\right)^2+...+\left(\frac{1}{(n+n)+1/2}\right)^2$$
$$>(n+1)\left(\frac{1}{2n+1/2}\right)^2>(4n)^{-1},$$

and obviously, $\Omega_{0+}(f_0,n^{-1})\le n^{-2}$.

The main result of this paper is as follows:

Theorem 1. *Let f be bounded on [0, 1], $f(x+)$ and $f(x-)$ exsit at a fixed point $x\in(0,1)$ and $r=x/(1-x)$. Then for $n\ge 2$ we have*

$$\left|M_n(f,x)-\frac{f(x+)+f(x-)}{2}-\frac{A_{f,n,x}}{3\sqrt{2\pi xn}}\right|\le\frac{4}{nx(1-x)}\sum_{k=1}^{n}\Omega(x,g_x,\sqrt{k})+O(n^{-1}),\tag{2}$$

where $g_x(t)$ is defined by

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leq t < x. \end{cases} \tag{3}$$

and

$$A_{f,n,x} = \begin{cases} 3(1-x)f(x) + (2x-1)f(x+) + (x-2)f(x-), & nr = [nr] \\ (2x - 1 + 3(1-x)(nr - [nr]))(f(x+) - f(x-)), & nr \neq [nr] \end{cases}, \tag{4}$$

in (4), $[nr]$ *denotes the greatest integer not exceeding* nr.

From Theorem 1 we get an interesting asymptotic formula as follows.

Corollary 1. *Under the conditions of Theorem* 1, *if* $\Omega(x, g_x, \lambda) = o(\lambda^{-1})$, *then we have the following asymptotic formula*

$$M_n(f,x) = \frac{f(x+) + f(x-)}{2} + \frac{A_{f,n,x}}{3\sqrt{2\pi xn}} + o(n^{-1/2}). \tag{5}$$

We point out that approximation of functions of bounded variation is the spacial case of Theorem 1. From Theorem 1 we get immediately

Corollary 2. *Let f be a function of bounded variation on* $[0,\ 1]$, *and let* $\bigvee_a^b(f)$ *denote the total variation of f on* $[a,b]$, $x \in (0,1)$ *and* $r = x/(1-x)$. *Then for* $n \geq 2$ *we have*

$$\left| M_n(f,x) - \frac{f(x+) + f(x-)}{2} - \frac{A_{f,n,x}}{3\sqrt{2\pi xn}} \right| \leq \frac{4}{nx(1-x)} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}) + O(n^{-1})$$

$$\leq \frac{4}{nx(1-x)} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) + O(n^{-1}), \tag{6}$$

where $A_{f,n,x}$ and $g_x(t)$ are defined as in Theorem 1.

2 A SET OF LEMMAS

Each of the following four lemmas will be required in the proof of Theorem 1.

Lemma 1. *For* $n \geq 2$, $x \in [0,1]$ *there holds*

$$M_n((t-x)^2, x) \leq \frac{2x(1-x)}{n}. \tag{7}$$

Proof. By [2, Lemma 2.1] and simple calculation, for $n \geq 2$ there holds

$$\Delta = \sum_{k=0}^{\infty} \left(\frac{k}{n+k} - x \right)^2 \frac{(n+k)!}{k!n!} x^k (1-x)^{n+1} \leq \left(1 + \frac{2x}{n-1} \right) \frac{x(1-x)^2}{n+1}.$$

Thus

$$M_n((t-x)^2, x) \leq \frac{\Delta}{1-x} \leq \frac{2x(1-x)}{n}.$$

Using Bojanic-Cheng-Khan's method [3, 5, 9] and Lemma 1 we obtain

Lemma 2. *For* $g_x(t)$ *defined in* (3) *we have*

$$|M_n(g_x, x)| \leq \frac{4}{nx(1-x)} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}). \tag{8}$$

Because the method of proof of Lemma 2 is well known (cf. [3, 5, 7, 9, 12]), we here omit the details of the proof.

Lemma 3. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of independent and identically distributed random variables with the expectation* $E(\xi_1) = a_1$, *the variance* $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E(\xi_1 - a_1)^4 < \infty$, *and let* F_n *stand for the distribution function of* $\sum_{k=1}^{n} (\xi_k - a_1)/\sigma\sqrt{n}$. *If* F_n *is a lattice distribution and* F_n^* *is a polygonal approximation of* F_n (see the following Definition 1), *then the following equation holds for all* $t \in (-\infty, +\infty)$

$$F_n^*(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du - \frac{E(\xi_1 - a_1)^3}{6\sigma^3 \sqrt{n}} (1-t^2) \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = O(n^{-1}). \tag{9}$$

The proof of Lemma 3 can be found in [6, p. 540-542].

Definition 1 ([6, p. 540, Definition]). *Let* F *be concentrated on the lattice of points* $b \pm nh$, *but on no sublattice* (*that is,* h *is the span of* F). *A polygonal approximation* F^* *to* F *is a distribution function with a polygonal graph with vertices at the midpoints* $b \pm (n + 1/2)h$ *lying on the graph of* F. *Thus*

$$F^*(t) = F(t) \qquad if\ t = b \pm (n + 1/2)h; \tag{10}$$

$$F^*(t) = 1/2[F(t) + F(t-)] \qquad if\ t = b \pm nh. \tag{11}$$

The following Lamma 4 is an accurate estimate technique to the basis functions of the operators M_n, which is a key auxiliary result in the proof of Theorem 1.

Lemma 4. *For* $x \in (0,1)$, $r = x/(1-x)$, *we have*

$$m_{n,[nr]}(x) = \frac{1-x}{\sqrt{2\pi xn}} + O\left((nx)^{-3/2}\right), \tag{12}$$

and

$$m_{n,[nr]+1}(x) = \frac{1-x}{\sqrt{2\pi xn}} + O\left((nx)^{-3/2}\right). \tag{13}$$

Proof. We first show that

$$\left(\frac{nr}{[nr]}\right)^{[nr]+1/2} \left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]-1/2} = 1 + O([nr]^{-1}). \tag{14}$$

Set $W_1(n,r) = \left(\frac{nr}{[nr]}\right)^{[nr]+1/2} \left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]-1/2}$, and write $nr = [nr] + \varepsilon$ $(0 \le \varepsilon < 1)$, then

$$W_1(n,r) = \left(1 + \frac{\varepsilon}{[nr]}\right)^{[nr]+1/2} \left(1 + \frac{\varepsilon}{n+[nr]}\right)^{-(n+[nr]-1/2)}.$$

Thus

$$\log W_1(n,r) = ([nr]+1/2)\log\left(1+\frac{\varepsilon}{[nr]}\right) - ([n+[nr]-1/2)\log\left(1+\frac{\varepsilon}{n+[nr]}\right)$$

$$= ([nr]+1/2)\left(\frac{\varepsilon}{[nr]} + O\left(\frac{\varepsilon}{[nr]}\right)^2\right) - ([n+[nr]-1/2)\left(\frac{\varepsilon}{n+[nr]} + O\left(\frac{\varepsilon}{n+[nr]}\right)^2\right)$$

$$= O([nr]^{-1}),$$

which implies that

$$W_1(n,r) = 1 + O([nr]^{-1}).$$

Using Stirling's formula:

$$n! = (n/e)^n \sqrt{2\pi n} e^{\theta_n/12n},\ 0 < \theta_n < 1,$$

and by direct calculations we find that

$$\begin{aligned}\frac{\sqrt{2\pi nx}}{1-x}m_{n,[nr]}(x) &= \frac{\sqrt{2\pi nx}}{1-x}\frac{(n+[nr]-1)!}{[nr]!(n-1)!}x^{[nr]}(1-x)^n\\ &= \frac{\sqrt{2\pi nx}}{1-x}\frac{n}{n+[nr]}\frac{(n+[nr])!}{[nr]!n!}x^{[nr]}(1-x)^n\\ &= \frac{(n+[nr])^{n+[nr]-1/2}}{[nr]^{[nr]+1/2}n^{n-1}}x^{[nr]+1/2}(1-x)^{n-1}e^{c(x,n)},\\ &= \left(\frac{nr}{[nr]}\right)^{[nr]+1/2}\left(\frac{n+[nr]}{n+nr}\right)^{n+[nr]-1/2}e^{c(x,n)},\end{aligned}$$

where

$$-\frac{1}{12n}-\frac{1}{12[nr]}\le c(x,n)\le\frac{1}{12n}.$$

Thus, it follows from Eq. (12) that

$$\frac{\sqrt{2\pi nx}}{1-x}m_{n,[nr]}(x)=1+O([nr]^{-1}),$$

which derives the estimation (14). Furthermore, note that

$$m_{n,[nr]+1}(x)-m_{n,[nr]}(x)=m_{n,[nr]}(x)\left(\frac{n+[nr]}{[nr]+1}x-1\right),$$

and since $nr=[nr]+\varepsilon$ $(0\le\varepsilon<1)$, then

$$\frac{n+[nr]}{[nr]+1}x-1=\frac{(1-x)([nr]+\varepsilon)+[nr]x-[nr]-1}{[nr]+1}=\frac{\varepsilon-\varepsilon x-1}{[nr]+1},$$

that is

$$m_{n,[nr]+1}(x)=m_{n,[nr]}(x)\left(\frac{\varepsilon-\varepsilon x-1}{[nr]+1}+1\right).$$

Thus, we get (13) directly from (12). The proof of Lemma 4 is completed.

3 PROOF OF MAIN RESULT

Proof of Theorem 1. For any $f\in I_B$, if $f(x+)$ and $f(x-)$ exist at x, by Bojanic-Cheng decomposition it follows that

$$M_n(f,x)-\frac{f(x+)+f(x-)}{2}=\frac{f(x+)-f(x-)}{2}M_n(sgn_x,x)$$

$$+\frac{2f(x)-f(x+)-f(x-)}{2}M_n(\delta_x,x)+M_n(g_x,x). \tag{15}$$

where $g_x(t)$ is defined in (3), $sgn_x(t)=\begin{cases}1, & t>x\\ 0, & t=x\\ -1, & t<x\end{cases}$, and $\delta_x(t)=\begin{cases}1, & t=x\\ 0, & t\neq x\end{cases}$.

We need to estimate every term in the right side of (15). The term $M_n(g_x,x)$ has been estimated in Lemma 2. Let $r=x/(1-x)$. Direct calculation gives

$$M_n(\delta_x,x)=\begin{cases}m_{n,[nr]}(x), & nr=[nr]\\ 0, & nr\neq[nr]\end{cases}. \tag{16}$$

Below we estimate $M_n(sgn_x,x)$.

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same geometric distribution $P(\xi_i=k)=x^k(1-x),\ k=0,1,2,\cdots,$ and $x\in(0,1)$ is a parameter. Direct computations give

$$E\xi_1=\frac{x}{1-x},\quad E(\xi_1-E\xi_1)^2=\frac{x}{(1-x)^2},$$

$$E(\xi_1-E\xi_1)^3=\frac{x^2+x}{(1-x)^3},\quad E(\xi_1-E\xi_1)^4=\frac{x^3+7x^2+x}{(1-x)^4}<\infty.$$

Let $\eta_n=\sum\limits_{i=1}^{n}\xi_i$ and F_n stand for the distribution function of $\sum\limits_{i=1}^{n}(\xi_i-E\xi_1)/\sigma\sqrt{n}$. Then the probability distribution of the random variable η_n is

$$P(\eta_n=k)=\begin{pmatrix}n+k-1\\ k\end{pmatrix}x^k(1-x)^n=m_{n,k}(x).$$

Thus

$$\begin{aligned}M_n(sgn_x,x)&=-\sum_{k<nx}m_{n,k}(x)+\sum_{k>nx}m_{n,k}(x)\\ &=1-\sum_{k<nx}m_{n,k}(x)-\sum_{k\le nx}m_{n,k}(x)\\ &=1-P(\eta_n<nx)-P(\eta_n\le nx)\\ &=1-F_n(0-)-F_n(0)\end{aligned} \tag{17}$$

On the other hand, for $F_n^*(t)$, the polygonal approximation of $F_n(t)$, from (10), (11), we obtain

$$F_n^*(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{-u^2/2}du+\frac{E(\xi_1-a_1)^3}{6\sigma^3\sqrt{n}}\frac{1}{\sqrt{2\pi}}+O(n^{-1}).$$

$$= \frac{1}{2} + \frac{1+x}{6\sqrt{2\pi x n}} + O(n^{-1}). \tag{18}$$

We need to estimate the terms $F_n^*(0) - F_n(0)$ and $F_n^*(0) - F_n(0-)$, write $r = x/(1-x)$. If nr just is a natural number, i.e $nr = [nr]$, then 0 is a lattice point of F_n. From (11) we get

$$2F_n^*(0) - F_n(0-) - F_n(0) = 0. \tag{19}$$

If $nr \neq [nr]$, then

$$F_n(0) = F_n(0-) = \sum_{k \leq [nr]} m_{n,k}(x),$$

which implies that $F_n(t) = \sum\limits_{k \leq [nr]} m_{n,k}(x)$ on the interval $\left[-\frac{nr-[nr]}{\sigma\sqrt{n}}, \frac{1+[nr]-nr}{\sigma\sqrt{n}}\right)$, since $F_n(t)$ is a step function.

We need to compute $F_n^*(0)$. If $0 < nr - [nr] \leq 1/2$, from (10) and (11) it is known that

$$F_n^*\left(-\frac{nr-[nr]}{\sigma\sqrt{n}}\right) = \frac{1}{2}\left(\sum_{k \leq [nr]-1} m_{nk}(x) + \sum_{k \leq [nr]} m_{nk}(x)\right),$$

and

$$F_n^*\left(\frac{[nr]-nr+1/2}{\sigma\sqrt{n}}\right) = \sum_{k \leq [nr]} m_{nk}(x).$$

Since $F_n^*(t)$ is a polygonal approximation of $F_n(t)$, by a simple calculation we get

$$F_n^*(0) = \sum_{k \leq [nr]} m_{nk}(x) + (nr - [nr] - 1/2)m_{n,[nr]}(x).$$

Thus for $0 < nr - [nr] < 1/2$

$$F_n^*(0) - F_n(0) = F_n^*(0) - F_n(0-) = (nr - [nr] - 1/2)m_{n,[nr]}(x). \tag{20}$$

Similarly, for $1/2 < nr - [nr] < 1$

$$F_n^*(0) - F_n(0) = F_n^*(0) - F_n(0-) = (nr - [nr] - 1/2)m_{n,[nr]+1}(x). \tag{21}$$

Collecting the above estimates (17)–(21) and by means of Lemma 4 and some simple computations we obtain

$$M_n(sgn_x, x) = \begin{cases} -\dfrac{1+x}{3\sqrt{2\pi x n}} + O(n^{-1}), & nr = [nr] \\[2ex] \dfrac{(6nr - 6[nr] - 2)(1-x) - 2}{3\sqrt{2\pi x n}} + O(n^{-1}), & nr \neq [nr]. \end{cases} \tag{22}$$

Theorem 1 now follows from (15), (16), (22) and Lemma 2 with the simple calculation.

Acknowledgement

The present investigation was supported by NSFC under Grant 10571145.

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